Chin. Ann. of Math.
23B:1(2002),37-53.

RELAXATION OF FUNCTIONALS INVOLVING HOMOGENEOUS FUNCTIONS AND INVARIANCE OF ENVELOPES

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Abstract

The authors compute the quasiconvex envelope of certain functions defined on the space M_{mn} of real $m \times n$ matrices via a homogeneous function on M_{mn}. They also deduce invariance properties for various convex envelopes from corresponding invariance properties satisfied by a function. Some applications related in particular to nonlinear elasticity are given.

Keywords Homogeneous functions, Quasiconvex envelope, Nonlinear elasticity
2000 MR Subject Classification 32T27, 74B20
Chinese Library Classification O174.13, O343.5 **Document Code** A
Article ID 0252-9599(2002)01-0037-16

§1. Introduction

We denote by M_{mn} the space of real $m \times n$ matrices. Let W be a function defined on M_{mn} with values in $\mathbb{R}$ and Ω be a bounded domain in $\mathbb{R}^n$. The basic problem of the Calculus of Variations consists in minimizing such energy functionals as

$$I(u) = \int_\Omega W(\nabla u(x))dx, \tag{1.1}$$

where u is a mapping from Ω into $\mathbb{R}^m$ belonging to some subset of a Sobolev space. In this context, ∇u designates the gradient of u, i.e., the $m \times n$ matrix with components $(\nabla u)_{ij} = \frac{\partial u_i}{\partial x_j}$, where $u_1, \cdots, u_m$ are the Cartesian components of u.

In applications to problems in Continuum Mechanices, in particular in nonlinear elasticity, the mapping u represents a deformation of a given body occupying the domain Ω in its reference configuration, ∇u is the deformation gradient of u and W is the stored energy function of a hyperelastic material of which the body is assumed to be made. Naturally, appropriate boundary conditions and loading terms must be added to give rise to an actual equilibrium problem.

As a general rule, without additional convexity assunptions on W, the functional I is not weakly lower semicontinuous on Sobolev spaces. The direct method of the Calculus of

Manuscript received March 1, 2001.

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Variations thus does not apply to minimize (1.1). One of the ways of getting around this difficulty is to turn to the so-called relaxed problem, which in this case consists in minimizing the relaxed energy

$$\bar{I}(u) = \int_\Omega QW(\nabla u(x))dx, \tag{1.2}$$

where QW denotes the quasiconvex envelope of W (see [11]).

For the reader's convenience, we recall below the various convexity notions that are relevant in the vectorial case of the Calculus of Variations (see [11] again).

Let $\tau(m,n)$ be the number of all minors of an $m\times n$ matrix F and $M(F)$ be the vector of all such minors. A function $W : M_{mn} \to \mathbb{R}$ is said to be polyconvex if there exists a convex function $\widehat{W} : \mathbb{R}^{\tau(m,n)} \to \mathbb{R}$ such that for all matrices F,

$$W(F) = \widehat{W}(M(F)). \tag{1.3}$$

A function $W : M_{mn} \to \mathbb{R}$ is said to be quasiconvex if

$$W(F) \le \frac{1}{|\Omega|}\int_\Omega W(F + \nabla v(x))dx \tag{1.4}$$

for all matrices F, any bounded domain $\Omega \subset \mathbb{R}^n$ and all functions $v \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$.

A function $W : M_{mn} \to \mathbb{R}$ is said to be rank-one-convex if for all couples of matrices (F,G) such that $\operatorname{rank}(F-G) \le 1$ and all $\lambda \in [0,1]$,

$$W(\lambda F + (1-\lambda)G) \le \lambda W(F) + (1-\lambda)W(G). \tag{1.5}$$

Quasiconvexity was introduced by Morrey[16] as a necessary and sufficient condition for the weak lower semicontinuity of I over Sobolev spaces, under appropriate assumptions of growth and bound below. Morrey also proved that rank-one-convexity is a necessary condition for such weak lower semicontinuity. Polyconvexity was introduced by Ball[1] to solve existence questions in nonlinear elasticity, for which the growth conditions required by Morrey's theorem are not satified. In particular, using polyconvexity, certain energy densities W that take the value $+\infty$ become amenable.

