

WEIERSTRASS POLYNOMIALS AND PLANE PSEUDO-HOLOMORPHIC CURVES***

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Abstract

For an almost complex structure J on $U \subset \mathbb{R}^4$ pseudo-holomorphically fibered over $\mathbb{C}$ a J -holomorphic curve $C \subset U$ can be described by a Weierstrass polynomial. The J -holomorphicity equation descends to a perturbed $\bar{\partial}$ -operator on the coefficients; the operator is typically $(0, 2/m)$ -Hölder continuous if m is the local degree of C over $\mathbb{C}$. This sheds some light on the problem of parametrizing pseudo-holomorphic deformations of J -holomorphic curve singularities.

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§0. Introduction

Many of the elementary properties of plane holomorphic curves have been established also for pseudo-holomorphic curves. These include isolatedness of critical points and of points of intersection, positivity of intersection indices, removable singularities, existence of (singular) limits under a volume bound (cf. [2] and references therein). Maybe even more strikingly, singularities of plane pseudo-holomorphic curves topologically look quite the same as holomorphic curve singularities. In fact, there is a local C^1 -diffeomorphism of the ambient space mapping the pseudo-holomorphic curve singularity to a holomorphic one ([5, Theorem 6.2]).

Surprisingly the situation is unclear when it comes to deformations of plane pseudo-holomorphic curve singularities. In the holomorphic world there is the notion of semi-universal deformation. It consists of the germ of a holomorphic deformation over some parameter space $(S, 0)$, its base. Its characterizing property is that up to isomorphism any deformation of $(C, 0)$, with parameter space $(T, 0)$ say, is obtained by pull-back via a holomorphic “classifying” map $(T, 0) \rightarrow (S, 0)$. The classifying map is unique only up to an

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isomorphism fixing the map on the tangent spaces $T_{T,0} \rightarrow T_{S,0}$. Explicitly, let $(C, 0)$ be the germ of a plane holomorphic curve given by $F \in \mathcal{O}_{\mathbb{C}^2,0} \simeq \mathbb{C}\{z, w\}$. Then $(S, 0) = (\mathbb{C}^\tau, 0)$ is a smooth space of dimension equal to the Tyurina number

$$\tau = \dim_{\mathbb{C}} \mathbb{C}\{z, w\} / (F, \partial_z F, \partial_w F).$$

The interest in a similar result for pseudo-holomorphic curves for us comes from a possible analytic treatment of the isotopy problem for symplectic submanifolds of $\mathbb{CP}^2$ or the two S^2 -bundles over S^2 (cf. [6]). There are some indications that on these spaces symplectic submanifolds are isotopic iff they are homologous. One crucial obstacle in proving this statement by the technique of J -holomorphic curves is the lack of understanding that we have for deformations of singular J -holomorphic curves. One question which we should answer and is related to the holomorphic deformation theory discussed above runs as follows.

Question I. *Let J be an almost complex structure on the unit ball $B \subset \mathbb{C}^2$ and $C \subset B$ a J -holomorphic curve with $0 \in C$. Does there exist an open neighbourhood $U \subset B$ of 0 and an open subset $\mathcal{M}$ in a Banach space parameterizing J -holomorphic curves in $\text{cl}U$ that are sufficiently close to $C \cap \text{cl}U$ in the Hausdorff topology?*

This is of course true holomorphically. For example, taking appropriate linear coordinates z, w on $\mathbb{C}^2$ the defining equation of C can be taken in Weierstrass form

$$F(z, w) = w^d - a_1(z)w^{d-1} + \cdots + (-1)^d a_d(z)$$

for (z, w) in a polycylinder $\Delta \times \Delta$ contained in B . Here d is the intersection multiplicity of the line $z = 0$ with C , and a_i are holomorphic functions on $\text{cl}\Delta$. Obviously, deformations of C are in one-to-one correspondence with deformations of the coefficients a_i . Introducing an appropriate Banach space completion of $\mathcal{O}(\text{cl}\Delta)$ answers the holomorphic analogue of Question I affirmatively.

A related question that is both relevant to the isotopy problem and interesting in its own right is the local isotopy problem for plane pseudo-holomorphic curves. Let $U \subset \mathbb{C}^2$ be an open set with piecewise smooth boundary. We call two submanifolds with boundary $(\Sigma, \partial\Sigma)$, $(\Sigma', \partial\Sigma')$ in $(\text{cl}U, \partial U)$ isotopic if there is a continuous family of submanifolds $(\Sigma_t, \partial\Sigma_t) \subset (\text{cl}U, \partial U)$, $t \in [0, 1]$, connecting Σ and Σ' ($\Sigma = \Sigma_0$, $\Sigma' = \Sigma_1$). Note that $\partial\Sigma_t$ is then a tame isotopy of the links $\partial\Sigma$, $\partial\Sigma' \subset \partial U$. In case Σ, Σ' are symplectic (or pseudo-holomorphic, J -holomorphic respectively) then the isotopy will be called symplectic (pseudo-holomorphic, J -holomorphic) if Σ_t can be chosen symplectic (pseudo-holomorphic, J -holomorphic) for all t . Here “pseudo-holomorphic” means J -holomorphic for some J .

