Chin. Ann. of Math.
22B: 4(2000),529-540.

NEUMANN PROBLEM FOR THE LANDAU-LIFSHITZ-MAXWELL SYSTEM IN TWO DIMENSIONS

GUO BOLING* DING SHIJIN*

Abstract

This paper is concerned with the global existence and the partial regularity for the weak solution of the Landau-Lifshitz-Maxell system in two dimensions with Neumann boundary conditions.

Keywords Landau-Lifshitz-Maxwell system, Neumann problem, Global existence
1991 MR Subject Classification 35K90, 35Q60
Chinese Library Classification O175.26 **Document Code** A
Article ID 0252-9599(2001)04-0529-12

§1. Introduction

In 1935, Landau-Lifshitz[1] proposed the following coupled system of the nonlinear evolution equation

$$\vec{Z}_T = -\alpha_1 \vec{Z} \times (\vec{Z} \times (\triangle \vec{Z} + \vec{H})) + \alpha_2 \vec{Z} \times (\triangle \vec{Z} + \vec{H}), \tag{1.1}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E}, \tag{1.2}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{H}}{\partial t} - \beta \frac{\partial \vec{Z}}{\partial t}, \tag{1.3}$$

$$\nabla \cdot \vec{H} + \beta \nabla \cdot \vec{Z} = 0, \quad \nabla \cdot \vec{E} = 0, \tag{1.4}$$

where $\alpha_1,\ \alpha_2,\ \sigma,\ \beta$ are constants, $\alpha_1 \geq 0,\ \sigma \geq 0,\ \vec{Z}(x,t) = (Z_1(x,t),\ Z_2(x,t),\ Z_3(x,t))$ denotes the microscopic magnetization field, $\vec{H} = (H_1(x,t),\ H_2(x,t),\ H_3(x,t))$ the magnetic field, $\vec{E}(x,t) = (E_1(x,t),\ E_2(x,t),\ E_3(x,t))$ the electric field, $\vec{H}^e = \triangle \vec{Z} + H$ the effective magnetic field, $\triangle = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $\nabla = (\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial x_2},\ \cdots,\ \frac{\partial}{\partial x_n})$, "$\times$" the cross product of the vector in R^3.

If $\vec{H} = 0,\ \vec{E} = 0$, we obtain the Landau-Lifshitz system with Gilbert term

$$\vec{Z}_t = -\alpha_1 \vec{Z} \times (\vec{Z} \times \triangle \vec{Z}) + \alpha_2 \vec{Z} \times \triangle \vec{Z}, \tag{1.5}$$

Manuscript received September 21, 1999.

*Center of Nonlinear Studies, Institute of Applied Physics and Computational Mathematics, P. O. Box 8009, Beijing 100088, China. **E-mail:** gbl@ mail.iapcm.ac.cn

where α_1 is the Gilbert damping coefficient. In [2–4], the properties of the solution for system (1.5) and the links between the solution and the harmonic map on the compact Riemann manifold have been studied extensively. When $\alpha_1 = 0$, the system (1.5) becomes

$$\vec{Z}_t = \alpha_2 \vec{Z} \times \triangle \vec{Z}. \tag{1.6}$$

In the case of $n = 1$, it is an integral system and has soliton solution. In [5–13], the authors have studied in detail the solutions for (1.6), the interaction among solitons, the infinite conservative laws, the inverse scattering method, and the relation with the nonlinear Schrödinger equations. As pointed out in [20], the system (1.6) is strongly coupled degenerate quasilinear parabolic system. In [14–22], the authors have investigated extensively the classical and generalized solutions to the initial value problem and other kinds of boundary value problem for the system (1.6). Some properties of the solutions and the existence of global generalized solutions for $n \geq 2$ were obtained.

In [23], the authors considered the following problem

$$\frac{1}{2}\vec{Z}_t - \frac{1}{2}\vec{Z} \times \vec{Z}_t = \triangle \vec{Z} + \vec{Z}|\nabla \vec{Z}|^2 + \vec{Z} \times (\vec{Z} \times f(\vec{Z},\ x,\ t)) \tag{1.7}$$

with the conditions

$$\vec{Z}(x,0) = \vec{Z}_0(x), \quad \vec{Z}|_{\partial\Omega} = Z_0(x)|_{\partial\Omega}, \quad |\vec{Z}_0(x)| = 1, \tag{1.8}$$

and proved that if $\vec{Z}_0(x) \in C^{2,\alpha}(\bar{\Omega})$, then problem (1.7)–(1.8) admits unique solution in $C_{loc}^{2+\alpha,\ 1+\alpha/2}(\bar{\Omega} \times [0,\infty) \setminus A)$, where A is a set consisting of only countably many lines under some assumptions on $f(p,\ x,\ t)$. If $f(p,\ x,\ t) \equiv 0$, then A consists of only finitely many points [24].

In [25,26] the existence and uniqueness of the global smooth solution for the periodic initial value problem and initial value problem of the Landau-Lifshitz-Maxwell system (1.1)–(1.4) (with or without dissipation) in one and two space dimensions are proved (when $N = 2$, the initial data is assumed small).

In [27], Guo and Su studied the existence of the global generalized solutions for the 3-dimensional Landau-Lifshitz-Maxwell system (1.1)–(1.4) with the periodic initial value condition or the initial value condition.

However, there has been little discussion of the boundary value problem so far.

In this paper, we shall study the two dimensional Landau-Lifshitz-Maxwell system (1.1)–(1.4) with the initial value condition:

$$\vec{Z}(x,0) = \vec{Z}_0(x), \quad \vec{H}(x,0) = \vec{H}_0(x), \quad \vec{E}(x,0) = \vec{E}_0(x) \quad (x \in \Omega \subset R^2) \tag{1.9}$$

and Neumann boundary conditions:

$$\frac{\partial \vec{Z}}{\partial \nu}|_{\partial\Omega} = 0, \quad \vec{H} \cdot \nu|_{\partial\Omega} = 0, \quad \vec{E} \times \nu|_{\partial\Omega} = 0, \tag{1.10}$$

which indicates that the energy current vanishes on the boundary, where $\Omega \subset R^2$ is a boundary smooth domain, ν is the unit outer normal vector to $\partial\Omega$, and $\vec{Z}_0 \in \vec{H}^1(\Omega;\ S^2)$, $\vec{E}_0 \in L^2(\Omega)$, $\vec{H}_0 \in L^2(\Omega)$ satisfy

$$\nabla \cdot \vec{H}_0 + \beta \nabla \cdot \vec{Z}_0, \quad \nabla \cdot \vec{E}_0 = 0, \quad \frac{\partial \vec{Z}_0}{\partial \nu}\Big|_{\partial\Omega} = 0,$$

so that (1.4) is automatically satisfied since we have from (1.2) and (1.3) that

$$0 = \nabla \cdot (\nabla \times \vec{H}) = \frac{\partial}{\partial t}(\nabla \cdot \vec{E}) + \sigma \nabla \cdot \vec{E} = e^{-\sigma t}\frac{\partial}{\partial t}(e^{\sigma t}\nabla \cdot \vec{E}),$$
$$0 = \nabla \cdot (\nabla \times \vec{E}) = \frac{\partial}{\partial t}(\nabla \cdot \vec{H} + \beta \nabla \cdot \vec{Z}).$$

Throughout this paper, we let $\alpha_1 = \alpha_2 = 1$.