It is well-known that, in the finite-valued case,

$$W\text{convex} \Longrightarrow W \text{ polyconvex} \Longrightarrow W \text{ quasiconvex} \Longrightarrow W \text{ rank-one-convex} \tag{1.6}$$

and that the reverse implication are false in general, except when $m = 1$ or $n = 1$, which is to say in the scalar case, because all the above convexity notions are then equivalent.

Associated with these notions are corresponding convex, polyconvex, quasiconvex and rank-one-convex envelopes defined by

$$\begin{aligned}
CW &= \sup\{Z; Z \text{ convex and } Z \le W\},\\
PW &= \sup\{Z; Z \text{ polyconvex and } Z \le W\},\\
QW &= \sup\{Z; Z \text{ quasiconvex and } Z \le W\},\\
RW &= \sup\{Z; Z \text{ rank-one-convex and } Z \le W\}.
\end{aligned}$$

In view of (1.6), we obviously have

$$CW \le PW \le QW \le RW \le W. \tag{1.7}$$

Clearly, the four envelopes coincide when RW happens to convex.

The relationship between the quasiconvex envelope and the relaxed energy functional is that minimizing sequences for the original energy (1.1) weakly converge to minimizers of the relaxed functional (1.2) under appropriate technical assumptions (see [11]). The converse is also true in the sense that all minimizers of (1.2) are weak limits of a minimizing sequence for (1.1). One of the goals of this article is to compute the quasiconvex envelope of certain functions W which depend on the gradient through a homogeneous function. The main tool is a rank-one decomposition result for matrices adapted to such functions. Another goal is to derive invariance properties for the various envelopes from similar invariance properties satified by the function. Examples of such invariances are homogeneity, frame indifference and isotropy. These results are applied to a selection of examples, among which are a generalization of the James-Ericksen stored energy function and a question arising in a multiple well problem. Both examples are relevant in the study of phase transitions in crystals. Let us note that the explicit computation of quasiconvex envelopes is a difficult problem in general (see [3,4,5,7,6,15] for a few examples).

§2. Decomposition Results for Matrices

In this section, we denote by Ψ a continuous homogeneous function of degree $p > 0$ defined on M_{mn},

$$\Psi : M_{mn} \to \mathbb{R}, \quad \Psi(tF) = t^p \Psi(F)$$

for all F in M_{mn} and all $t \geq 0$.

Theorem 2.1. *Let us consider $F \in M_{mn}$ and $\alpha \in \mathbb{R}$ such that $\Psi(F) \leq \alpha$. Assume that there exists $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$ satisfying $\Psi(a \otimes b) > 0$ and $\Psi(-a \otimes b) > 0$. Then there exists $\lambda \in [0,1]$ and $B, C \in M_{mn}$ such that*

$$F = \lambda B + (1-\lambda)C, \tag{2.1}$$

$$\operatorname{rank}\{B - C\} \leq 1, \tag{2.2}$$

$$\Psi(B) = \Psi(C) = 0. \tag{2.3}$$

Proof. For all $t \geq 0$, let us set $B_t = F + ta \otimes b$. Clearly, $\Psi(B_0) = \Psi(F) \leq \alpha$. Since Ψ is continuous and homogeneous, and since $\Psi(a \otimes b) > 0$, we have

$$\lim_{t \to +\infty} \Psi(B_t) = \lim_{t \to +\infty} t^p \Psi\Big(\frac{F}{t} + a \otimes b\Big) = +\infty.$$

Therefore, by continuity we can select $t_0 \geq 0$ such that

$$\Psi(B_{t_0}) = \alpha.$$

If $t_0 = 0$, we are done. Let us assume that $t_0 > 0$ and let us now set $C_s = F - \frac{st_0}{1-s} a \otimes b$ for $0 \leq s < 1$. Again, we have $\Psi(C_0) = \Psi(F) \leq \alpha$. Furthermore,

$$\lim_{s \to 1^-} \Psi(C_s) = \lim_{s \to 1^-} \Big(\frac{st_0}{1-s}\Big) \Psi\Big(\frac{1-s}{st_0} F - a \otimes b\Big) = +\infty.$$

By continuity, we can select $s_0 \in [0,1]$ such that

$$\Psi(C_{s_0}) = \alpha.$$

Consequently, if we set

$$\lambda = s_0, \quad B = B_{t_0}, \quad C = C_{s_0},$$

the result follows at once.