Question II. *Let $C \subset B$ be a J -holomorphic curve with singular locus $C_{\text{sing}} = \{0\}$. If $\{\Sigma_n\}$ and $\{\Sigma'_n\}$ are two sequences of J -holomorphic curves in B with Hausdorff limit C , then are Σ_n and Σ'_n (symplectically, pseudo-holomorphically, J -holomorphically) isotopic for n sufficiently large?*

In the holomorphic category this again has a positive answer, for the set of tuples $(a_1, \dots, a_d) \in \mathcal{M}$ parametrizing singular holomorphic curves in Weierstrass form does not disconnect $\mathcal{M}$. On a technical level this follows by a straightforward application of the Sard-Smale theorem on an appropriate space of paths in $\mathcal{M}$.

In the almost complex setting it is still possible to bring C and all small deformations of C into Weierstrass form. To do this we may assume by a real, linear change of coordinates that

J_0 is the standard complex structure on $\mathbb{C}^2$. Let (z, w) be the standard linear coordinates on $\mathbb{C}^2$. Possibly after another (now complex-) linear change of coordinates we may assume the tangent lines of smooth irreducible components of C at 0 to be disjoint from $z = 0$, and that the closed polycylinder $|z| \leq 1, |w| \leq 1$ maps to the domain of definition of J and C . In [9, Lemma 5.4], it is shown that possibly after shrinking the polycylinder there is a local diffeomorphism of the form

$$\Theta : (z, w) \longmapsto (z, w + \varphi(z, w))$$

such that $w \mapsto \Theta(z, w)$ is an embedded J -holomorphic disk with $\Theta(z, 0) = (z, 0)$ for every z . Moreover, $\nabla\varphi$ can be made arbitrarily small by considering a sufficiently small polycylinder, that is by rescaling z and w . Changing coordinates by Θ we may therefore assume that for every $z \in \text{cl}\Delta$ the disk $\{z\} \times \Delta$ is J -holomorphic and not contained in C . The antiholomorphic tangent space may now be written as

$$T_{\mathbb{C}^2, J}^{0,1} = \langle \partial_{\bar{w}}, \partial_{\bar{z}} - a\partial_z - b\partial_w \rangle \quad (0.1)$$

for complex valued functions a and b . The point here is of course that ∂_w is contained in a J -holomorphic disk and hence lies in the holomorphic tangent space. Now let $d > 0$ be the intersection index of the disk $z = 0$ with the disjoint J -holomorphic curve C at 0. Then possibly after rescaling w and z , for every $z \in \Delta$ there are exactly d points of intersection of $\{z\} \times \Delta$ with C , counted with multiplicities. We obtain a map from the domain of z to the d -fold symmetric product $S^d\Delta$ of w , which is an open subset of $S^d\mathbb{C} \simeq \mathbb{C}^d$. Explicitely, to a zero cycle $\sum_{i=1}^d \lambda_i$ we associate the complex polynomial

$$(w - \lambda_1) \cdots (w - \lambda_d) = w^d - a_1 w^{d-1} + \cdots + (-1)^d a_d$$

with $a_i = \sigma_i(\lambda_1, \dots, \lambda_d)$ the i -th elementary symmetric polynomial. This yields d complex functions $a_1(z), \dots, a_d(z)$ with

$$C = \{(z, w) \in \Delta \times \Delta \mid w^d - a_1(z)w^{d-1} + \cdots + (-1)^d a_d(z) = 0\}.$$

The same argument holds for J -holomorphic curves that are sufficiently close to C in the Hausdorff topology. Moreover, using local representatives as given in [5] it is not hard to check that the a_i are continuous. In turn, the C^0 -topology on the a_i induces the Hausdorff topology on the space of J -holomorphic curves.