Similarly to [2], for $|\vec{Z}_0| = 1$, we have the following equivalent form of (1.1)–(1.4) in the classical sense

$$\frac{1}{2}\vec{Z}_t = \frac{1}{2}\vec{Z} \times \vec{Z}_t = \triangle \vec{Z} + |\nabla \vec{Z}|^2 \vec{Z} - \vec{Z} \times (\vec{Z} \times \vec{H}), \tag{1.11}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E}, \tag{1.12}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{H}}{\partial t} - \beta \frac{\partial \vec{Z}}{\partial t}. \tag{1.13}$$

Therefore, it is natural to consider the following penalty system

$$\frac{1}{2}\vec{Z}_{\varepsilon t} - \frac{1}{2}\vec{Z}_\varepsilon \times \vec{Z}_{\varepsilon t} = \triangle \vec{Z}_\varepsilon + \frac{1}{\varepsilon^2}\vec{Z}_\varepsilon(1 - |\vec{Z}_\varepsilon|^2) - \vec{Z}_\varepsilon \times (\vec{Z}_\varepsilon \times \rho\sigma * \vec{H}_\varepsilon), \tag{1.14}$$

$$\nabla \times \vec{H}_\varepsilon = \frac{\partial \vec{E}_\varepsilon}{\partial t} + \sigma \vec{E}_\varepsilon, \tag{1.15}$$

$$\nabla \times \vec{E}_\varepsilon = -\frac{\partial \vec{H}_\varepsilon}{\partial t} - \beta \frac{\partial \vec{Z}_\varepsilon}{\partial t} \tag{1.16}$$

with the initial and boundary conditions (1.9)–(1.10) in which $\rho\delta$ is the usual mollifier. Here and in the following we assume that $\vec{H}_\varepsilon$ is identically zero outside Ω.

We shall prove that the weak solution of (1.14)–(1.16) approximates the solution of (1.11)–(1.13). Besides the existence of the weak solution, we also obtain the higher (partial) regularity for the spin vector $Z(x,\ t)$ than that in [25].

A key estimate in [23, 24] is the gradient estimate $\sup\limits_{\bar{\Omega}\times[0,\infty)} |\nabla \vec{Z}_\varepsilon| \le C\varepsilon^{-1}$ under the assumption that $|f(p,x,t)| \le g(x,t)$ in $\{p \in R^3 :\ |p| \le 1\} \times \bar{\Omega} \times [0,\infty)$ with $g(x,t) \in L^\infty(\Omega \times [0,\infty))$. This is the reason we mollify $\vec{H}_\varepsilon$ in (1.14) so that for $\vec{H}_\varepsilon \in L^\infty(0,T;L^2(\Omega))$ we have $\|u \times (u \times \rho\delta * \vec{H}_\varepsilon)\|_{L^\infty(\Omega\times[0,T])} \le C\|\vec{H}_\varepsilon\|_{L^\infty(0,T;T^2(\Omega))}$.

In the sequel, we denote $\Omega_T = \{(x,t),\ x \in \bar{\Omega},\ 0 \le t \le T\}$, $B_r(x) = B(x,r)$ a disk centered at x with radius r, $\Omega_T = \Omega \times [0,T]$ and $\Omega(T) = \Omega \times \{T\}$. We also denote

$$V_T = \Big\{(\vec{Z},\ \vec{H},\ \vec{E})\ :\ \Omega_T \longrightarrow R^3 \times R^3 \times R^3 \Big| \int_0^T \int_\Omega |\vec{Z}_t|^2 + \sup_{[0,T]} \int_\Omega (|\nabla \vec{Z}|^2 + |\vec{H}|^2 + |\vec{E}|^2) < \infty \Big\},$$

$$V_T^1 = \Big\{(\vec{Z},\ \vec{H},\ \vec{E})\ :\ \Omega_T \longrightarrow S^3 \times R^3 \times R^3 \Big| \int_0^T \int_\Omega |\vec{Z}_t|^2 + \sup_{[0,T]} \int_\Omega (|\nabla \vec{Z}|^2 + |\vec{H}|^2 + |\vec{E}|^2) < \infty \Big\},$$

Our main results are the following

Main Theorem. *Let* $(\vec{Z}_0,\ \vec{E}_0,\ \vec{H}_0) \in (H^1(\Omega;S^2),\ L^2(\Omega),\ L^2(\Omega))$. *Then the problem* (1.9)–(1.13) *admits at least one solution* $(\vec{Z},\ \vec{E},\ \vec{H})$ *in* V_T^1. *Moreover, for any* $\alpha \in (0,1)$, $\vec{Z}(x,t) \in C_{loc}^{1+\alpha,(1+\alpha)/2}(\bar{\Omega} \times (0,T] \setminus A)$ *where* A *consists of at most countably many lines in* Ω_T.

§2. Weak Solution to (1.9)–(1.13)

In this section, we shall prove the existence of weak solution for the problem (1.9)–(1.13), and derive some uniform estimates for the problem (1.14)–(1.16), (1.9)–(1.10) for the need in the next section.

By Galerkin method, it is not difficult to prove as in [23] the following lemmas.

Lemma 2.1. *Assume $\vec{Z}_0(x) \in H^1(\Omega; S^2)$, $\vec{f} \in L^2(Q_T)$. For any given $\delta > 0$ and $\varepsilon > 0$, there exists $\vec{Z}_\varepsilon(x,t) \in L^\infty(0,T : H^1(\Omega))$, $\vec{Z}_{\varepsilon t} \in L^2(Q_T)$ solves the following problem*

$$\frac{1}{2}\vec{Z}_{\varepsilon t} - \frac{1}{2}\vec{Z}_\varepsilon \times \vec{Z}_{\varepsilon t} = \triangle \vec{Z}_\varepsilon + \frac{1}{\varepsilon^2}\vec{Z}_\varepsilon(1 - |\vec{Z}_\varepsilon|^2) - \vec{Z}_\varepsilon \times (\vec{Z}_\varepsilon \times \rho\sigma * \vec{f}), \tag{2.1}$$

$$\vec{Z}_\varepsilon(x,0) = \vec{Z}_0, \quad \frac{\partial \vec{Z}_\varepsilon}{\partial \nu}|_{\partial\Omega} = 0, \tag{2.2}$$

and the following estimates hold

$$|\vec{Z}_\varepsilon(x,t) \le 1, \quad \forall t > 0, \tag{2.3}$$

$$\int_0^t \int_\Omega |\vec{Z}_{\varepsilon t}|^2 + \int_\Omega |\nabla \vec{Z}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\int_\Omega (1 - |\vec{Z}_\varepsilon|^2)^2 \le C\left(1 + \int_1^t \int_\Omega |\vec{f}|^2\right), \tag{2.4}$$

where C is independent of δ and ε

Moreover, for any given $\delta > 0$ and $\varepsilon > 0$, $\vec{Z}_\varepsilon$ is smooth on $\vec{\Omega} \times (0,\infty)$.

