Chin. Ann. of Math.
23B:1(2002),11-24.

EXISTENCE AND UNIQUENESS OF A SOLUTION TO AN AEROACOUSTIC MODEL

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Abstract

A linear modelling of aeroacoustic waves propagation is discussed. The first point is an existence and uniqueness theorem. But restrictive assumptions are required on the velocity of the flow. Then a counter example proves that they are necessary.

Keywords Fluid-structure, Stability, Wave propagation, Localization of energy, Numerical scheme.
1991 MR Subject Classification 35F25, 35D05
Chinese Library Classification O175.29, O175.26 **Document Code** A
Article ID 0252-9599(2000)01-0011-14

§0. Introduction

A new challenge is arising in the mechanical engineering community. It concerns the modelling of aeroacoustic waves when fluid-structure interactions occur. As a matter of fact the main difficulty is due to a localization of the energy at the interface between the structure and the fluid. This phenomenon is well know in geophysics and is usually attributed to the so-called Stonley waves. But it can only appear if the sound celerity in the structure is smaller than the one in the fluid. Concerning the existence of solutions, some analogous conditions have to be discussed but the velocity of the steady flow has also to be taken into account. Our goal is to formulate a mathematical model in order to separate the wave propagation from the diffusion phenomenon which is induced by the viscosity of the fluid. It could be objected that the physical system that we analyze is only an approximation of the reality. But it has the huge advantage of allowing precise mathematical results.

The steady flow in which we consider the acoustic wave propagation is obtained from inviscid and incompressible flow hypothesis. Here again this is a simplification which enables us to derive nice mathematical properties. It is not obvious that similar results could be obtained with compressible or/and viscous flow.

§1. Modelling of Pressure Wave in an Air Flow

Let us consider a three dimensional open set denoted by Ω. Its boundary contains three parts. One—say Γ_0—corresponds to a vanishing acoustic pressure. The complementary of Γ_0 is Γ_1 and it corresponds to a structure. A part of it is rigid—say Γ_R—and the rest is

Manuscript received June 11, 2001.

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assumed to be flexible. It can be a plate, a membrane or a shell. It is denoted by Γ_s. But for the evaluation of the steady flow, all the boundary Γ_1 is assumed to be rigid.

Let us introduce the potential function φ^0 which is solution of the Neumann problem:

$$\begin{cases} -\triangle\varphi^0 = 0, & \text{in } \Omega,\ \int_\Omega \varphi^0 = 0, \\ \dfrac{\partial\varphi^0}{\partial\nu} = 0, & \text{on } \Gamma_1 = (\Gamma_s \cup \Gamma_R), \\ \dfrac{\partial\varphi^0}{\partial\nu} = g, & \text{on } \Gamma_0,\ (\int_{\Gamma_0} g = 0). \end{cases} \tag{1.1}$$

Then the steady velocity field is given by the gradient of φ^0. A very particular case, which will be helpful in the following, corresponds to a uniform flow (Γ_1 is a flat boundary).

If the magnitude of the velocity is U and the direction is e (parallel to Γ_1), then one has

$$\varphi^0 = (x - xg) \bullet eU,$$

where $x = (x_1, x_2, x_3)$ are the coordinates of a point of Ω and xg its center of inertia. Finally the dot stands for the scalar product in $\mathbb{R}^3$. One important question for our study is the regularity of φ^0. As a matter of fact it is very classical to prove the $C^\infty(\Omega)$ regularity of φ^0. But it is not true up to the boundary of Ω.

When $\partial\Omega$ has corners, there exist singularities which restrict the smoothness of φ^0. Nevertheless we shall assume that φ^0 is sufficiently regular in order to justify the following calculus. But the case mentioned on Fig.1.1 is also very important and some physical phenomena can appear near the corner on Γ_s.

Case 1. $\varphi^0 \in C^1(\overline{\Omega})$ Case 2. $\varphi^0 \notin C^1(\overline{\Omega})$

Fig. 1.1 Singular Geometries

We discuss in the text the difficulty which arises when φ^0 is not in the space $C^2(\bar{\Omega})$.

Let us set, assuming $\varphi^0 \in C^2(\bar{\Omega})$,

$$\begin{cases} U_s = \sup\limits_{x\in\bar{\Gamma}_s} |\nabla_s\varphi^0|(x), \\ U = \sup\limits_{x\in\bar{\Omega}} |\nabla\varphi^0|(x), \\ H = \sup\limits_{\substack{x\in\bar{\Omega} \\ i,j\in\{1,2,3\}}} \Big|\dfrac{\partial^2\varphi^0}{\partial x_i \partial x_j}\Big|(x). \end{cases} \tag{1.2}$$

It is worth to notice that $U_s \le U$ and that $H = 0$ for a uniform flow. Furthermore only one component of the second order derivatives will be used in the following. Therefore the condition $\varphi^0 \in C^2(\bar{\Omega})$ can certainly be weakened.

Let us now assume that the unsteady waves in the fluid can also be represented by a potential function denoted by ϕ. It is dependent on both x and t. Furthermore, we assume that the fluid is barotropic (i.e. the pressure only depends on the mass density). Then setting (ρ_f is the mass density in the fluid)

$$F(\rho_f) = \int_{\rho_f^0}^{\rho_f} \frac{1}{\rho}\frac{\partial p}{\partial \rho}(\rho)d\rho,$$

where p is the pressure, one formulates Bernoulli theorem as follows:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + F(\rho) = 0 \qquad \text{in } \Omega\times]O,T[. \tag{1.3}$$

Furthermore the mass conservation principle is

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\nabla\phi) = 0 \qquad \text{in } \Omega\times]O,T[. \tag{1.4}$$

These two equations are the general nonlinear aeroacoustic models. But it is necessary to specify the constitutive relationship between the mass density and the pressure. As a matter of fact this is a tough question which has been widely discussed by many authors (see for instance [9]). Actually the adiabatic hypothesis is commonly accepted, at least for the acoustic pressure. Obviously it is necessary to add boundary conditions and initial values for ϕ and ρ.