The crucial point is then to characterize those tuples $a_1, \dots, a_d$ actually corresponding to J -holomorphic curves. We will find that this can be done by a nonlinear $\bar{\partial}$ -equation provided the projection $(z, w) \mapsto z$ is holomorphic, in the almost complex sense. This is the case iff $a \equiv 0$ in (0.1). In other words we require that J is given by only one complex function b instead of two:

$$T_{\mathbb{C}^2, J}^{0,1} = \langle \partial_{\bar{w}}, \partial_{\bar{z}} - b\partial_w \rangle. \quad (0.2)$$

For the precise statement we assume that b is extended to all of $\Delta \times \mathbb{C}$ with uniformly bounded C^1 -norm. From b we will construct d complex functions $b_1, \dots, b_d$ on $\Delta \times S^d\mathbb{C}$. Let $D \subset S^d\mathbb{C}$ be the discriminant locus. The b_r are smooth away from $\Delta \times D$ and Hölder of some exponent $0 < \alpha \leq 1$ depending only on d (Lemma 1.2). We are now ready to state our first theorem.

Theorem 0.1. *There is a one-to-one correspondence between the sets*

$$\left\{ J\text{-holomorphic curves } C \subset \Delta \times \mathbb{C} \text{ that are proper of degree } d \text{ over } \Delta \right\}$$

and

$$\left\{ \mathbf{a} = (a_1, \dots, a_d) \in W_{\text{loc}}^{1,p}(\Delta; \mathbb{C}) \mid \delta(\mathbf{a}) \neq 0, \partial_{\bar{z}} a_r = b_r(z, \mathbf{a}), r = 1, \dots, d \right\}$$

(any finite $p > 2$). The J -holomorphic curve belonging to $(a_1, \dots, a_d)$ is

$$\{(z, w) \in \Delta \times \mathbb{C} \mid w^d - a_1(z)w^{d-1} + \dots + (-1)^d a_d(z) = 0\}.$$

Moreover, the a_r are even of class $C^{1,\alpha}(\Delta; \mathbb{C})$ for some $\alpha = \alpha(d) > 0$.

Remark 0.1. By dropping the requirement $\delta(\mathbf{a}) \neq 0$, the correspondence extends to any J -holomorphic cycles. To see this we first observe that the map from the first set to the second is still well-defined: Given a J -holomorphic cycle $C = \sum_i m_i C_i$, where C_i is a branched cover over Δ of degree d_i , let f_i be the Weierstrass polynomial of C_i constructed in Theorem 0.1. Put $f = \prod_i f_i^{m_i}$. Its coefficients $\mathbf{a}_j$ define a map to $W_{\text{loc}}^{1,p}(\Delta; \mathbb{C}^d)$. This map can be also regarded as a pseudo-holomorphic section of the d -fold relative symmetric product of $\Delta \times \mathbb{C}$ over Δ . This latter symmetric product has a pseudo-holomorphic stratification according to partitions of d . The section which arises from $C = \sum_i m_i C_i$ belongs to the stratum associated to $d = \sum_i m_i d_i$. In order to prove that the extension gives rise to a one-to-one correspondence, we need to show that a pseudo-holomorphic section $\mathbf{a}$ stays in one stratum except at finitely many points. This is true but a bit delicate. It follows from a unique continuation theorem for pseudo-holomorphic sections of the above type, that is, if the intersection of a pseudo-holomorphic curve with the closure of a stratum has an accumulation point, then it lies in the closure of the stratum. By induction, one can reduce it to the case that the stratum is a hypersurface. If the stratum is the hypersurface corresponding to $d = 2 + 1 + \dots + 1$, then it amounts to check that the discriminant either is identically zero or has only finitely many zeroes. In general, one can have a function which plays the role of the discriminant.

If the b_r are Lipschitz, we can easily proceed to parametrize solutions $(a_1, \dots, a_r)$ of the nonlinear PDE by a Banach space of d holomorphic functions on Δ , possibly after shrinking the domain of z . Unfortunately this is not generally true. If one stratifies the discriminant locus D according to partitions of d , then at a point of a stratum indexed by $d = d_1 + \dots + d_l$, one expects b_r to be generally not better than C^α with $\alpha = 2/\max\{d_1, \dots, d_l\}$. The exception is if b is indeed holomorphic in w , or for $d = 2$.

Theorem 0.2. *Let an almost complex structure J on $\Delta \times \mathbb{C}$ be given of the form (0.2). Let $C \subset \Delta \times \mathbb{C}$ be a J -holomorphic curve mapping properly to Δ . Put $d = \deg(C \rightarrow \Delta)$. Assume that*

- (a) either $d \leq 2$,
- (b) or $\bar{\partial}_w b \equiv 0$.