Sending $\delta \to 0$ and $\varepsilon_n \to 0$ for some subsequence of $\varepsilon > 0$, noting that $\lim\limits_{\delta\to 0} \rho\delta * \vec{f} = f$ strongly in $L^2(Q_T)$, we obtain

Lemma 2.2. *Assume $\vec{Z}_0(x) \in H^1(\Omega; S^2)$, $\vec{f} \in L^2(Q_T)$. There exists $\vec{Z}(x,t) \in L^\infty(0,T; H^1(\Omega; S^2))$, $\vec{Z}_t \in L^2(Q_T)$ solves the following problem*

$$\frac{1}{2}\vec{Z}_t - \frac{1}{2}\vec{Z} \times \vec{Z}_t = \triangle \vec{Z} + \vec{Z}|\nabla \vec{Z}|^2 - \vec{Z} \times (\vec{Z} \times \vec{f}), \tag{2.5}$$

$$\vec{Z}(x,0) = \vec{Z}_0, \quad \frac{\partial \vec{Z}}{\partial \nu}|_{\partial\Omega} = 0, \tag{2.6}$$

and the following estimate holds

$$\int_0^t \int_\Omega |\vec{Z}_t|^2 + \int_\Omega |\nabla \vec{Z}|^2 \le C\left(1 + \int_0^t \int_\Omega |\vec{f}|^2\right). \tag{2.7}$$

Lemma 2.3.[28] *Let $\vec{g}(x,t) \in L^2(Q_T)$, $\vec{H}_0(x) \in L^2(\Omega)$, and $\vec{E}_0(x) \in L^2(\Omega)$. There exists $(\vec{H}(x,t),\ \vec{E}(x,t)) \in L^\infty(0,T; L^2(\Omega)) \times L^\infty(0,T; L^2(\Omega))$ satisfying*

$$\nabla \times \vec{H} = \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E}, \tag{2.8}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{H}}{\partial t} + \vec{g}, \tag{2.9}$$

$$\vec{E}(x,0) = \vec{E}_0(x), \quad \vec{H}(x,0) = \vec{H}_0(x), \quad \vec{E} \times \nu|_{\partial\Omega} = 0, \quad \vec{H} \cdot \nu|_{\partial\Omega} = 0. \tag{2.10}$$

Combining Lemma 2.1–Lemma2.3, we obtain the following two lemmas.

Lemma 2.4. *For every $T > 0$ the problem* (1.14)–(1.16) *and* (1.9)–(1.10) *admits a solution $(\vec{Z}_\varepsilon, \vec{E}_\varepsilon . \vec{H}_\varepsilon) \in V_T$ and there holds*

$$\int_0^T \int_\Omega |\vec{Z}_{\varepsilon t}|^2 + \int_\Omega |\nabla \vec{Z}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\int_\Omega (1 - |\vec{Z}_\varepsilon|^2)^2 + \sigma \int_0^T \int_\Omega |\vec{E}_\varepsilon|^2 + \int_\Omega |\vec{E}_\varepsilon|^2 + \int_\Omega |\vec{H}_\varepsilon|^2 \le M, \tag{2.11}$$

where $M > 0$ depends only on the data and T.

Proof. The existence follows from Lemma 2.1 and Lemma 2.3 with $\vec{f}$ and $\vec{g}$ replaced by $\vec{H}_\varepsilon$ and $\vec{Z}_{\varepsilon t}$ respectively. We only need to prove (2.11).

Multiplying (1.14) by $\vec{Z}_{\varepsilon t}$ and integrating it over Ω we obtain

$$\frac{1}{2}\int_\Omega |\vec{Z}_{\varepsilon t}|^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega \left(|\nabla \vec{Z}_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|\vec{Z}_\varepsilon|^2)^2\right) = \int_\Omega (\vec{Z}_\varepsilon \times (\vec{Z}_\varepsilon \times \rho\delta * \vec{H}_\varepsilon))\vec{Z}_{\varepsilon t}. \quad (2.12)$$

Integrate (2.12) over $(0,T)$ to give

$$\begin{aligned}&\frac{1}{2}\int_0^T\int_\Omega |\vec{Z}_{\varepsilon t}|^2 + \frac{1}{2}\int_\Omega |\nabla \vec{Z}_\varepsilon|^2 + \frac{1}{4\varepsilon^2}\int_\Omega (1-|\vec{Z}_\varepsilon|^2)^2\\ &= \frac{1}{2}\int_\Omega |\nabla \vec{Z}_0|^2 + \int_0^T\int_\Omega (\vec{Z}_\varepsilon \times (\vec{Z}_\varepsilon \times \rho\delta * \vec{H}_\varepsilon))\vec{Z}_{\varepsilon t}, \end{aligned} \quad (2.13)$$

$$\frac{1}{4}\int_0^T\int_\Omega |\vec{Z}_{\varepsilon t}|^2 + \frac{1}{2}\int_\Omega |\nabla \vec{Z}_\varepsilon|^2 + \frac{1}{4\varepsilon^2}\int_\Omega (1-|\vec{Z}_\varepsilon|^2)^2 \le C_0 + C\int_0^T\int_\Omega |\vec{H}_\varepsilon|^2. \quad (2.14)$$

Multiplying (1.15) by $\vec{E}$, (1.16) by $\vec{H}$ and integrating over Ω and noting that from (1.10) $(\vec{E}_\varepsilon \times \nu|_{\partial\Omega} = 0)$

$$\int_\Omega \nabla\cdot(\vec{H}_\varepsilon \times \vec{E}_\varepsilon) = \int_{\partial\Omega} (\vec{H}_\varepsilon \times \vec{E}_\varepsilon)\cdot\nu = -\int_{\partial\Omega} (\vec{E}_\varepsilon \times \nu)\cdot\vec{H} = 0,$$

we have

$$\begin{aligned} 0 &= \int_\Omega \nabla\cdot(\vec{H}_\varepsilon \times \vec{E}_\varepsilon) = \int_\Omega [(\nabla\times\vec{H})\cdot\vec{E} - \nabla\times\vec{E})\cdot\vec{H}]\\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega |\vec{E}_\varepsilon|^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega |\vec{H}_\varepsilon|^2 + \sigma\int_\Omega |\vec{E}_\varepsilon|^2 + \beta\int_\Omega \vec{Z}_{\varepsilon t}\vec{H}_\varepsilon; \end{aligned}$$

this combined with (2.12) yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega [|\vec{E}_\varepsilon|^2 + |\vec{H}_\varepsilon|^2 + |\nabla\vec{Z}_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|\vec{Z}_\varepsilon|^2)^2] + \frac{1}{4}\int_\Omega |\vec{Z}_{\varepsilon t}|^2 + \sigma\int_\Omega |\vec{Z}_\varepsilon|^2 \le C\int_\Omega \vec{H}_\varepsilon|^2.$$

Then (2.11) following from Gronwall inequality. This competes the proof.

Lemma 2.5. *The problem* (1.9)–(1.13) *admits a solution* $(\vec{Z},\vec{E},\vec{H}) \in V_T^1$ *for every given* $T>0$ *and there holds*

$$\int_0^T\int_\Omega |\vec{Z}_t|^2 + \int_\Omega |\nabla\vec{Z}|^2 + \sigma\int_0^T\int_\Omega |\vec{E}|^2 + \int_\Omega |\vec{E}|^2 + \int_\Omega |\vec{H}|^2 \le M, \quad (2.15)$$

where $M>0$ *depends only on the data and* T. *Moreover, there holds*

$$\frac{1}{2}\int_0^T\int_\Omega |\vec{Z}_t|^2 + \frac{1}{2}\int_\Omega |\nabla\vec{Z}|^2 = \frac{1}{2}\int_\Omega |\nabla\vec{Z}_0|^2 + \int_0^T\int_\Omega (\vec{Z}\times(\vec{Z}\times\vec{H}))\vec{Z}_t. \quad (2.16)$$

Proof. The existence follows from the estimate (2.11) and the fact that if $\vec{H}_\varepsilon \to \vec{H}$ weakly in $L^2(\Omega_T)$ then

$$\rho\delta * \vec{H}_\varepsilon \to \vec{H} \quad \text{strongly in} \quad L^2(\Omega_T). \quad (2.17)$$

The proof of (2.15) and (2.16) is similar to that of (2.11).