But, first of all we linearize (1.3) and (1.4). Therefore we set

$$\begin{cases} \phi(x,t) = \varphi^0(x) + \varphi(x,t), \\ \rho(x,t) = \rho^0 + \delta\rho(x,t), \end{cases} \tag{1.5}$$

which leads to

$$\begin{cases} \dfrac{\partial \delta\rho}{\partial t} + \rho^0\triangle\varphi + \nabla\varphi^0 \bullet \nabla\delta\rho = 0, & \text{in } \Omega\times]O,T[, \\ \dfrac{\partial \varphi}{\partial t} + \nabla\varphi^0 \bullet \nabla\varphi + \dfrac{1}{\rho^0}\dfrac{\partial p}{\partial \rho}(\rho^0)\delta\rho = 0, & \text{in } \Omega\times]O,T[. \end{cases} \tag{1.6}$$

Setting (sound celerity)

$$c_f = \sqrt{\frac{\partial p}{\partial \rho}(\rho_0)}, \tag{1.7}$$

we deduce that

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} + \nabla\varphi^0 \bullet \nabla\varphi + \dfrac{c_f^2}{\rho^0}\delta\rho = 0, & \text{in } \Omega\times]O,T[. \\ \dfrac{\partial \delta\rho}{\partial t} + \rho^0\triangle\varphi + \nabla\varphi^0 \bullet \nabla\delta\rho = 0, & \text{in } \Omega\times]O,T[, \end{cases} \tag{1.8}$$

Finally by applying the operator $\dfrac{\partial}{\partial t}(\bullet) + \nabla\varphi^0 \bullet \nabla(\bullet)$ to the first equation and using the second one, we obtain the classical wave equation

$$\frac{\partial^2 \varphi}{\partial t^2} + 2\nabla\varphi^0 \bullet \nabla\frac{\partial \varphi}{\partial t} + \nabla\varphi^0 \bullet \nabla(\nabla\varphi^0 \bullet \nabla\varphi) - c_f^2\triangle\varphi = 0. \tag{1.9}$$

Once φ is known, $\delta\rho$ can be computed by solving the second equation (1.8), which is a linear advection model. The streamlines are those of the steady flow described by the potential function φ^0. Let us now discuss the boundary conditions which should be satisfied by φ. Because there are three different boundaries, there are three different conditions.

(a) The rigid wall boundaries condition (on $\boldsymbol{\Gamma_R}$)

The normal component of the velocity is zero because the air particles cannot enter the wall. Hence the boundary condition is

$$\frac{\partial \varphi}{\partial t} = 0 \qquad \text{on } \Gamma_R \times]O, T[. \tag{1.10}$$

(ν is the unit normal to Γ_R).

(b) The flexible structure boundary condition (on $\boldsymbol{\Gamma_s}$)

This is certainly the most difficult condition to write correctly. This is due to the rotation of the normal to the structure. Let us denote by z the normal displacement of the structure along the unit normal. Then a simple calculus using differential geometry enables one to derive the following relation

$$\frac{\partial \varphi}{\partial \nu} = \nabla \varphi \bullet \nu = \frac{\partial z}{\partial t} + \nabla_s \varphi^0 \bullet \nabla_s z, \tag{1.11}$$

where ∇_s is the gradient operator along the surface Γ_s.

Remark 1.1. The convection term $\nabla_s \varphi^0 \bullet \nabla_s z$ is omitted in most papers. Then there is no energy conservation and the existence theorem seems to be false.

(c) Boundary condition on the inner boundary ($\boldsymbol{\Gamma_0}$).

Let us consider a point m on Γ_0 . There are several ways to prescribe a boundary condition at m. One of them consists in assuming that the movements of the particles are normal to Γ_0. Hence we set $\varphi = 0$ on Γ_0. This condition is very convenient for our analysis. But sometimes people prefer to formulate a transparency condition. It traduces that the coming in or out waves are not modified by the boundary Γ_0:

$$\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi^0}{\partial \nu} \bullet \frac{\partial \varphi}{\partial \nu} = 0 \qquad \text{on } \Gamma_0 \times]O, T[. \tag{1.12}$$

When the steady flow is uniform and for instance parallel to the axis x_1, one has

$$\frac{\partial \varphi}{\partial t} + U \frac{\partial \varphi}{\partial x_1},$$

on each boundary orthogonal to x_1. As a matter of fact the condition (1.12) is really meaningful if Γ_0 is a potential line for φ^0. Then (1.12) is equivalent to

$$\frac{\partial \varphi}{\partial t} + \nabla \varphi^0 \bullet \nabla \varphi = 0,$$

which simply traduces that the acoustics pressure is vanishing on Γ_0.

But another possibility, which maybe is more realistic, consists in prescribing that, on Γ_0, one has

$$\varphi(x, t) = 0 \qquad \forall\ (x, t) \in \Gamma_0 \times]O, T[, \tag{1.13}$$

which corresponds to the hypothesis that the acoustic waves which are reaching Γ_0 are radial (no tangential velocity). We use that one in the following. But similar results could be obtained with other boundary conditions if the boundary Γ_0 is correctly chosen. This will be discussed in a forthcoming paper.

The last point concerns the initial conditions satisfied by φ. We set

$$\varphi(x, 0) = \varphi_0(x), \quad \frac{\partial \varphi}{\partial t}(x, 0) = \varphi_1(x) \quad \forall x \in \Omega, \tag{1.14}$$

and the regularity of φ_0 and φ_1 will be discussed in the following.

§2. The Structural Model

The flexible structure occupies the portion Γ_s of the boundary of Ω. It can be a plate or a shell. But just in order to simplify the writings, we consider that it is a flat membrane.

Therefore the deflection z is solution of the following model:

$$\begin{cases} \dfrac{\partial^2 z}{\partial t^2} - c_s^2 \triangle_s z = -\dfrac{\rho_f}{\rho_s}\big(\dfrac{\partial \varphi}{\partial t} + \nabla_s \varphi^0 \bullet \nabla_s \varphi\big) + q, & \text{on } \Gamma_s \times]O,T[, \\ z = 0, & \text{on } \partial\Gamma_s \times]O,T[, \\ z(x,t) = z_0(x), \ \ \dfrac{\partial z}{\partial t}(x,0) = z_1(x), & \forall x \in \Gamma_s, \end{cases} \tag{2.1}$$

where $z_0,\ z_1$ and q are given functions. The first term in the right hand side of the previous equation is the acoustic pressure due to the fluid and applied to the flexible structure. As a matter of fact, the term $\dfrac{\partial \varphi^0}{\partial \nu}\dfrac{\partial \varphi}{\partial \nu}$ is vanishing because $\dfrac{\partial \varphi^0}{\partial \nu} = 0$ on Γ_s. The coefficients ρ_f and ρ_s are respectively the mass density of the fluid and of the structure. Finally c_s is the sound celerity in the structure.

The coupling between the two mechanical models appears in the terms at the right hand side of (2.1) and on the one of (1.11).

The function q which is at the right hand side of (2.1) represents an external force applied directly to the structure. It can be a control for instance as we have studied it in [6] or [7].