Then for sufficiently small $\varepsilon > 0$ the space

$$\mathcal{M}_\varepsilon = \{C' \subset B_\varepsilon(0) \times \mathbb{C} \mid J\text{-holomorphic curve}\}$$

is a Banach manifold at $C \cap (B_\varepsilon(0) \times \mathbb{C})$. It is modelled on the Banach space

$$\mathcal{O}^{1,p}(\Delta; \mathbb{C}^d) := W^{1,p}(\Delta; \mathbb{C}^d) \cap \mathcal{O}(\Delta; \mathbb{C}^d)$$

endowed with the $W^{1,p}$ -norm, where d is the degree of the projection $C \rightarrow \Delta$.

While this is only a partial result we would like to point out that it includes singularities of arbitrarily high Milnor number. Moreover, by computing the Nijenhuis tensor one can check that the assumption $\bar{\partial}_w b \equiv 0$ is equivalent to integrability of the almost complex structure. As we remarked before the parametrization by a Banach manifold is no surprise in this case. However, this description is useful for the global parametrization problem for pseudo-holomorphic curves on S^2 -bundles (see [7]).

To describe our result in the general case, that is without the restrictive assumptions (a) or (b), we remind the reader of the decomposition $W^{1,p}(\Delta, \mathbb{C}^d) = \mathcal{O}^{1,p}(\Delta, \mathbb{C}^d) \oplus L^p(\Delta, \mathbb{C}^d)$ provided by a right-inverse T (see (2.1) and the discussion following it below). A pair $(\mathbf{h}, \boldsymbol{\xi})$ on the right yields the function $\mathbf{h} + T\boldsymbol{\xi}$ of Sobolev class $(1, p)$. Using a Leray-Schauder fixed point theorem we obtain the following result.

Theorem 0.3. *Let an almost complex structure J on $\Delta \times \mathbb{C}$ be given of the form (0.2) and let $d > 0$. For every $\mathbf{h} \in \mathcal{O}^{1,p}(\Delta, \mathbb{C}^d)$ there exists an $\boldsymbol{\xi} \in L^p(\Delta, \mathbb{C}^d)$ with $\mathbf{a} = \mathbf{h} + T\boldsymbol{\xi}$ corresponding to a (possibly non-reduced) J -holomorphic curve in the Weierstrass picture. In other words, the projection map*

$$\text{Id} - T \circ \partial_{\bar{z}} : \{\mathbf{a} \in W^{1,p}(\Delta, \mathbb{C}^d) \mid \partial_{\bar{z}} a_r = b_r(z, \mathbf{a})\} \longrightarrow \mathcal{O}^{1,p}(\Delta, \mathbb{C}^d)$$

is surjective.

Note that there is no assumption on the smallness of b , so this last theorem is in fact a global result.

The simplicity of the describing PDE also clearly exhibits the analytical difficulty that parametrizing deformations of C poses. The Weierstrass picture provides a uniform formulation for all deformations of C with only the nonlinear, zero order term being sensitive to the change of topology.

One note on conventions: Throughout the text, a J -holomorphic curve in an almost complex manifold (M, J) is always understood as a closed subset of M . Viewed as a 2-cycle we therefore assume all components to have multiplicity one.

§1. Proof of Theorem 0.1

In this section we derive the PDE and prove Theorem 0.1. For the equation we observe first that in view of (0.2) the graph of a function $\lambda : \Delta \rightarrow \mathbb{C}$ is a pseudo-holomorphic curve with respect to $J = J(b)$ iff

$$\partial_{\bar{z}} \lambda = b(z, \lambda(z)). \quad (1.1)$$

Now if $C \subset \Delta \times \mathbb{C}$ is a J -holomorphic curve that maps properly to Δ , the special form of J implies that the projection $C \rightarrow \Delta$ is a finite holomorphic map, hence a branched covering, of covering degree d say. Away from the discrete critical set $D_C \subset \Delta$, locally C is the union of the graphs of d functions $\lambda_1, \dots, \lambda_d$. As noted in the introduction C is then given in Weierstrass form

$$w^d - a_1(z)w^{d-1} + \dots + (-1)^d a_d(z) = 0$$

with

$$a_r = \sigma_r(\lambda_1, \dots, \lambda_d).$$

Taking $\partial_{\bar{z}}$ of a_r yields

$$\partial_{\bar{z}} a_r = \partial_{\bar{z}} \sum_{\{i_1, \dots, i_r\} \subset \{1, \dots, d\}} \lambda_{i_1} \cdots \lambda_{i_r} = \sum_{\nu=1}^d \sigma_{r-1}(\lambda_1, \dots, \hat{\lambda}_\nu, \dots, \lambda_d) \cdot b(z, \lambda_\nu),$$

where the entry with a hat is to be omitted. The right-hand side of this equation is a function that is invariant under the action of the symmetric group on the branches, and hence can be expressed as function $b_r(z; a_1, \dots, a_d)$ in z and the a_r . Conversely, the union of the graphs of $\lambda_1, \dots, \lambda_d$ with $\lambda_i(z) \neq \lambda_j(z)$ for $i \neq j$ and all z is J -holomorphic iff the functions $a_r := \sigma_r(\lambda_1, \dots, \lambda_d)$ fulfill

$$\partial_{\bar{z}} a_r = b_r(z; a_1, \dots, a_d), \quad r = 1, \dots, d. \quad (1.2)$$