Lemma 2.6. *Let* $\vec{Z}_\varepsilon$ *and* $\vec{Z}$ *be as above. Then we have a subsequence denoted by* $\vec{Z}_{\varepsilon_n}$ *such that*

$$\vec{Z}_{\varepsilon_n t} \to \vec{Z}_t \quad \textit{strongly in} \quad L^2(0,T;L^2(\Omega)), \quad (2.18)$$

$$\nabla\vec{Z}_{\varepsilon_n}(\cdot,t) \to \nabla\vec{Z}(\cdot,t) \quad \textit{strongly in} \quad L^2(\Omega), \quad \forall t\ge 0. \quad (2.19)$$

Proof. Since we have from Lemma 2.4 that there is some sequence of $\vec{Z}_\varepsilon$, denoted by

$\vec{Z}_{\varepsilon_n}$, such that

$$\begin{aligned}\vec{Z}_{\varepsilon_n} &\to \vec{Z} \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)),\\ \vec{Z}_{\varepsilon_n t} &\to \vec{Z}_t \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)),\\ \nabla\vec{Z}_{\varepsilon_n} &\to \nabla\vec{Z} \quad \text{weakly}^* \text{ in} \quad L^2(0,T;L^2(\Omega)),\end{aligned}$$

letting $\omega_{\varepsilon_n} = \vec{Z}_{\varepsilon_n} - \vec{Z}$ we obtain from these relations, (2.13), (2.16) that as $\varepsilon_n \to 0$,

$$\begin{aligned}&\int_0^T\int_\Omega |\omega_{\varepsilon_n t}|^2 + \int_\Omega |\nabla\omega_{\varepsilon_n}|^2\\ &= \int_0^T\int_\Omega [|\vec{Z}_t|^2 + |\vec{Z}_{\varepsilon_n t}|^2] + \int_\Omega [\nabla\vec{Z}|^2 + |\nabla\vec{Z}_{\varepsilon_n}|^2] - 2\int_0^T\int_\Omega \vec{Z}_{\varepsilon_n t}\vec{Z}_t - 2\int_\Omega \nabla\vec{Z}_{\varepsilon_n}\cdot\nabla\vec{Z}\\ &\le 2\int_0^T\int_\Omega |\vec{Z}_t|^2 + 2\int_\Omega |\nabla\vec{Z}|^2 - 2\int_0^T\int_\Omega \vec{Z}_{\varepsilon_n t}\vec{Z}_t - 2\int_\Omega \nabla\vec{Z}_{\varepsilon_n}\cdot\nabla\vec{Z}\\ &\quad + 2\int_0^T\int_\Omega [\vec{Z}_{\varepsilon_n t}(\vec{Z}_{\varepsilon_n} \times (\vec{Z}_{\varepsilon_n} \times \rho\delta * \vec{H}_{\varepsilon_n})) - \vec{Z}_t(\vec{Z} \times (\vec{Z} \times \vec{H}))]\\ &= 2\int_0^T\int_\Omega (\vec{Z}_t - \vec{Z}_{\varepsilon_n t})\vec{Z}_t + 2\int_\Omega \nabla\vec{Z}\cdot(\nabla\vec{Z} - \nabla\vec{Z}_{\varepsilon_n})\\ &\quad + 2\int_0^T\int_\Omega [\vec{Z}_{\varepsilon_n t}(\vec{Z}_{\varepsilon_n} \times (\vec{Z}_{\varepsilon_n} \times \rho\delta * \vec{H}_{\varepsilon_n})) - \vec{Z}_t(\vec{Z} \times (\vec{Z} \times \vec{H}))]\\ &= o(1) + 2\int_0^T\int_\Omega [\vec{Z}_{\varepsilon_n t}(\vec{Z}_{\varepsilon_n} \times (\vec{Z}_{\varepsilon_n} \times \rho\delta * \vec{H}_{\varepsilon_n})) - \vec{Z}_t(\vec{Z} \times (\vec{Z} \times \vec{H}))].\end{aligned}$$

Therefore, (2.18) and (2.19) can be proved if we have

$$\int_0^T\int_\Omega [\vec{Z}_{\varepsilon_n t}(\vec{Z}_{\varepsilon_n} \times (\vec{Z}_{\varepsilon_n} \times \rho\delta * \vec{H}_{\varepsilon_n})) - \vec{Z}_t(\vec{Z} \times (\vec{Z} \times \vec{H}))] = o(1). \tag{2.20}$$

Now we prove (2.20). In fact, the left-hand side of (2.20) equals to

$$\begin{aligned}&\int_0^T\int_\Omega (\vec{Z}_{\varepsilon_n t} - \vec{Z}_t)(\vec{Z}_{\varepsilon_n} \times (\vec{Z}_{\varepsilon_n} \times \rho\delta * \vec{H}_{\varepsilon_n}))\\ &\quad + \int_0^T\int_\Omega \vec{Z}_t[\vec{Z}_{\varepsilon_n} \times (\vec{Z}_{\varepsilon_n} \times \rho\delta * \vec{H}_{\varepsilon_n}) - \vec{Z} \times (\vec{Z} \times \vec{H})]\\ &= \int_0^T\int_\Omega (\vec{Z}_{\varepsilon_n t} - \vec{Z}_t)[\vec{Z}_{\varepsilon_n} \times (\vec{Z}_{\varepsilon_n} \times \rho\delta * \vec{H}_{\varepsilon_n}) - \vec{Z} \times (Z_{\varepsilon_n} \times \rho\delta * \vec{H}_{\varepsilon_n})]\\ &\quad + \int_0^T\int_\Omega (\vec{Z}_{\varepsilon_n t} - \vec{Z}_t)[\vec{Z} \times (\vec{Z}_{\varepsilon_n} \times \rho\delta * \vec{H}_{\varepsilon_n}) - \vec{Z} \times (Z \times \rho\delta * \vec{H}_{\varepsilon_n})]\\ &\quad + \int_0^T\int_\Omega (\vec{Z}_{\varepsilon_n t} - \vec{Z}_t)[\vec{Z} \times (\vec{Z} \times \rho\delta * \vec{H}_{\varepsilon_n}) - \vec{Z} \times (\vec{Z} \times \vec{H})]\\ &\quad + \int_0^T\int_\Omega (\vec{Z}_{\varepsilon_n t} - \vec{Z}_t)(\vec{Z} \times (\vec{Z} \times \vec{H}))\\ &\quad + \int_0^T\int_\Omega \vec{Z}_t[\vec{Z}_{\varepsilon_n} \times (\vec{Z}_{\varepsilon_n} \times \rho\delta * \vec{H}_{\varepsilon_n}) - \vec{Z} \times (\vec{Z} \times \vec{H})]\\ &=: I_1 + I_2 + I_3 + I_4 + I_5.\end{aligned}$$

For I_1, we have from (2.11) that

$$|I_1| \le \|\rho\delta * \vec{H}_{\varepsilon_n}\|_{L^\infty}(\Omega_T) \cdot \Big\{ \int_0^T \int_\Omega (|\vec{Z}_{\varepsilon_n t}|^2 + |\vec{Z}_t|^2) \Big\}^{\frac{1}{2}} \Big\{ \int_0^T \int_\Omega |\vec{Z}_{\varepsilon_n} - \vec{Z}|^2 \Big\}^{\frac{1}{2}} = o(1).$$

The estimates for the other terms cas be done in the similar manner. In the proof, we have used (2.17) and the fact that if $\vec{H}_\varepsilon$ is uniformly bounded in $L^2(Q_T)$ then

$$\|\rho\delta * \vec{H}_\varepsilon\|_{L^\infty(Q_T)} \le C\|\vec{H}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \le C. \tag{2.21}$$

§3. The a Priori Estimates for the Penalty Problem

In this section we shall give some uniform estimate for the solution of the penalty problem. In the following we denote by $(\vec{Z}_\varepsilon, \vec{E}_\varepsilon, \vec{H}_\varepsilon)$ the solution of (1.14)–(1.16) and (1.9)–(1.10) obtained in Lemma 2.4. Our aim is to obtain higher regularity for the spin vector $\vec{Z}(x,t)$, where $(\vec{Z}, \vec{E}, \vec{H})$ is the solution obtained in Lemma 2.5.