§3. An a Priori Estimate for the Coupled Model

Let us start with few notions. First of all we recall that it is assumed that $\varphi^0 \in C^2(\bar\Omega)$ (even if it is possible to slightly weaken this hypothesis). Then we introduce the Steklov problem which consists in finding $(y,\eta) \in V \times \mathbb{R}^+$, $\Big(V = \Big\{v \in H^1(\Omega),\ v = 0 \text{ on } \Gamma_0\Big\}\Big)$, such that

$$\begin{cases} -c_f^2 \triangle y + \nabla\varphi^0 \bullet \nabla(\nabla\varphi^0 \bullet \nabla y) = 0 \qquad \text{in } \Omega, \\ \dfrac{\partial y}{\partial \nu} = 0 \qquad \text{on } \Gamma_R, \qquad c_f^2 \dfrac{\partial y}{\partial \nu} = \eta y \qquad \text{on } \Gamma_s. \end{cases} \tag{3.1}$$

This is a very classical spectral problem as far as $c_f > U\big(= \max\limits_{x \in \bar\Omega} |\nabla \varphi^0|\big)$. From the min-max theorem, we know that the smallest eigenvalue—say η_0—satisfies the following inequality

$$\forall v \in V, \quad \eta_0 \int_{\Gamma_s} v^2 \le c_f^2 \int_\Omega |\nabla v|^2 - \int_\Omega (\nabla\varphi^0 \bullet \nabla v)^2 \overset{\text{def}}{=} a^f(v,v). \tag{3.2}$$

Let us now assume just for a while that (φ, z) is a smooth enough solution of the coupled aeroacoustic system. Then multiplying (1.9) by $\dfrac{\partial \varphi}{\partial t}$ and (2.1) by $\dfrac{\partial z}{\partial t}$ and by integrating them over Ω and along Γ_s, we obtain the following identity:

$$\begin{aligned} &\frac{\partial}{\partial t}\Big\{\frac{1}{2}\int_\Omega \Big(\frac{\partial \varphi}{\partial t}\Big)^2 + \frac{c_f^2}{2}\int_\Omega |\nabla\varphi|^2 - \frac{1}{2}\int_\Omega (\nabla\varphi^0 \bullet \nabla\varphi)^2 + c_f^2 \frac{\rho_s}{\rho_f}\Big(\frac{1}{2}\int_{\Gamma_s}\Big(\frac{\partial z}{\partial t}\Big)^2 \\ &+ \frac{c_s^2}{2}\int_{\Gamma_s} |\nabla_s z|^2\Big)\Big\} \\ &= c_f^2 \Big\{\int_{\Gamma_s} \nabla_s\varphi^0 \bullet \Big(\nabla_s z \frac{\partial \varphi}{\partial t} - \nabla_s \varphi \frac{\partial z}{\partial t}\Big) + \frac{\rho_s}{\rho_f}\int_{\Gamma_s} q \frac{\partial z}{\partial t}\Big\}. \end{aligned}$$

Let us set

$$A = \int_0^T \int_{\Gamma_s} \nabla_s\varphi^0 \bullet \Big(\nabla_s z \frac{\partial \varphi}{\partial t} - \nabla_s\varphi \frac{\partial z}{\partial t}\Big) = \int_0^T \int_{\Gamma_s} \triangle_s \varphi^0 \varphi \frac{\partial z}{\partial t} + \Big[\int_{\Gamma_s} (\nabla_s\varphi^0 \bullet \nabla_s z)\varphi\Big]_0^T;$$

therefore, for any number $\alpha > 0$ and $\beta > 0$ (H is defined in (1.2)),

$$\begin{aligned}|A| \leq & \frac{\rho_f 2H\sqrt{\alpha}}{c_f^2\rho_s}\Big[\frac{1}{2}c_f^2\frac{\rho_s}{\rho_f}\int_0^T\int_{\Gamma_s}(\frac{\partial z}{\partial t})^2 + \frac{c_f^2\rho_s}{2\rho_f\alpha}\int_0^T\int_{\Gamma_s}\varphi^2\Big]\\ &+\frac{U_s\sqrt{\beta}\rho_f}{c_s^2c_f^2\rho_s}\Big[\frac{1}{2}\frac{c_s^2c_f^2\rho_s}{\rho_f}\int_{\Gamma_s}|\nabla_s z|^2 + \frac{\rho_s c_s^2c_f^2}{2\beta\rho_f}\int_{\Gamma_s}\varphi^2\Big](0)\\ &+\frac{U_s\sqrt{\beta}\rho_f}{c_s^2c_f^2\rho_s}\Big[\frac{1}{2}\frac{c_s^2c_f^2\rho_s}{\rho_f}\int_{\Gamma_s}|\nabla_s z|^2 + \frac{\rho_s c_s^2c_f^2}{2\beta\rho_f}\int_{\Gamma_s}\varphi^2\Big](T).\end{aligned}$$

Then because of (3.2),

$$\begin{aligned}|A| \leq & \frac{\rho_f 2H\sqrt{\alpha}}{c_f^2\rho_s}\Big[\frac{1}{2}c_f^2\frac{\rho_s}{\rho_f}\int_0^T\int_{\Gamma_s}(\frac{\partial z}{\partial t})^2 + \frac{c_f^2\rho_s}{2\eta_0\rho_f\alpha}\int_0^T a^f(\varphi,\varphi)\Big]\\ &+\frac{U_s\sqrt{\beta}\rho_f}{c_s^2c_f^2\rho_s}\Big[\frac{1}{2}\frac{c_s^2c_f^2\rho_s}{\rho_f}\int_{\Gamma_s}|\nabla_s z|^2 + \frac{\rho_s c_s^2c_f^2}{2\beta\rho_f\eta_0}a^f(\varphi,\varphi)\Big](0)\\ &+\frac{U_s\sqrt{\beta}\rho_f}{c_s^2c_f^2\rho_s}\Big[\frac{1}{2}\frac{c_s^2c_f^2\rho_s}{\rho_f}\int_{\Gamma_s}|\nabla_s z|^2 + \frac{\rho_s c_s^2c_f^2}{2\beta\rho_f\eta_0}a^f(\varphi,\varphi)\Big](T).\end{aligned}$$

Let us set

$$\alpha = \frac{\rho_s c_f^2}{\eta_0\rho_f}, \qquad \beta = \frac{c_s^2c_f^2\rho_s}{\eta_0\rho_f},$$

and we introduce the energy of the coupled system:

$$\varepsilon(t) = \frac{1}{2}\int_\Omega(\frac{\partial\varphi}{\partial t})^2 + \frac{1}{2}a^f(\varphi,\varphi) + \frac{c_f^2\rho_s}{\rho_f}\Big[\frac{1}{2}\int_{\Gamma_s}(\frac{\partial z}{\partial t})^2 + \frac{c_s^2}{2}\int_{\Gamma_s}|\nabla z|^2\Big]; \tag{3.3}$$

we obtain

$$|A| \leq \frac{2H}{c_f\sqrt{\eta_0}}\sqrt{\frac{\rho_f}{\rho_s}}\int_0^T\varepsilon(t)dt + \frac{U_s}{c_fc_s\sqrt{\eta_0}}\sqrt{\frac{\rho_f}{\rho_s}}[\varepsilon(T)+\varepsilon(0)]. \tag{3.4}$$