This follows by applying the linear map

$$A: (v_r)_{r=1, \dots, d} \mapsto \left(\sum_{\nu=1}^d \sigma_{r-1}(\lambda_1, \dots, \hat{\lambda}_\nu, \dots, \lambda_d) \cdot v_\nu \right)_{r=1, \dots, d}$$

to $v_r = \partial_{\bar{z}} \lambda_r - b(z, \lambda_r)$ and noting the following elementary fact.

Lemma 1.1. *If $\lambda_i \neq \lambda_j$ for all $i \neq j$, then A is invertible.*

Proof. An explicit inverse can be seen by writing

$$\begin{aligned} & (\lambda_r - \lambda_1) \cdots (\widehat{\lambda_r - \lambda_r}) \cdots (\lambda_r - \lambda_d) \cdot v_r \\ &= \left(\sum_{\nu} (w - \lambda_1) \cdots (\widehat{w - \lambda_\nu}) \cdots (w - \lambda_d) v_\nu \right) \Big|_{w=\lambda_r} \\ &= \sum_{\nu, \mu} (-1)^\mu \sigma_\mu(\lambda_1, \dots, \hat{\lambda}_\nu, \dots, \lambda_d) \cdot v_\nu \cdot \lambda_r^{d-1-\mu} \\ &= \sum_{\mu} (-1)^\mu \lambda_r^{d-1-\mu} (A \cdot v)_\mu. \end{aligned}$$

To extend over the critical set $D_C \subset \Delta$ we interpret b_r as functions on $\Delta \times S^d \mathbb{C} \simeq \Delta \times \mathbb{C}^d$. Recall that the isomorphism $S^d \mathbb{C} \simeq \mathbb{C}^d$ is given by the elementary symmetric functions $\sigma_1, \dots, \sigma_d$, which in fact provide global holomorphic coordinates on $S^d \mathbb{C}$. We henceforth endow $S^d \mathbb{C}$ with the differentiable structure thus inherited.

Lemma 1.2. *The functions on $\Delta \times S^d \mathbb{C}$ induced by*

$$b_r: \Delta \times \mathbb{C}^d \longrightarrow \mathbb{C}, \quad (z, \lambda_1, \dots, \lambda_d) \longmapsto \sum_{\nu} \sigma_r(\lambda_1, \dots, \hat{\lambda}_\nu, \dots, \lambda_d) \cdot b(z, \lambda_\nu)$$

are of Hölder class C^α for some $\alpha = \alpha(d) > 0$. They are smooth away from Δ times the discriminant locus $D \subset S^d \mathbb{C}$. Moreover, if b depends holomorphically on w , or if $d = 2$, then the b_r are Lipschitz. The Lipschitz constant tends to zero with $\|\nabla b\|_\infty$.

Proof. It is clear by the definition of the topology on $S^d \mathbb{C} = \mathbb{C}^d / S_d$ that b_r and all partial derivatives in z are continuous. For the Hölder property we consider $\sum_{\nu=1}^d \sigma_{r-1}(\lambda_1, \dots, \hat{\lambda}_\nu, \dots, \lambda_d) \cdot b(z, \lambda_\nu)$ as smooth function in $(z; \lambda_1, \dots, \lambda_d) \in \Delta \times \mathbb{C}^d$. Now b_r equals $1/d!$ times the trace of this function under the branched cover $\Delta \times \mathbb{C}^d \rightarrow \Delta \times S^d \mathbb{C}$. Functions of this

type have been studied by Barlet. Among other things he proved that after blowing up the base $\Delta \times S^d\mathbb{C}$ to make the branch divisor simple normal crossing, traces of smooth functions are locally of class $C^{\tilde{\alpha}}$ with $\tilde{\alpha} = 2/\beta$ with β the maximal ramification index (see [3, Theorem 3 together with Lemma 4] and [4]). Now quite generally, if f is a function on some $U \subset \mathbb{C}^d$ and the pull-back σ^*f under some blowing-up σ is Hölder, then f is also Hölder, but possibly of smaller exponent.