In view of (2.21), by the same method in [23,24] we have the following two lemmas.

Lemma 3.1. *There exists a constant $C > 0$ independent of ε and δ such that*

$$|\vec{Z}_\varepsilon| \le 1, \quad |\nabla \vec{Z}_\varepsilon| \le C\varepsilon^{-1}, \quad \forall (x,t) \in \bar{\Omega} \times [0,\infty). \tag{3.1}$$

Lemma 3.2. *There exist constant $\lambda_0 > 0$, $\mu_0 > 0$ independent of ε and t such that if*

$$\frac{1}{\varepsilon^2} \int_{\Omega \bigcap B_{2l}} (1 - |\vec{Z}_\varepsilon|^2)^2 \le \mu_0, \tag{3.2}$$

provided that $l/s \ge \lambda_0$, $0 < l \le 1$, then

$$|\vec{Z}_\varepsilon| \ge \frac{1}{2}, \quad \forall x \in \Omega \bigcap B_l, \tag{3.3}$$

where B_l is any sphere in R^2 with radius l.

According to Lemma IV.1 of [29], we have a family of disks $\{B(x_i, \lambda_0\varepsilon)\}_{i\in I}$ such that $x_i \in \Omega$, $B(x_i, \lambda_0\varepsilon/4) \bigcap B(x_j, \lambda_0\varepsilon/4) = \phi\,(i \ne j)$ and $\Omega \subset \bigcup_{i\in I} B(x_i, \lambda_0\varepsilon)$. We call $B(x_i, \lambda_0\varepsilon)$ "good disk" if $\frac{1}{\varepsilon^2}\int_{\Omega \bigcap B(x_0, 2\lambda_0\varepsilon)} (1 - |\vec{Z}_\varepsilon|^2)^2 \le \mu_0$. Otherwise, we call it "bad disk". Denote $J = \{j \in I,\ B(x_j, \lambda_0\varepsilon)$ is a bad disk$\}$. Then we can prove the following version of Lemma IV.2 of [29].

Lemma 3.3. *There exists a positive integer N independent of ε and t such that* Card$J \le$ *N and*

$$|\vec{Z}_\varepsilon| \ge \frac{1}{2} \quad on \quad \Omega \setminus \bigcup_{j\in J} B(x_j, \lambda_0)\varepsilon). \tag{3.4}$$

Moreover, we can choose (see [29, Section IV.2]) $J' :\ J' \subset J$ and $\lambda \ge \lambda_0$ such that

$$\begin{cases} |x_i - x_j| \ge 8\lambda\varepsilon, \quad i \ne j, \quad i, j \in J', \\ \bigcup_{j\in J} B(x_j,\ \lambda_0\varepsilon) \subset \bigcup_{j\in J'} B(x_j,\ \lambda\varepsilon), \\ |\vec{Z}_\varepsilon| \ge \frac{1}{2} \quad \text{on } \Omega \setminus \bigcup_{j\in J'} B(x_j,\ \lambda\varepsilon). \end{cases} \tag{3.6}$$

In the following of this section, we want to derive some estimates uniformly in ε for $\vec{Z}_\varepsilon$ determined in Lemma 2.4.

Lemma 3.4. *Let $x_0 \in \bar{\Omega}$, $P_{2r} = B_{2r}(x_0) \times [t_0,\ t_0 + 4r^2]$ for $x_0 \in \Omega$, $P_{2r} = (B_{2r}(x_0) \bigcap \Omega) \times [t_0, t_0 + 4r^2]$ for $x_0 \in \partial\Omega$. If $|\vec{Z}_\varepsilon| \ge \alpha_0 > 0$ on P_{2r}, then there exists a constant $C > 0$ independent of ε such that $\int_{P_r} |D^2\vec{Z}_\varepsilon|^2 \le C$.*

Proof. Let ξ be the standard cut-off function of $B_{2r}(x_0)$. Multiplying (1.14) by $\triangle\vec{Z}_\varepsilon$ and integrating over P_{2r} by part, we get (dropping the footnote ε for simplicity)

$$\int_{P_{2r}}\xi^2|\triangle\vec{Z}|^2\le\int_{P_{2r}}\xi^2|\triangle\vec{Z}|\,|u_t|+\int_{P_{2r}}\xi^2|\triangle\vec{Z}||\rho\delta*\vec{H}|$$
$$+\int_{P_{2r}}\frac{1}{\varepsilon^2}\xi^2|\triangle\vec{Z}|^2(1-|\vec{Z}|^2)+2\int_{P_{2r}}\frac{1}{\varepsilon^2}\xi(\triangle\vec{Z}\cdot\nabla\xi)(1-|\vec{Z}|^2).$$

Replacing $\frac{1}{\varepsilon^2}(1-|\vec{Z}|^2)$ by $|\vec{Z}|^{-1}|\frac{1}{2}\vec{Z}_t-\frac{1}{2}\vec{Z}_t\times\vec{Z}_t-\triangle\vec{Z}+\vec{Z}(\vec{Z}\times\rho\delta*\vec{H})|$ and using Hölder inequality we have

$$\frac{1}{2}\int_{P_{2r}}\xi^2|\triangle\vec{Z}|^2\le C\int_{P_{2r}}\xi^2|u_t|^2+\int_{P_{2r}}\xi^2|\rho\delta*\vec{H}|^2+\int_{P_{2r}}\xi^2|\triangle\vec{Z}|^4.$$

This combined with (2.11) and the theorem of parabolic equations implies that

$$\frac{1}{2}\int_{P_{2r}}\xi^2|D^2\vec{Z}|^2\le C+C\int_{P_{2r}}\xi^2|\nabla\vec{Z}|^4. \tag{3.7}$$

We have from embedding theorem that

$$\int_{P_{2r}}|D\vec{Z}|^4\xi^2\le C+C\int_{P_{2r}}|\nabla\vec{Z}|^2\int_{P_{2r}}\xi^2|D^2\vec{Z}|^2.$$

It following from (3.7) that

$$\int_{P_{2r}}\xi^2|D^2\vec{Z}|^2\le C+C\int_{P_{2r}}|\nabla\vec{Z}|^2\int_{P_{2r}}\xi^2|D^2\vec{Z}|^2.$$

Note that from (2.11), $C\int_{P_{2r}}|\nabla\vec{Z}|^2\le 1/2$ if r is small enough. The Lemma is proved.