Then from (3.3) and (3.4) we derive the following inequality:

$$|\varepsilon(t)-\varepsilon(0)| \leq C_1\int_0^t\varepsilon(s)ds + C_2(\varepsilon(t)+\varepsilon(0)) + C_3, \tag{3.5}$$

$$C_1 = \sqrt{\frac{\rho_f}{\rho_s}}\frac{c_f2H}{\sqrt{\eta_0}} + \frac{C_f^2}{2}\frac{\rho_s}{\rho_f}, \quad C_2 = \frac{c_fU_s}{c_s\sqrt{\eta_0}}\sqrt{\frac{\rho_f}{\rho_s}}, \quad C_3 = c_f^2\frac{\rho_s}{\rho_f}\|q\|_{L^2(]O,T[\times\Gamma_s)}. \tag{3.6}$$

From (3.5), we deduce that

(i) $(1-C_2)\varepsilon(t) \leq (1+C_2)\varepsilon(0) + C_1\int_0^t\varepsilon(s)ds + C_3$,

(ii) $(1+C_2)\varepsilon(t) \geq (1-C_2)\varepsilon(0) - C_1\int_0^t\varepsilon(s)ds - C_3$.

Therefore (let us assume that $C_2 < 1$ which is discussed in the following)

$$\frac{C_1\varepsilon(t)}{(1+C_2)\varepsilon(0)+C_3+C_1\int_0^t\varepsilon(s)ds} \leq \frac{C_1}{1-C_2}.$$

Then

$$\frac{(1+C_2)\varepsilon(0)+C_3+C_1\int_0^t\varepsilon(s)ds}{(1+C_2)\varepsilon(0)+C_3} \leq e^{\frac{C_1t}{1-C_2}},$$

which enables one to write

$$C_1\int_0^t\varepsilon(s)ds \leq [C_3+(1+C_2)\varepsilon(0)]\Big[e^{\frac{C_1t}{1-C_2}}-1\Big].$$

Finally

$$\varepsilon(t) \leq \Big(\frac{1+C_2}{1-C_2}\Big)\varepsilon(0) + \frac{C_3}{1-C_2} + \Big(\frac{C_3+(1+C_2)\varepsilon(0)}{1-C_2}\Big)\Big[e^{(\frac{C_1 t}{1-C_2})} - 1\Big]. \tag{3.7}$$

In a similar way one has also the following lower bound which is meaningful if $C_3 = 0!$ $(q \equiv 0)$:

$$\varepsilon(t) \geq \Big(\frac{1-C_2}{1+C_2}\Big)\varepsilon(0) - \frac{C_3}{1-C_2} - \Big(\frac{C_3+(1+C_2)\varepsilon(0)}{1+C_2}\Big)\Big[e^{(\frac{C_1 t}{1-C_2})} - 1\Big]. \tag{3.8}$$

Lemma 3.1. *Let us summarize the previous results:*

Assume that $U < c_f$, $U_s < \sqrt{\frac{\rho_s}{\rho_f}\eta_0(\frac{c_s}{c_f})}$, $\varphi^0 \in C_2(\bar{\Omega})$. *Then the energy* $\varepsilon(t)$ *defined in* (3.3) *is such that*

$$\begin{aligned}&\Big(\frac{1-C_2}{1+C_2}\Big)\varepsilon(0) - \frac{C_3}{1-C_2} - \Big(\frac{C_3+(1+C_2)\varepsilon(0)}{1+C_2}\Big)\Big[e^{(\frac{C_1 t}{1-C_2})} - 1\Big]\\ &\leq \varepsilon(t) \leq \Big(\frac{1+C_2}{1-C_2}\Big)\varepsilon(0) + \frac{C_3}{1-C_2} + \Big(\frac{C_3+(1+C_2)\varepsilon(0)}{1-C_2}\Big)\Big[e^{(\frac{C_1 t}{1-C_2})} - 1\Big],\end{aligned}$$

where C_1, C_2 *and* C_3 *are defined in* (3.6).

Remark 3.1. The condition on U_s can be differently traduced. Let us denote by η_{01} the smallest eigenvalue of the Steklov problem for $U = 0$ ($\varphi^0 \equiv 0!$). Then, Schwarz inequality, applied to (3.2), enables one to derive the following inequality:

$$\forall v \in V,\ \eta_{01}\int_{\Gamma_s} v^2 \leq c_f^2 \int_\Omega |\nabla v|^2,$$

and therefore from the fact that v_0 is the eigenvector associated to η_0,we have

$$\eta_0 = c_f^2\int_\Omega |\nabla v_0|^2 - \int_\Omega (\nabla\varphi^0 \bullet \nabla v_0)^2 \geq (c_f^2 - U^2)\int_\Omega |\nabla v_0|^2 \geq \eta_{01}\Big(\frac{c_f^2 - U^2}{c_f^2}\Big).$$

(We used the normalization $\int_{\Gamma_s} v_0^2 = 1$.) Hence the restriction on U_s can be ensured by a more restrictive one:

$$U_s < \sqrt{1-M^2}\,\frac{c_s}{c_f}\sqrt{\frac{\eta_{01}\rho_s}{\rho_f}}, \tag{3.9}$$

where $M = \dfrac{U}{c_f}$ is the Mach number. Furthermore η_{01} can be estimated by a simple calculus as one does for the Poincaré constant.

Thus one obtains (D is the diameter of Ω)

$$\eta_{01} \simeq \frac{c_f^2}{d},$$

and finally the inequality (3.9) is approximately equivalent to

$$U_s < \sqrt{1-M^2}\,c_s\sqrt{\frac{\rho_s}{D\rho_f}}. \tag{3.10}$$

Remark 3.2. One could object that the restrictive condition on U_s is sufficient but not necessary. As a matter of fact, it is certainly not optimal but one can prove that there is a restriction on the tangential velocity of the steady flow near the flexible structure. Let us discuss this point with a two dimensional example. The geometry is the one represented in Fig. 3.1.

Fig. 3.1 The Two Dimensional Geometry for the Example Concerning the Restriction on the U_s

Let us set for arbitrary elements $X = (\varphi, z) \in V \times H_0^1(\Gamma_s)$,

$$a_0(X,X) = c_f^2 \int_\Omega |\nabla\varphi|^2 - \int_\Omega (\nabla\varphi^0 \bullet \nabla\varphi)^2 + c_f^2 \frac{\rho_s}{\rho_f} c_s^2 \int_{\Gamma_s} |\nabla_s z|^2 - 2c_f^2 \int_{\Gamma_s} (\nabla_s \varphi^0 \bullet \nabla_s z)\varphi.$$

Setting $\varphi^0 = x_1 U$ (uniform steady flow), one has

$$a_0(X,X) = c_f^2 \int_\Omega |\nabla\varphi|^2 - U^2 \int_\Omega \Big(\frac{\partial\varphi}{\partial x_1}\Big)^2 + c_f^2 \frac{\rho_s}{\rho_f} c_s^2 \int_0^L |\nabla_s z|^2 - 2c_f^2 U \int_0^L \frac{\partial z}{\partial x_1}\varphi.$$