The smoothness statement follows since D is the branch locus of the covering $\mathbb{C}^d \rightarrow S^d\mathbb{C}$.

If b is holomorphic in w then also the b_r are holomorphic in w by the Riemann Extension Theorem. Hence in this case the b_r are even smooth. Finally, for $d = 2$ the discriminant locus is smooth and the Lipschitz property follows from Barlet's results. Alternatively, one can do a simple explicit computation.

We are now ready to prove Theorem 0.1.

Proof of Theorem 0.1. Given a tuple of functions $\mathbf{a} = (a_1, \dots, a_d)$ with $\partial_{\bar{z}}a_r = b_r(\mathbf{a})$ and discriminant $\delta(\mathbf{a}) \neq 0$, let $C \subset \Delta \times \mathbb{C}$ be the associated curve. We first discuss regularity. The Sobolev embedding $W^{1,p}(\Delta) \subset C^{1-\frac{2}{p}}(\Delta)$ shows local boundedness of the a_r . Then under our hypothesis on b , by the preceding lemma $b_r(\mathbf{a})$ is Hölder of some exponent $\alpha > 0$ too. Now for any smooth function ρ on Δ with compact support we can reconstruct ρa_r from

$$\partial_{\bar{z}}(\rho a_r) = \partial_{\bar{z}}\rho \cdot a_r + \rho \cdot b_r(\mathbf{a}) \in C^\alpha(\Delta)$$

by application of the Cauchy integral operator. From the standard estimates for the latter (cf. [11]), we obtain $\rho a_r \in C^{1,\alpha}(\Delta)$ and hence $a_r \in C_{\text{loc}}^{1,\alpha}(\Delta)$.

Now let us assume we are given $\mathbf{a}$ fulfilling (1.2). Near any $P \in \Delta \setminus D_C$ there exist $\lambda_1, \dots, \lambda_d$ with $a_r = \sigma_r(\lambda_1, \dots, \lambda_d)$. From our discussion following Equation (1.2) it follows that C is indeed J -holomorphic over a neighbourhood of P . To investigate J -holomorphicity near D_C we need a lemma.

Lemma 1.3. *The zero set of the discriminant $\delta(\mathbf{a})$ is discrete.*

Proof. In a neighbourhood of some $P \in \Delta \setminus D_C$ we may write $a_r = \sigma_r(\lambda_1, \dots, \lambda_d)$. Using the equation $\partial_{\bar{z}}\lambda_i = b(z, \lambda_i)$ we compute

$$\begin{aligned} \partial_{\bar{z}}\delta(\mathbf{a}) &= \partial_{\bar{z}} \prod_{i < j} (\lambda_i - \lambda_j)^2 = 2 \sum_{i < j} (\lambda_i - \lambda_j) (\partial_{\bar{z}}\lambda_i - \partial_{\bar{z}}\lambda_j) \prod_{\substack{k < l \\ (k,l) \neq (i,j)}} (\lambda_k - \lambda_l)^2 \\ &= 2 \sum_{i < j} (\lambda_i - \lambda_j) (b(z, \lambda_i) - b(z, \lambda_j)) \prod_{\substack{k < l \\ (k,l) \neq (i,j)}} (\lambda_k - \lambda_l)^2. \end{aligned}$$

Therefore, the discriminant fulfills the linear $\bar{\partial}$ -equation $\partial_{\bar{z}}\delta(\mathbf{a}) = f \cdot \delta(\mathbf{a})$ with coefficient

$$f = 2 \sum_{i < j} \frac{b(z, \lambda_i) - b(z, \lambda_j)}{\lambda_i - \lambda_j}.$$

Because $\nabla_w b$ is uniformly bounded, $f \in L^\infty$. A standard trick now reduces to the case of holomorphic functions^[11]: Let $g \in W^{1,p}(\Delta)$ solve $\partial_{\bar{z}}g = -f$; then $e^g \cdot \delta(\mathbf{a})$ is holomorphic. Hence the claim.

For any $P \in \Delta$ we may thus choose a domain $U \subset \Delta$ with $U \cap D_C = \{P\}$. By J -holomorphicity of the projection $p : \Delta \times \mathbb{C} \rightarrow \Delta$, the map $p : C \cap p^{-1}(U \setminus \{P\}) \rightarrow U \setminus \{P\}$ is a holomorphic, finite, unbranched cover. The curve C thus decomposes over $U \setminus \{P\}$

into a finite disjoint union of pointed disks Δ^* such that in appropriate local holomorphic coordinates

$$p|_{\Delta^* \subset C} : t \mapsto z = t^m.$$

On this branch C is thus the image of a map of the form

$$\Delta^* \longrightarrow C, \quad t \longmapsto (t^m, \lambda(t))$$

for some smooth function λ . Pseudo-holomorphicity is expressed in terms of λ by

$$\partial_{\bar{t}}\lambda(t) = \partial_{\bar{z}}\lambda \cdot \partial_{\bar{t}}\bar{z} = m\bar{t}^{m-1}b(t^m, \lambda(t)). \quad (1.3)$$