Lemma 3.5. *Let* $Q_{r,s}=B_r(x_0)\times[t_0-s,\ t_0+s]$ *for* $x_0\in\Omega,\ Q_{r,s}=(B_r(x_0)\bigcap\Omega)\times[t_0-s,\ t_0+s]$ *for* $x_0\in\partial\Omega$. *If* $|\vec{Z}_\varepsilon|\ge\alpha_0>0$ *on* $Q_{r,s}$, *then for any* $q>2$ *there is a constant* $C_q>0$ *independent of* ε *such that*

$$\|\vec{Z}_\varepsilon\|_{W^{2,1}_q(Q_{r/2,\,s/2})}\le C_q. \tag{3.8}$$

Proof. First of all, we have from Lemma 3.4 that $\|\vec{Z}_\varepsilon\|_{L^q_{(Q_{r,s})}}\le C_q$. Moreover we have for $\Psi=\frac{1}{\varepsilon^2}(1-|\vec{Z}_\varepsilon|^2)$ that

$$\frac{1}{2}\varepsilon^2\Psi_t-\varepsilon^2\triangle\Psi+2\alpha_0^2\Psi\le 2|\triangle\vec{Z}_\varepsilon|^2\quad\text{in } Q_{r,s}. \tag{3.9}$$

Take cut-off function $\xi(x)\in C_0^\infty(B_r(x_0)),\ \xi\equiv 1$ in $B_{r/2}(x_0),\ \eta(t)\in C_0^\infty([t_0-s,\ t_0+s]),\ \eta\equiv 1$ in $[t_0-s/2,\ t_0+s/2],\ |\nabla\xi|\le C/r,\ |\eta_t|\le C/s,\ 0\le\xi\le 0,\ 0\le\eta\le 1$. Multiply (3.9) by $\xi^2(x)\eta^2(t)\Psi^{q-1}$ and integrate it over $Q_{r,s}$ to give

$$\frac{\varepsilon^2}{2q}\int_{B_r}\xi^2(x)\eta^2(t)\Psi^q|_{t_0-s}^{t_0+s}-\varepsilon^2\int_{Q_{r,s}}\xi^2\eta^2\Psi^{q-1}\triangle\Psi+2\alpha_0\int_{Q_{r,s}}\xi^2\eta^2\Psi^q$$
$$\le 2\int_{Q_{r,s}}\xi^2\eta^2|\nabla\vec{Z}_\varepsilon|^2\Psi^{q-1}+\frac{\varepsilon^2}{q}\int_{Q_{r,s}}\xi^2\eta|\eta_t|\Psi^q,$$

i.e.

$$\frac{1}{2}\varepsilon^2(q-1)\int_{Q_{r,s}}\xi^2\eta^2\Psi^{q-2}|\nabla\Psi|^2+2\alpha_0^2\int_{Q_{r,s}}\xi^2\eta^2\Psi^q$$
$$\le\sigma\int_{Q_{r,s}}\xi^2\eta^2\Psi^q+C_\sigma\int_{Q_{r,s}}\xi^2\eta^2|\nabla\vec{Z}_\varepsilon|^{2q}$$
$$+\frac{\varepsilon^2}{q}\int_{Q_{r,s}}\xi^2\eta|\eta_t|\Psi^q+\frac{2\varepsilon^2}{q-1}\int_{Q_{r,s}}\eta^2|\nabla\xi|^2\Psi^q.$$

Set $\sigma=\alpha_0^2$ in above inequality. We have

$$\alpha_0^2\int_{Q_{r,s}}\xi^2\eta^2\Psi^q\le C\int_{Q_{r,s}}\xi^2\eta^2|\nabla\vec{Z}_\varepsilon|^{2q}+\frac{2\varepsilon^2}{q-1}\int_{Q_{r,s}}\eta^2|\nabla\xi|^2\Psi^q+\frac{\varepsilon^2}{q}\int_{Q_{r,s}}\xi^2\eta|\eta_t|\Psi^q.$$

Hence

$$\alpha_0^2\int_{Q_{r,s}}\xi^2\eta^2\Psi^q\le C_q+C\varepsilon^2\int_{Q_{r,s}\setminus Q_{r/2,s/2}}\left(\frac{1}{r^2}\Psi^q+\frac{1}{s}\Psi^q\right).$$

Fixing r,s and taking ε small enough such that $\frac{C\varepsilon^2}{r^2}\le\frac{1}{4}\alpha_0^2$, $\frac{C\varepsilon^2}{s}\le\frac{1}{4}\alpha_0^2$, we obtain

$$\alpha_0^2\int_{Qr,s}\xi^2\eta^2\Psi^q\le C^q+\frac{\alpha_0^2}{2}\int_{Q_{r,s}\setminus Q_{r/2,s/2}}\Psi^q.$$

It follows from hole-filling method that

$$\int_{Q_{r/2,s/2}}\Psi^q\le C_q\qquad\forall q>2. \tag{3.10}$$

It is concluded from (3.10) and L^q theory of parabolic system that (3.8) holds.

Corollary 3.1. *Under the assumption of Lemma 3.5, we have for any* $\gamma\in(0,1)$

$$\|\nabla\vec{Z}_\varepsilon\|_{L^\infty(Q_{r,s})}\le C, \tag{3.11}$$

$$\|\vec{Z}_\varepsilon\|_{C^{1+\gamma,\ (1+\gamma)/2}(Q_{r,s})}\le C \tag{3.12}$$

with C independent of ε.

§4. The Partial Regularity

It is easy to see that the main theorem is a consequence of the following

Theorem 4.1. *There exists $0<T_1<T_2<\cdots$ and $a_j^i\in\bar\Omega$, $j=1,\cdots,N_i$, $i=1,2,\cdots$, $N_i\le N$, such that, $\forall\gamma\in(0,1)$, we have for some sequence $\{\vec{Z}_{\varepsilon n}\}$ that*

$$\vec{Z}_{\varepsilon n}\to\vec{Z}\qquad in\ C^{1+\gamma,(1+\gamma)/2}_{\rm loc}(\bar\Omega\times(0,T]\setminus A),$$

where $A=\bigcup_i\bigcup_{j=1}^{N_i}(\{a_j^i\}\times[T_i,T])$, $\vec{Z}$ is determined by Lemma 2.5.

According to Section 3, it suffices to give $C^{1+\gamma,(1+\gamma)/2}$-estimates ($\forall\gamma\in(0,1)$) uniformly in ε for $\vec{Z}_\varepsilon$ on any compact subset of $(\bar\Omega\times(0,T]\setminus A)$.

Lemma 4.1. *There exists $\tilde{T}_1>0$ independent of ε such that*

$$|\vec{Z}_\varepsilon|\le\frac{1}{2}\qquad on\ \ \bar\Omega\times[0,\tilde{T}_1]. \tag{4.1}$$

Proof. $\forall x_0\in\bar\Omega$, let ξ be the standard cut-off function on $B_{2R}(x_0)$ such that $0\le\xi\le$

$1,\ \xi \equiv 1$ on $B_R(x_0),\ |\nabla\xi| \le \frac{1}{R}$. Test (1.14) by $\xi^2\vec{Z}_{\varepsilon t}$ to give for any $\beta > 0$,

$$\frac{1}{2}\int_0^t\int_\Omega \xi^2|\vec{Z}_{\varepsilon t}|^2 + \sup_{0\le\tau\le t}\Big[\frac{1}{2}\int_{\Omega(\tau)}\xi^2|\nabla\vec{Z}_\varepsilon|^2 + \frac{1}{4\varepsilon^2}\int_{\Omega(\tau)}\xi^2(1-|\vec{Z}_\varepsilon|^2)^2\Big]$$
$$\le \frac{1}{2}\int_\Omega \xi^2|\nabla\varphi|^2 + \beta\int_0^t\int_\Omega \xi^2|\vec{Z}_{\varepsilon t}|^2 + C_\beta\int_0^T\int_\Omega |\nabla\xi|^2|\nabla\vec{Z}_\varepsilon|^2 + C_\beta\int_0^T\int_\Omega \xi^2|\rho\delta * \vec{H}_\varepsilon|^2.$$