Let us denote by $(y_1,\ \eta_1)$ the second eigenmode of the Steklov model (3.1). One has the identity :

$$c_f^2 \int_\Omega |\nabla y_1|^2 - U^2 \int_0^L \Big|\frac{\partial y_1}{\partial x_1}\Big|^2 = \eta_1 \int_0^L |y_1|^2,$$

and therefore, setting $X = (y_1, z)$, one obtains

$$a_0(X,X) = c_f^2 \frac{\rho_s}{\rho_f} c_s^2 \int_0^L \Big|\frac{\partial z}{\partial x_1}\Big|^2 + \eta_1 \int_0^L |y_1|^2 - 2c_f^2 U \int_0^L \Big(\frac{\partial z}{\partial x_1}\Big) y_1.$$

But, because of the simplified geometry (rectangle), it is possible to compute analytically y_1 and η_1. One obtains the following expressions:

$$\begin{cases} y_1(x_1,x_2) = A_1 \sin\Big(\dfrac{2\Pi x_1}{L}\Big) \mathrm{sh}\Big(\dfrac{2\Pi}{L}(x_2+\ell)\sqrt{1-M^2}\Big), \\ \eta_1 = \dfrac{2\Pi}{L} c_f^2 \sqrt{1-M^2} \coth\Big(\dfrac{2\Pi\ell}{L}\sqrt{1-M^2}\Big), \end{cases}$$

where A_1 is a constant (it is arbitrary unless we choose a normalization condition). Let us now set for any arbitrary constant B:

$$z_1(x_1) = B \int_0^{x_1} y_1(s,0)ds.$$

One has the boundary conditions: $z_1(0) = z_1(L) =0$. Then for $X_1 = (y_1, z_1)$, we deduce the following relation

$$a_0(X_1,X_1) = c_f^2 \Big[c_s^2 \frac{\rho_s}{\rho_f} B^2 - 2BU + \frac{\eta_1}{c_f^2}\Big] \int_0^L |y_1(s,0)|^2 ds.$$

Let us assume that ($U = U_s$ in this example)

$$U > \frac{c_s}{c_f}\sqrt{\frac{\rho_s \eta_1}{\rho_f}}.$$

Then, there exist values of B such that

$$a_0(X_1, X_1) < 0.$$

In other words the bilinear form a_0 can be negative and no stability result can be obtained for the coupled fluid-structure model that we are considering. When

$$U = U_0 = \frac{c_s}{c_f}\sqrt{\frac{\rho_s \eta_1}{\rho_f}}, \tag{3.11}$$

the bilinear form a_0 has a kernel which is different from zero, but remains non-negative. It is also worth noting that for $M \to 1$, one has

$$U_0 \to c_s\sqrt{\frac{\rho_s}{\ell \rho_f}}.$$

But because we assumed that $U < c_f$, it is necessary that

$$c_s\sqrt{\frac{\rho_s}{\ell \rho_f}} \simeq c_f, \qquad \text{(because } \ M \simeq 1).$$

This relation is also the upper limit for c_s such that there exist Stoneley stationary waves near the flexible structure. One can find further details in [6].

§4. Approximation of the Aeroacoustic Model

Let us first introduce two bases of functions for the fluid and the structure separately.

(a) Basis for the fluid. It is solution of

$$\begin{cases} \text{find } (w, \lambda^f) \in V \times \mathbb{R} \text{ such that} \\ -c_f^2 \triangle w + \nabla\varphi^0 \bullet \nabla(\nabla\varphi^0 \bullet \nabla w) = \lambda^f w, \text{ in } \Omega, \\ w = 0 \text{ on } \Gamma_0, \ \dfrac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma_1 = (\Gamma_R \cup \Gamma_s). \end{cases} \tag{4.1}$$

(b) Basis for the structure. It is solution of

$$\begin{cases} \text{find } (z, \lambda^s) \in H_0^1(\Gamma_s) \times \mathbb{R} \text{ such that} \\ -c_s^2 \triangle_s z = \lambda^s z, \quad \text{on } \Gamma_s. \end{cases} \tag{4.2}$$

The existence of solutions to the system (4.1) and (4.2) is classical from the general spectral theory. But the hypothesis that $U < c_f$ is really necessary in order to prove the coerciveness of the bilinear form representing the fluid energy. Let us denote by $\{w_n\}$ (respectively $\{z_n\}$) the eigenvectors of the fluid (respectively the structure). Then we introduce the following finite dimensional spaces:

$$\begin{cases} V^N = \Big\{ y = \displaystyle\sum_{n=1,N} \alpha_n w_n, \quad \alpha_n \in \mathbb{R} \Big\}, \\ Z^N = \Big\{ z = \displaystyle\sum_{n=1,N} \beta_n z_n, \quad \beta_n \in \mathbb{R} \Big\}. \end{cases} \tag{4.3}$$

The approximate model is formulated from a weak formulation of the coupled system. It is defined by

$$
\left\{
\begin{array}{l}
\text{find } (\varphi^N, z^N) \in C^1\Big([O,T];\ V^N\Big) \times C^1\Big([O,T];\ Z^N\Big), \text{ such that} \\
\forall \Psi \in V^N,\ \displaystyle\int_\Omega \frac{\partial^2 \varphi^N}{\partial t^2}\Psi + 2\int_\Omega \nabla\varphi^0 \bullet \nabla \frac{\partial \varphi^N}{\partial t}\Psi + c_f^2 \int_\Omega \nabla\varphi^N \bullet \nabla\Psi \\
\qquad\qquad -\displaystyle\int_\Omega (\nabla\varphi^0 \bullet \nabla\varphi^N)(\nabla\varphi^0 \bullet \nabla\Psi) = c_f^2 \int_{\Gamma_s} \Big(\frac{\partial z^N}{\partial t} + \nabla_s z^n \bullet \nabla\varphi^0\Big)\Psi, \\
\forall v \in Z^N,\ \int_{\Gamma_s} \frac{\partial^2 z^N}{\partial t^2} v + c_s^2 \int_{\Gamma_s} \nabla_s z^N \bullet \nabla_a v = -\frac{\rho_f}{\rho_s}\int_{\Gamma_s}\Big(\frac{\partial\varphi^N}{\partial t} + \nabla_s\varphi^0 \bullet \nabla_s\varphi^N\Big)v \\
\qquad\qquad + \int_{\Gamma_s} \varphi v.
\end{array}
\right.
\tag{4.4}
$$

Because (4.4) is a finite dimensional linear differential system, it has a unique solution as soon as initial conditions on $(\varphi^N, z^N)(0)$ and $(\dot\varphi^N,\ \dot z^N)(0)$ are prescribed. For the sake of brevity let us choose the following conditions :

$$
\left\{
\begin{array}{ll}
z^N(0) = P_s^N z_0, & \dfrac{\partial z^n}{\partial t}(0) = P_s^N z_1, \\
\varphi^N(0) = P_f^N \varphi_0, & \dfrac{\partial \varphi^N}{\partial t}(0) = P_f^N \varphi_1.
\end{array}
\right.
$$

Here P_s^N (respectively P_f^N) is the projection from V (respectively $H_0^1(\Gamma_s)$), onto V^N (respectively onto Z^N). The next result gives an a priori estimate on (φ^N, z^N) independent of N.