This shows that $|\partial_{\bar{t}}\lambda|$ is uniformly bounded. Moreover, properness of the map $\mathbb{C}^d \rightarrow S^d\mathbb{C}$ plus continuity of the a_r show that λ has a continuous extension to Δ . It is a well-known fact that this implies $\lambda \in W^{1,2}(\Delta)$ (cf. e.g. [8, Lemma 2.4.2]). By elliptic bootstrapping λ is smooth and hence $t \mapsto (t^k, \lambda(t))$ is indeed J -holomorphic.

Conversely, starting from a J -holomorphic curve C we checked at the beginning of this section that on $\Delta \setminus D_C$ there are uniquely defined smooth functions a_r fulfilling equation (1.2). To extend over the branch points let $\varphi : \Sigma \rightarrow \Delta \times \mathbb{C}$ be the J -holomorphic map with image C . So Σ is a union of Riemann surfaces with boundary, and the composition of φ with the projection $\Delta \times \mathbb{C} \rightarrow \Delta$ exhibits Σ as a branched cover of the unit disk. Properness of the projection $C \rightarrow \Delta$ implies boundedness of the a_r . In particular, $\partial_{\bar{z}}a_r = b_r(z, \mathbf{a})$ is bounded in L^p . Now elliptic regularity as above shows $a_r \in W_{\text{loc}}^{1,p}(\Delta)$ for every $2 < p < \infty$.

Remark 1.1. While this was not a stimulus for this paper, we would like to point out that the use of symmetric polynomials in the study of pseudo-holomorphic curves is not entirely new. It has been used by Taubes in a static picture (for just one curve) to investigate the singularities of an almost everywhere pseudo-holomorphic current ([9, proof of Lemma 6.13]).

§2. Proof of Theorems 0.2 and 0.3

In this section we give sufficient conditions under which the solution space to Equation (1.2) is a Banach manifold. As this PDE becomes singular near multiple points, we certainly want to restrict to tuples $\mathbf{a}$ with discriminant $\delta(\mathbf{a})$ not vanishing identically. Note that rescaling $z \rightarrow \varepsilon^{-1}z$ leads merely to a change $b \rightarrow \varepsilon b$ in the describing Equation (1.1). Since we are only interested in the local behaviour, we may thus work over the unit disk and assume that the C^1 -norm of b is as small as we want. Notice also that rescaling w does not have any effect in that regard. The cases we can treat are singularities of multiplicity 2 ($d = 2$) and the more artificial case that b is holomorphic fiberwise (in the w -direction).

Proof of Theorem 0.2. We view (1.2) as a nonlinear map

$$\Phi : \mathcal{B} \longrightarrow \mathcal{E}, \quad \mathbf{a} = (a_1, \dots, a_d) \longmapsto (\partial_{\bar{z}}a_r - b_r(z, \mathbf{a}))_{r=1, \dots, d}$$

between function spaces

$$\mathcal{B} := \{\mathbf{a} \in W^{1,p}(\Delta, \mathbb{C}^d) \mid \delta(\mathbf{a}) \not\equiv 0\}, \quad \mathcal{E} := L^p(\Delta, \mathbb{C}^d),$$

where we choose some $2 < p < \infty$. Let $\mathbf{a}$ fulfill

$$\Phi(\mathbf{a}) = 0, \quad \delta(\mathbf{a}) \not\equiv 0.$$

We want to apply a fixed point method to find a bijection between small holomorphic perturbations of $\mathbf{a}$ and solutions of $\Phi = 0$. To this end we need an approximate right-inverse to the linearization of Φ . The latter can easily be checked to exist and to be a zero order perturbation of the $\partial_{\bar{z}}$ -operator. We thus simply take the right-inverse T to

$$\partial_{\bar{z}} : W^{1,p}(\Delta; \mathbb{C})^d \rightarrow L^p(\Delta; \mathbb{C})^d$$

provided by the Cauchy integral operator for our approximate right inverse:

$$T : L^p(\Delta; \mathbb{C})^d \longrightarrow W^{1,p}(\Delta; \mathbb{C})^d, \quad (T\xi)(z) = \left(\frac{1}{2\pi i} \int_{\Delta} \frac{\xi_r(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta} \right)_r. \quad (2.1)$$