Taking $\beta = \frac{1}{4}$ in above inequality, we have from (2.11) and (2.21) that

$$\frac{1}{4}\int_0^t\int_\Omega \xi^2|\vec{Z}_{\varepsilon t}|^2 + \sup_{0\le\tau\le t}\Big[\frac{1}{2}\int_{\Omega(\tau)}\xi^2|\nabla\vec{Z}_\varepsilon|^2 + \frac{1}{4\varepsilon^2}\int_{\Omega(\tau)}\xi^2(1-|\vec{Z}_\varepsilon|^2)^2\Big]$$
$$\le \frac{1}{2}\int_\Omega \xi^2|\nabla\varphi|^2 + C\int_0^t\int_\Omega |\nabla\xi|^2|\nabla\vec{Z}_\varepsilon|^2 + C\int_0^T\int_\Omega \xi^2|\rho\delta * \vec{H}_\varepsilon|^2$$
$$\le \frac{1}{2}\int_\Omega \xi^2|\nabla\varphi|^2 + \frac{4Ct}{R^2}\int_\Omega |\nabla\varphi|^2 + CR^2t. \tag{4.2}$$

Fixing $R = R_0 > 0,\ t = \tilde{T}_1 > 0$ in (4.2) so that

$$\frac{1}{2}\int_{B_{2R_0}}|\nabla\varphi|^2 \le \mu_0/8, \qquad CR_0^2\tilde{T}_1 + \frac{4C\tilde{T}_1}{R_0}\int_\Omega |\nabla\varphi|^2 \le \mu_0/8,$$

we deduce

$$\sup_{0\le t\le\tilde{T}_1}\frac{1}{\varepsilon^2}\int_{B_{R_0}(x_0)}(1-|\vec{Z}_\varepsilon|^2)^2 \le \mu_0.$$

It following from this and Lemma 3.2 that $|\vec{Z}_\varepsilon| \ge \frac{1}{2}$ on $B_{R_0}(x_0)\times[0,\tilde{T}_1]$. This implies the desired result.

Now we define $T_1 \ge \tilde{T}_1$ by

$$T_1 = \inf\{T|T>0, \ \text{ there is } \ x_0\in\Omega \text{ such that } \lim_{\varepsilon\to 0}\inf|\vec{Z}_\varepsilon(x_0,\ T)| = 0\}. \tag{4.3}$$

From the definition of T_1 we know that there is no bad disk on $\Omega(t)$ if $0 \le t < T_1$ and for any $0 < T < T_1$ there holds $\|\vec{Z}_\varepsilon\|_{C^{1+\gamma,(1+\gamma)/2}_{\rm loc}(\bar{\Omega}\times(0,T])} \le C$.

Denote the bad disks on $\Omega(T_1)$ by $\{B(x_i^\varepsilon,\ \lambda\varepsilon)\times\{T_1\}\},\ i=1,\cdots,\tilde{N}_1$, where$\tilde{N}_1 \le N$, N is determined by Lemma 3.3. Passing to a subsequence, we assume

$$x_i^{\varepsilon_n} \to a_j^1, \quad j=1,\cdots,N_1, \quad N_1 \le \tilde{N}_1, \quad a_l^1 \ne a_k^1 \quad (l\ne k).$$

At this time, on any compact subset of $\bar{\Omega}\times(0,T_1]\setminus\bigcup_{j=1}^{N_1}(\{a_j^1\}\times\{T_1\})$, we have $|\vec{Z}_{\varepsilon_n}| \ge 1/2$ if n is large enough. Therefore the conclusion of Corollary 3.1 holds on such compact subset.

Now we work starting from $t = T_1$. We first prove

Lemma 4.2. *For the function Ψ defined in* (3.9) *we have*

$$\Psi \in L^\infty_{\rm loc}(\bar{\Omega}\times[0,T_1]\setminus\bigcup_{j=1}^{N_1}(\{a_j^1\}\times\{T_1\})). \tag{4.4}$$

Proof. The interior estimates and the estimates near the boundary are done in the following one step. Denote

$$K = B_{2r}(x_0) \times [0, T_1] \subset \Omega \times [0, T_1] \setminus \bigcup_{j=1}^{N_1} (\{a_j^1\} \times \{T_1\}), \quad x_0 \in \Omega,$$

$$\tilde{K} = (B_{2r}(x_0) \bigcap \Omega) \times [0, T_1] \setminus \bigcup_{j=1}^{N_1} (\{a_j^!\} \times \{T_1\}), \quad x_0 \in \partial\Omega.$$

Again denote by ξ the standard cut-off function of $B_{2r}(x_0)$. We get

$$\varepsilon_n^2 \frac{\partial}{\partial t}(\xi\Psi) - 2\varepsilon_n^2 \triangle(\xi\Psi) + \xi\Psi \le 4\xi|\nabla \vec{Z}_{\varepsilon_n}|^2 - 4\varepsilon_n^2 \nabla\xi \cdot \nabla\Psi - 2\varepsilon_n^2 \Psi \triangle \xi. \tag{4.5}$$

It following from Section 3 that on the compact subsets K and $\tilde{K}$, the right-hand side of (4.5) is bounded uniformly in n. Then Lemma 4.3 follows from the maximum principle (see also [24]).

Lemma 4.3. *There exists $\tilde{T}_2 > T_1$ independent of ε_n such that on any compact subset M of $\bar{\Omega} \times [T_1, \tilde{T}_2] \setminus \bigcup_{j=1}^{N_1} (\{a_j^1\} \times [T_1, \tilde{T}_2])$,*

$$|\vec{Z}_{\varepsilon_n}| \ge \frac{1}{2} \qquad on \quad M. \tag{4.6}$$

Proof. For any $x_0 \in \bar{\Omega} \setminus \bigcup_{j=1}^{N_1} \{a_j^1\}$, take $R > 0$ so small that $B_{2R}(x_0)$ does not contain $a_j^1 (1 \le j \le N_1)$. Let $\xi(x)$ be the cut-off function of $B_{2R}(x_0)$ and define

$$E_\xi(\vec{Z}) = \frac{1}{2}\int_\Omega \xi^2 |\vec{Z}|^2 + \frac{1}{4\varepsilon^2}\int_\Omega \xi^2 (1 - |\vec{Z}|^2)^2.$$

It following from simple computations that for $t > T_1$,

$$\begin{aligned} E_\xi(\vec{Z}_{\varepsilon_n}(x,t)) &\le E_\xi(\vec{Z}_{\varepsilon_n}(x, T_1)) + C\int_{T_1}^t \int_\Omega |\nabla\xi|^2 |\nabla \vec{Z}_{\varepsilon_n}|^2 + CR^2(t - T_1) \\ \le \frac{1}{2}&\int_{B_{2R}(x_0)\times\{T_1\}} \xi^2 |\nabla \vec{Z}_{\varepsilon_n}|^2 + \frac{1}{4\varepsilon_n^2}\int_{B_{2R}(x_0)\times\{T_1\}} \xi^2 (1 - |\vec{Z}_{\varepsilon_n}|^2)^2 \\ &+ \frac{C}{R^2}\int_{T_1}^t \int_{B_{2R}(x_0)} |\nabla \vec{Z}_{\varepsilon_n}|^2 + CR^2(t - T_1). \end{aligned}$$

Hence we have from this inequality, (2.11), Lemma 2.6 and Lemma 4.3 that

$$E_\xi(\vec{Z}_{\varepsilon_n}(x,t)) \le o(1) + CR^2 + \frac{C(t - T_1)}{R^2} + CR^2(t - T_1).$$

Now the desired conclusion follows from Lemma 3.2 if one fixes $R = R_0,\ t = \tilde{T}_2 > T_1$ so that

$$o(1) + CR_0^2 + \frac{C(\tilde{T}_2 - T_1)}{R_0^2} + CR_0^2(\tilde{T}_2 - T_1) \le \frac{\mu_0}{4}.$$