Lemma 4.1. *Let us assume that* $U < c_f$ *and* $U_s < \dfrac{c_s}{c_f}\sqrt{\dfrac{\rho_s\eta_0}{\rho_f}}$ *where* η_0 *is the smallest eigenvalue of the Steklov problem defined in* (3.1). *Furthermore we consider that*

$$
q \in L^2(]O,T[\times\Gamma_s),\quad \varphi_0 \in V,\quad \varphi_1 \in L^2(\Omega),\quad z_0 \in H_0^1(\Gamma_s),\quad z_1 \in L^2(\Gamma_s).
$$

Then there exists a constant c which is independent of N and such that

$$
\Big[\Big\|\frac{\partial\varphi^N}{\partial t}\Big\|^2_{O,\Omega} + \|\varphi^N\|^2_{1,\Omega} + \Big\|\frac{\partial z^N}{\partial t}\Big\|^2_{0,\Gamma_s} + \|z^N\|^2_{1,\Gamma_s}\Big](t) \le c, \qquad \forall \in [O,T].
$$

Proof. Let us set $\Psi = \dot\varphi^N$ and $v = \dot z^N$ in (4.4). We obtain the following equality:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\Big\{\frac12\int_\Omega\Big(\frac{\partial\varphi^N}{\partial t}\Big)^2 + \frac{c_f^2}{2}\frac{\rho_s}{\rho_f}\int_{\Gamma_s}\Big(\frac{\partial z^N}{\partial t}\Big)^2 + \frac{c_f^2}{2}\int_\Omega |\nabla\varphi^N|^2 \\
&-\frac12\int_\Omega(\nabla\varphi^0 \bullet \nabla\varphi^N)^2 + \frac{c_f^2 c_s^2 \rho_s}{2\rho_f}\int_{\Gamma_s}|\nabla_s z^N|^2\Big\} \\
=\ & c_f^2\Big[\int_{\Gamma_s}(\nabla_s\varphi^0 \bullet \nabla_s z^N)\frac{\partial\varphi^N}{\partial t} - \int_{\Gamma_s}\Big(\nabla_s\varphi^0 \bullet \nabla_s\varphi^N - \frac{\rho_s}{\rho_f}q\Big)\frac{\partial z^N}{\partial t}\Big].
\end{aligned}
$$

Following the same method as we did in Section 3 and using Lemma 3.1, we derive the next estimate (φ^N and z^N are smooth functions):

$$
\begin{aligned}
&\frac12\int_\Omega\Big(\frac{\partial\varphi^N}{\partial t}\Big)^2 + \frac{c_f^2}{2}\frac{\rho_s}{\rho_f}\int_{\Gamma_s}\Big(\frac{\partial z^N}{\partial t}\Big)^2 + \frac{c_f^2}{2}\int_\Omega|\nabla\varphi^N|^2 \\
&-\frac12\int_\Omega\Big(\nabla\varphi^0 \bullet \nabla\varphi^N\Big)^2 + c_f^2\frac{c_s^2\rho_s}{2\rho_f}\int_{\Gamma_s}|\nabla_s z^N|^2 \le c_2,
\end{aligned}
$$

where c_2 is a constant which is independent of N.

§5. Existence and Uniqueness of a Solution

From Lemma 4.1, we deduce that there exists subsequence, say $(\varphi^{N'},\ z^{N'})$, such that

$$\begin{cases} \varphi^{N'} \to \varphi^* & \text{in } L^2(]O,T[;\ V) - \text{weak}, \\ z^{N'} \to z^* & \text{in } L^2\Big((]O,T[;\ H_0^1(\Gamma_s)\Big)) - \text{weak}, \\ \dfrac{\partial \varphi^{N'}}{\partial t} \to \chi^* & \text{in } L^2\Big((]O,T[;\ L^2(\Omega)\Big) - \text{weak}, \\ \dfrac{\partial z^{N'}}{\partial t} \to h^* & \text{in } L^2\Big((]O,T[;\ L^2(\Gamma_s)\Big) - \text{weak}. \end{cases} \tag{5.1}$$

It is classical to prove that

$$\chi^* = \frac{\partial \varphi^*}{\partial t}, \quad h^* = \frac{\partial z^*}{\partial t}.$$

Then one can use a weak formulation satisfied by the approximate solution (φ^N, z^N) in order to characterize (φ^*, z^*). One has $(\forall N \geq N_0\ )$

$$\begin{cases} -\displaystyle\int_0^T \int_\Omega \Big(\frac{\partial \varphi^N}{\partial t} + \nabla\varphi^0 \bullet \nabla\varphi^N\Big)\Big(\frac{\partial \Psi}{\partial t} + \nabla\varphi^0 \bullet \nabla\Psi\Big) - \Big[\int_\Omega \Big(\frac{\partial \varphi^N}{\partial t} + \nabla\varphi^0 \bullet \nabla\varphi^N\Big)\Psi\Big](0) \\ \quad +c_f^2 \displaystyle\int_0^T \int_\Omega \nabla\varphi^N \bullet \nabla\Psi = c_f^2 \int_0^T \int_{\Gamma_s} \Big(\frac{\partial \varphi^N}{\partial t} + \nabla_s\varphi^0 \bullet z^N\Big)\Psi, \quad \forall \Psi \in \mathcal{D}([O,T[;\ V^{N_0}), \\ -\displaystyle\int_0^T \int_{\Gamma_s} \frac{\partial z^N}{\partial t}\frac{\partial v}{\partial t} + c_s^2 \int_0^T \int_{\Gamma_s} \nabla_s z^N \bullet \nabla_s v \\ \quad = -\dfrac{\rho_f}{\rho_s} \displaystyle\int_0^T \int_{\Gamma_s} \Big(\frac{\partial \varphi^N}{\partial t} + \nabla_s\varphi^0 \bullet \nabla_s\varphi^N\Big)v + \int_0^T \int_{\Gamma_s} qv, \quad \forall v \in \mathcal{D}([O,T[;\ Z^{N_0}). \end{cases} \tag{5.2}$$

Then the weak limit (5.1) set into (5.2) proves that $(\varphi^*,\ z^*)$ is a solution of the weak formulation for any $(\Psi,\ v) \in \mathcal{D}\ ([O,T[;\ V^{N_0} \times Z^{N_0})$ for all N_0.