From T we obtain the decomposition

$$W^{1,p}(\Delta)^d = \mathcal{O}^{1,p}(\Delta)^d \oplus L^p(\Delta)^d$$

with $\mathcal{O}^{1,p}(\Delta)$ the space of holomorphic functions on the unit disk of Sobolev class $(1, p)$. The correspondence is

$$\mathcal{O}^{1,p}(\Delta)^d \oplus L^p(\Delta)^d \ni (\mathbf{h}, \xi) \mapsto \mathbf{h} + T\xi \in W^{1,p}(\Delta)^d,$$

with inverse $f \mapsto (f - T\partial_{\bar{z}}f, \partial_{\bar{z}}f)$. We therefore want to find solutions of Equation (1.2) of the form

$$\Phi(\mathbf{a} + \mathbf{h} + T\xi(\mathbf{h})) = 0.$$

For any $\mathbf{h}$ we claim that the map

$$K_{\mathbf{h}}(\xi) = \xi - \Phi(\mathbf{a} + \mathbf{h} + T\xi)$$

is contractive. Expanding and using $\Phi(\mathbf{a}) = 0$ we obtain

$$K_{\mathbf{h}}(\xi) = b(z, \mathbf{a} + \mathbf{h} + T\xi) - b(z, \mathbf{a}).$$

We therefore have to estimate

$$\|K_{\mathbf{h}}(\xi) - K_{\mathbf{h}}(\zeta)\|_p = \|b(z, \mathbf{a} + \mathbf{h} + T\xi) - b(z, \mathbf{a} + \mathbf{h} + T\zeta)\|_p$$

by the L^p -distance of ξ and ζ . Under our hypothesis b is Lipschitz, with arbitrarily small Lipschitz constant $q = q(\|\nabla b\|_{\infty})$ (Lemma 1.2). We obtain

$$\|K_{\mathbf{h}}(\xi) - K_{\mathbf{h}}(\zeta)\|_p \leq a\|T\| \cdot \|\xi - \zeta\|_p.$$

Assuming $q < 1/\|T\|^{-1}$ we thus see that K is indeed contractive. Therefore, for any $\mathbf{h}$ the equation $K_{\mathbf{h}}(\xi) = \xi$ has a unique solution $\xi(\mathbf{h})$. Moreover, the norm of ξ tends to zero with $\mathbf{h}$.

Proof of Theorem 0.3. Without bounds on the linearization of the equation we need to use stronger functional analytic methods. For our existence problem the following version of the Leray-Schauder fixed point theorem is custom made.

Theorem 2.1. ^[10, Theorem 14.B.5] *Let $\mathcal{B}$ be a Banach space and let $F : [0, 1] \times \mathcal{B} \rightarrow \mathcal{B}$ be a continuous and compact map. Putting $F_{\sigma}(\xi) = F(\sigma, \xi)$ for $\sigma \in [0, 1]$, we assume that $F_0 \equiv \mathbf{b}_0$ for some $\mathbf{b}_0 \in \mathcal{B}$, and that the fixed points of F_{σ} are uniformly bounded for all $\sigma \in [0, 1]$:*

$$F_{\sigma}(\xi) = \xi \Rightarrow \|\xi\| < M.$$

Then F_1 has a fixed point.

With the notations of the proof of Theorem 0.2 above, we apply this theorem with $\mathcal{B} = L^p(\Delta)^d$ to

$$F_\sigma(\xi) := \sigma \mathbf{b}(z, \mathbf{h} + T\xi).$$

A point $\xi \in \mathcal{B}$ is a fixed point of F_1 iff $\xi = \partial_{\bar{z}}(\mathbf{h} + T\xi)$ equals $\mathbf{b}(z, \mathbf{h} + T\xi)$, so these are in one-to-one correspondence with solutions of (1.1) of the form $\mathbf{h} + T\xi$. Moreover, $F_0 \equiv 0$, and F is a compact map as composition of the compact operator

$$\text{Id} \times T : [0, 1] \times L^p(\Delta)^d \rightarrow [0, 1] \times C^0(\Delta)^d$$

with the continuous map

$$[0, 1] \times C^0(\Delta)^d \longrightarrow L^p(\Delta)^d, \quad (\sigma, \mathbf{v}) \longmapsto \sigma \mathbf{b}(z, \mathbf{h} + \mathbf{v}).$$

Finally, the uniform estimate for fixed points $\xi = F_\sigma(\xi)$ is

$$\|\xi\|_p = \|\sigma \mathbf{b}(z, \mathbf{h} + T\xi)\|_p \leq \pi^{1/p} \|\mathbf{b}\|_\infty =: M.$$

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