As before, we define $T_2 > T_1$ by

$$T_2 = \inf\{T | T > T_1, \ \text{ there is } \ x_0 \in \Omega \setminus \bigcup_{j=1}^{N_1} \{a_j^1\} \text{ such that } \liminf_{\varepsilon\to 0} |\vec{Z}_\varepsilon(x_0, T)| = 0\}. \tag{4.7}$$

Denote the bad disks on $\Omega(T_2)$ by $B(x_k^\varepsilon, \lambda\varepsilon),\ k = 1, \cdots, \tilde{N}_2, \tilde{N}_2 \le N$. Passing to a further subsequence, still denoted by $\vec{Z}_{\varepsilon_n}$, we assume $x_k^{\varepsilon_n} \to a_l^2,\ l = 1, \cdots, N_2 \le \tilde{N}_2$ with

a_l^2 different from each other. On the compact subset of

$$\bar{\Omega} \times [T_1, T_2] \setminus \Big(\bigcup_{j=1}^{N_1} \{a_j^1\} \times [T_1, T_2] \bigcup_{l=1}^{N_2} \{a_l^2\} \times \{T_2\} \Big),$$

repeating above proof, we obtain

Lemma 4.4. *For any $\gamma \in (0, 1)$ and any compact subset M of*

$$\bar{\Omega} \times [T_1, T_2] \setminus \Big(\bigcup_{j=1}^{N_1} \{a_j^1\} \times [T_1, T_2] \bigcup \bigcup_{l=1}^{N_2} \{a_l^2\} \times \{T_2\} \Big)$$

we have for some constant $C > 0$ independent of n that

$$\|\vec{Z}_{\varepsilon_n}\|_{C^{1+\gamma,(1+\gamma)/2}(M)} \leq C.$$

Summing up, we have proved Theorem 4.1 by virtue of Lemma 4.4.

Remark. It is clear that the energy $E_\varepsilon(\vec{Z}_\varepsilon(x, \cdot))$ need not be non-increasing. Therefore, we can only get the same smoothness for $\vec{Z}(x, t)$ as in [22].

References

[1] Landau, L. & Lifshitz, E., Electrodynamique des milieux continus [M], Cours de Physique Théorique, tome VIII, Ed. Mir, Moscow, 1969.

[2] Guo, B. L. & Hong, M. C., The Landau-lifeshitz equation of ferromagnetic spin chain and harmonic maps [J], *Cal. Var. PDE.,* **1**(1993), 311–334.

[3] Guo, B. L. & Wang, Y., Generakized Kandau-lifshitz systems of the ferromagnetic spin chain and harmonic maps [J], *Science in China,* **26A**(1996), 800–810.

[4] Chen, Y. & Guo, B. L., Two dimensional Landau-lifshitz equation [J], *J. PDE.,* **9**(1996), 313–332.

[5] Makamura, K. & Sasada, T., Soliton and wave trains in ferromagnets [J], *Phys. Let.,* **48A**(1974), 321–322.

[6] Lakshmanan, M., Ruijgrok, T. W. & Thompson, C. J., On the dynamics of e continuum spin system [J], *Phys. A*, **84A**(1976), 577–590.

[7] Tjon, J. & Wright, T., Soliton in the continuous Heisenberg spin chain [J], *Phys. Rev.,* **B15**(1977), 3470–3476.

[8] Fogedby, H. C., Lect. Notes in Phys., Springer-Verlag, Berlin, Heidberg, New York, 1980, 131.

[9] Takhtajan, L. A., Integration the continuous Heisenberg spin chain through the inverse scattering mathod [J], *Phys. Lett.* **64A**(1977), 235–237.

[10] Zakharov, V. E. & Takhtajan, L. A., Equation of nonlinear Schrödinger equ. and Heisenbery ferromgnet [J], *Theor. Mat. Phys.,* **38**(1979), 17–20.

[11] Lakshmanana, M. & Daniel, M., Soliton damping and energy loss in the classical continuous Heisenberg spin chain [J], *Phys, Rev. B,* **24:**11(1981), 6751–6754.

[12] Daniel, M. & Lakshmanana, M., Perturbation of solitons in the classical continuous isotropic Heisenberg spin system, *Phys.,* **120A**(1983), 125–152.

[13] Lakshmanana, M. & Naksmurs, K., Landau-lifshitz equations of ferro-magnetism: Exact treatment of the Gilbert damping, *Phys. Rev. Letts.,* **53:**6(1984) 2497–2499.

[14] Zhou, Y. L. & Guo, B. L., The solvability of the inital value problem for the quasilinear degenerate parabolic system [A], Proceeding of DD-3 symposium [C], 1982, 713–732.

[15] Zhou, Y. L. & Guo, B. L., Existence of weak solution for boundary problem of systems of ferromagnetic chain [J], *Scientia Sinica,* **27**(1984) 779–811.

[16] Zhou, Y. L. & Guo, B. L., Finite difference solutions of the boundary problems for systems of ferromagnetic chain [J], *J. Comp. Math.,* **1**(1983), 294–302.

[17] Zhou, Y. L. & Guo, B. L., Some boundary problem of the spin system and the systems of ferromagnetic chain I, Nonlinear boundary problems [J], *Acta Math. Scientia,* **6:**4(1986), 321–337.

[18] Zhou, Y. L. & Guo, B. L., Some boundary problem of the spin system and the systems os ferromagnetic chain II, Mixed problems and others [J], *Acta Math. Scientia,* **7:**2(1987), 121–132.

[19] Zhou, Y. L. & Guo, B. L., Weak solution of system of ferromagnetic chain with several variables [J], *Scientia Sinica,* **30A**(1987), 1251–1266.

[20] Zhou, Y. L., Guo, B. L. & Tan, S., Existence and uniquebess of smooth solution for system of ferromagnetic chain [J], *Science in China, Ser. A,* **54:**3(1991), 257–266.

[21] Zhou, Y. L., Sun, H. S. & Guo, B. L., The Cauchy problem for one class of the system of Landau-lifshitz equations [J], *Science in China,* **23:**4(1993), 352–362.

[22] Zhou, Y. L. & Sun, H. S. & Guo, B. L., Multidimensional system of ferro-magnetic chain type [J], *Science in China, Ser. A,* **36:**12(1993), 1422–1434.

[23] Guo, B. L. & Ding, S., Initial-boundary value problem for the unsaturated Landau-Lifshitz system with applied filed [R], Prepint 1996.

[24] Guo, B. L. & Ding, S., Initial-boundary value problem for the Landau-Lifshitz system I, II [J], *Progress in Nat. Sci.,* **8:**1(1998), 11; **8:**2(1998), 147–151.

[25] Guo, B. L. & Su, F., The global solution for Landau-Lifshitz-Maxwell equations, *J. PDE,* **14:**2(2001), 133–148.

[26] Su, F. & Guo, B. L., The global smooth solution for Landau-Lifshitz-Maxwell equations without dissipation [J], *J. PDE.,* **11:**3(1998), 193–208.

[27] Guo, B. L. & Su, F., Global weak solution for Landau-Lifshitz-Maxwell equations in three space dimensions [J], *J. Math. Anal. Appl.,* **211**(1997), 326–346.

[28] Cooper, J. & Strauss, W., The initial boundary problem for the Maxwell equations in the presence of a moving boundary [J], *SIAM J. Math. Anal.,* **16:**6(1985), 1165–1179.

[29] Brezis, H. & Bethuel, H. & Hëelein, H., Ginzburg-Landau vortices [M], Birkhauser, 1993.