But because $(V^{N_0} \times Z^{N_0})$ is dense in the space $V \times H_0^1(\Gamma_s)$, one can choose any function in the space $\mathcal{D}\Big(([O,T[;\ V \times H_0^1(\Gamma_s)\Big)$ in the equalities (5.2). The interpretation of the weak formulation proves that $(\varphi^*,\ z^*)$ is a solution of the coupled model but in a distribution space.

The uniqueness would be easily derived from the energy estimate as soon as a sufficient regularity is assumed on the time derivative of the solutions. It could be obtained by deriving the equations with respect to time. Then the regularity is a consequence of the one of the initial data and of the right hand side. But it can more generally be proved directly with the variational formulation even for non smooth solutions (in the space where the existence has been proved). Let us set (see [10])

$$\begin{cases} \Psi(t) = \begin{cases} -\int_t^s \varphi^*(\sigma)d\sigma, & t < s < T, \\ 0, & \text{elsewhere}, \end{cases} \\ v(t) = \begin{cases} -\int_t^s z^*(\sigma)d\sigma, & t < s < T, \\ 0, & \text{elsewhere}, \end{cases} \end{cases} \tag{5.3}$$

where $(\varphi^*,\ z^*)$ is a weak solution of the coupled model with both homogeneous right hand side and initial conditions. Then introducing these test functions in the weak formulation, one deduces the following relations :

$$\begin{aligned}
&-\int_0^s\int_\Omega \frac{\partial\varphi^*}{\partial t}\frac{\partial\Psi}{\partial t}+c_f^2\int_0^s\int_\Omega \nabla\varphi^*\bullet\nabla\Psi-\int_0^s\int_\Omega(\nabla\varphi^*\nabla\varphi^0)(\nabla\varphi^0\bullet\nabla\Psi)\\
&-c_f^2\frac{\rho_s}{\rho_f}\int_0^s\int_{\Gamma_s}\frac{\partial z^*}{\partial t}\frac{\partial v}{\partial t}+c_f^2c_s^2\frac{\rho_s}{\rho_f}\int_0^s\int_{\Gamma_s}\nabla_s z^*\bullet\nabla_s v\\
&-c_f^2\int_0^s\int_{\Gamma_s}\Big[\Big(\frac{\partial z^*}{\partial t}+\nabla_s\varphi^0\bullet\nabla_s z^*\Big)\Psi-\Big(\frac{\partial\varphi^*}{\partial t}+\nabla_s\varphi^0\bullet\nabla_s\varphi^*\Big)v\Big]=0,
\end{aligned}$$

or else

$$\begin{aligned}
&-\int_0^s\int_\Omega\frac{\partial\varphi^*}{\partial t}\varphi^*-c_f^2\frac{\rho_s}{\rho_f}\int_0^s\int_{\Gamma_s}\frac{\partial z^*}{\partial t}z^*+c_f^2\int_0^s\int_\Omega\nabla\frac{\partial\Psi}{\partial t}\bullet\nabla\Psi\\
&-\int_0^s\int_\Omega\Big(\nabla\varphi^0\bullet\nabla\frac{\partial\Psi}{\partial t}\Big)(\nabla\varphi^0\bullet\nabla\Psi)+c_f^2c_s^2\frac{\rho_s}{\rho_f}\int_0^s\int_\Omega\nabla_s\frac{\partial v}{\partial t}\bullet\nabla_s v\\
&-c_f^2\int_0^s\int_{\Gamma_s}\Big(\frac{\partial z^*}{\partial t}\Psi-\frac{\partial\varphi^*}{\partial t}v\Big)-c_f^2\int_0^s\int_{\Gamma_s}\Big[\nabla_s\varphi^0\bullet\nabla_s z^*)\Psi-\Big(\nabla_s\varphi^0\bullet\nabla_s\varphi^*\Big)v\Big]=0,
\end{aligned}$$

and therefore

$$\begin{aligned}
A=&-\frac12\int_\Omega|\varphi^*|^2(s)-\frac{c_f^2}{2}\frac{\rho_s}{\rho_f}\int_{\Gamma_s}|z^*|^2(s)\\
&-\Big[\frac{c_f^2}{2}\int_\Omega|\nabla\Psi|^2(0)-\frac12\int_\Omega(\nabla\varphi^0\bullet\nabla\Psi)^2(0)\Big]-\frac{c_s^2c_f^2}{2}\frac{\rho_s}{\rho_f}\int_{\Gamma_s}|\nabla_s v|^2(0)\\
=&\,c_f^2\int_0^s\int_{\Gamma_s}\Big(\frac{\partial z^*}{\partial t}\Psi-\frac{\partial\varphi^*}{\partial t}v\Big)+c_f^2\int_0^s\int_{\Gamma_s}\Big[\nabla_s\varphi^0\bullet\nabla_s z^*)\Psi-(\nabla_s\varphi^0\bullet\nabla_s\varphi^*\Big)v\Big].
\end{aligned}$$

But from

$$\int_0^s\int_{\Gamma_s}\frac{\partial z^*}{\partial t}\Psi=-\int_0^s\int_{\Gamma_s}z^*\varphi^*=\int_0^s\int_{\Gamma_s}\frac{\partial\varphi^*}{\partial t}v$$

and because

$$\begin{aligned}
&c_f^2\int_0^s\int_{\Gamma_s}\Big[(\nabla_s\varphi^0\bullet\nabla_s z^*)\Psi-(\nabla_s\varphi^0\bullet\nabla_s\varphi^*)v\Big]\\
&=-c_f^2\Big[\int_{\Gamma_s}(\nabla_s\varphi^0\bullet\nabla_s v)\Psi\Big](0)-c_f^2\int_0^s\int_{\Gamma_s}\Delta_s\varphi^0z^*\Psi
\end{aligned}$$

(we used the property that the function v is zero on the boundary $\partial\Gamma_s$ and the fact that $v(s)=0,\ \Psi(s)=0,\ z^*(0)=\varphi^*(0)=0$), the quantity A defined previously is thus such that

$$A=c_f^2\Big[\int_{\Gamma_s}(\nabla_s\varphi^0\bullet\nabla_s v)\Psi\Big](0)-c_f^2\int_0^s\int_{\Gamma_s}\Delta_s\varphi^0z^*\psi^*.$$

Let us recall that we assumed that φ^0 is $C^2(\bar\Omega)$ (see (1.2)). Thus

$$|A|\le c_f^2\Big\{U_s\int_{\Gamma_s}|\nabla_s v|\cdot|\Psi|(0)+2H\int_0^s\int_{\Gamma_s}|z^*|\cdot|\Psi|(\xi)d\xi\Big\},$$

where U_s and H have been defined in (1.2). From Cauchy-Schwarz inequality we deduce that

$$\begin{aligned}
|A|\le c_f^2\Big\{&\frac{\alpha}{2}U_s\int_{\Gamma_s}|\nabla_s v|^2(0)+\frac{U_s}{2\alpha}\int_{\Gamma_s}|\Psi|^2(0)\\
&+H\beta\int_0^s\int_{\Gamma_s}|z^*|^2(\xi)d\xi+\frac{H}{\beta}\int_0^s\int_{\Gamma_s}\Psi^2(\xi)d\xi\Big\}.
\end{aligned}\tag{5.4}$$

Furthermore, this relation is true for any $s \geq 0$ and $s \leq T$. Let us now recall that η_0 is the smallest eigenvalue of the Steklov model that has been introduced in (3.1). Thus from (5.4), one obtains

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\varphi^*|^2(s) + \frac{c_f^2}{2}\frac{\rho_s}{\rho_f}\int_{\Gamma_s}|z^*|^2(s) + \frac{1}{2}\Big(1-\frac{U_s c_f^2}{\alpha\eta_0}\Big)\Big[c_f^2\int_\Omega|\nabla\Psi|^2(0)\\
&-\int_\Omega(\nabla\varphi^0\bullet\nabla\Psi)^2(0)\Big] + \frac{c_f^2}{2}\Big(c_s^2\frac{\rho_s}{\rho_f}-\alpha U_s\Big)\int_{\Gamma_s}|\nabla_s v|^2(0)\\
\leq\ &\frac{H}{\beta\eta_0}\int_0^s\Big[c_f^2\int_\Omega|\nabla\Psi|^2-\int_\Omega(\nabla\varphi^0\bullet\nabla\Psi)^2\Big](\xi)d\xi + H\beta\int_0^s\int_{\Gamma_s}|z^*|^2(\xi)d\xi.
\end{aligned}$$

From the assumption formulated on the boundary velocity U_s in Lemma 3.1, it is always possible to find $\alpha > 0$ such that all the terms on the left hand side of the previous inequality are positive.

Let us now make a basic remark in order to complete the proof of the uniqueness. The functions v and Ψ are both dependent on s and t. Let us write for instance (just from the definitions (5.3))

$$\Psi(t) = \Psi(s,t) = \Psi(s,0) - \Psi(t,0).$$

Thus (using the bilinear form a^f defined in (3.2) in order to shorten the expressions),

$$a^f(\Psi,\Psi)(\xi) = a^f(\Psi(s,0),\Psi(s,0)) + a^f(\Psi(\xi,0),\Psi(\xi,0)) - 2a^f(\Psi(s,0),\Psi(\xi,0));$$

therefore, there exists a positive constant c such that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega|\varphi^*|^2(s) + \frac{c_f^2}{2}\frac{\rho_s}{\rho_f}\int_{\Gamma_s}|z^*|^2(s) - H\beta\int_0^s\int_{\Gamma_s}|z^*|^2(\xi)d\xi + \frac{1}{2}(c-\frac{4Hs}{\eta_0\beta})a^f\Psi(s,0),\ \Psi(s,0))\\
&-\frac{H}{\eta_0\beta}\int_0^s a^f(\Psi(\xi,0),\ \Psi(\xi,0))d\xi + \frac{c_f^2}{2}c\int_{\Gamma_s}|\nabla_s v|^2(0) \leq 0.
\end{aligned}$$

The coefficient β can always be chosen such that $c - 2Hs/\eta_0\beta > 0$. We obtain

$$\frac{d}{ds}\Big[e^{-c_3}\int_0^s\int_{\Gamma_s}|z^*|^2(\xi)d\xi\Big] + \frac{d}{ds}\Big[e^{-c_3}\int_0^s a^f(\Psi(\xi,0),\ \Psi(\xi,0))d\xi\Big] \leq 0,$$

where

$$c_1 = \frac{2H\beta\rho_f}{c_f^2\rho_f},\qquad c_2 = \frac{2H}{\eta_0\beta c - 4Hs},\quad c_3 = \max(c_1,c_2)$$

and thus, integrating from 0 to s, we have

$$e^{-c_3 s}\int_0^s\int_{\Gamma_s}|z^*|^2(\xi)d\xi + e^{-c_3 s}\int_0^s a^f(\Psi(\xi,0),\ \Psi(\xi,0))d\xi \leq 0.$$

Finally we obtain for any $s \leq T$,

$$z^*(s) = 0\ \ \forall s\in[0,T]\ \text{ and }\ \Psi(\xi,0) = 0\ \ \forall\xi\in[0,T],$$

and from (5.4),

$$\varphi^*(s) = 0\ \ \forall s\in[0,T],$$

which implies the uniqueness of a solution to the coupled system. The obtained results are summarized in the following statement.

Theorem 5.1. *Let us assume that φ^0 is such that*

$$U < c_f,\quad U_s < \frac{c_s}{c_f}\sqrt{\frac{\eta_0\rho_s}{\rho_f}}$$

and that $q \in L^2(]O,T[\times\Gamma_s)$, $\varphi_0 \in V$, $\varphi_1 \in L^2(\Omega)$, $z_0 \in H_0^1(\Gamma_s)$, $z_1 \in L^2(\Gamma_s)$. *Then, the coupled model* (1.9)-(2.1)-(1.10)-(1.13)-(1.14)-(1.11) *has a unique solution* (φ, z) *in the space* $\mathcal{C}^0([O,T];\ V\times H_0^1(\Gamma_s))\cap\mathcal{C}^1([O,T];\ L^2(\Omega)\times L^2(\Gamma_s))$.

Remark 5.1. As a matter of fact the restriction that has been used on the maximum of the boundary velocity U_s can be overcome. It only proves that the solution is stable (i.e. bounded). The previous theorem can be extended to a general subsonic case ($U_s < c_f$) just by replacing the initial unknowns by

$$(\varphi, z)(x,t) = e^{\lambda t}(\bar{\varphi}, \bar{z})(x,t), \quad \text{where} \quad \lambda > 0 \quad \text{is chosen large enough.}$$

The coefficient λ takes into account the exponential growth with respect to time. This method can be applied because the terms which are at the origin of the instability are "compact" compared to higher order terms. This point will be discussed in a forthcoming paper in which a nonlinear interaction between the structure and the fluid is considered. Furthermore, the critical value of U_s at which flutter instability occurs can be computed precisely from a standard eigenvalue problem.

§6. Conclusions

The propagation of waves in a steady flow can be modelled by a potential function. But, when a part of the boundary of the open set containing the flow is flexible, the existence of a stable solution is not obvious. Restrictions on the velocity of the flow are necessary. But it is also useful that the remaining boundary conditions around the flow could enable one to derive an a priori estimate on the potential function. This is necessary along the flexible structure with respect to the unsteady velocity inside the flow (from a Steklov eigenvalue problem). The results obtained in this paper seem to be new. They give a first answer to the question of existence and uniqueness of a solution to a three dimensional aeroacoustic model coupled with a flexible structure. But additional studies should be carried out for flutter phenomenon.

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