Remark 2.1. If Ψ is assumed to be positively homogeneous, that is to say, if $\Psi(tF) = |t|^p\Psi(F)$ for all $F \in M_{mn}$ and $t \in \mathbb{R}$, then condition $\Psi(-a \otimes b) > 0$ becomes redundant.

Remark 2.2. If we do not assume Ψ to be homogeneous, then condition $\Psi(\pm a \otimes b) > 0$ in Theorem 2.1 may be replaced by $\Psi(G) \to +\infty$ when $l(G) \to \pm\infty$ and l is a linear form on M_{mn} such that $l(a \otimes b) > 0$. With the choice $l(G) = \mathrm{tr}(G^T a \otimes b)$ for example, this means that the function Ψ is assumed to be coercive in the direction of some rank one matrix.

Theorem 2.2. *Let Ψ be a continuous homogeneous function of degree $p > 0$. Let us consider $F \in M_{mn}$ and $\alpha \in \mathbb{R}$ such that $\Psi(F) \neq \alpha$. Assume there exist $a \in \mathbb{R}^m,\ b \in \mathbb{R}^n$ satisfying $\Psi(a \otimes b) > 0$ and $\Psi(-a \otimes b) > 0$ and $c \in \mathbb{R}^m,\ d \in \mathbb{R}^n$ satisfying $\Psi(c \otimes d) < 0$ and $\Psi(-c \otimes d) < 0$. Then there exists $\lambda \in [0,1]$ and $B, C \in M_{mn}$ such that*

$$F = \lambda B + (1-\lambda)C, \tag{2.4}$$

$$\mathrm{rank}\{B - C\} \leq 1, \tag{2.5}$$

$$\Psi(B) = \Psi(C) = \alpha. \tag{2.6}$$

Proof. Assume first that $\Psi(F) < \alpha$. Then Theorem 2.1 applies directly.

Next, assume that $\Psi(F) > \alpha$. Let us introduce the function $\Phi : M_{mn} \to \mathbb{R}$ defined by $\Phi(F) = -\Psi(F)$. Then Φ is a homogeneous function, $\Phi(F) < -\alpha$, $\Phi(c \otimes d) > 0$ and $\Phi(-c \otimes d) > 0$. We again apply Theorem 2.1 to get the result.

Let us now consider $\Theta : \mathbb{R}^m \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m \to \mathbb{R}^p$ an antisymmetric n-linear function, and denote by $\widehat{\Theta}$ the function defined on M_{mn} by $\Theta(F) = \widehat{\Theta}(F^1, \cdots, F^n)$, where F^j is the j-th column of the matrix F.

Theorem 2.3. *Let Ψ be a continuous homogeneous function of degree $p > 0$. Let us consider $F \in M_{mn}$ and $\alpha \in \mathbb{R}$ such that $\Psi(F) \leq \alpha$. Assume there exist $j \in \{1, \cdots, n\}$ and $b \in \mathbb{R}^n$ satisfying*

$$\Psi(F^j \otimes b) > 0, \quad \Psi(-F^j \otimes b) > 0 \quad \textit{and} \quad b_j = 0, \tag{2.7}$$

where b_j denotes the j-th component of the vector b. Then there exists $\lambda \in [0,1]$ and $B, C \in M_{mn}$ such that

$$F = \lambda B + (1-\lambda)C, \tag{2.8}$$

$$\mathrm{rank}\{B - C\} \leq 1, \tag{2.9}$$

$$\Psi(B) = \Psi(C) = \alpha, \tag{2.10}$$

$$\Theta(B) = \Theta(C) = \Theta(F). \tag{2.11}$$

Proof. This is similar to the proof of Proposition 2.1 in [5]. It suffices to observe that the k-th column of $F^j \otimes b$ is either 0 if $k = j$ or a scalar multiple of F^j if $k \neq j$.

Remark 2.3. If $m = n$, we can take $\Theta(F) = \det F$. If $m = n + 1$, we can take $\Theta(F) = \mathrm{adj}_n F$.

Let us now turn to a slightly different type of decomposition result.

Theorem 2.4. *Let U be any open, convex, strict subset of M_{mn}. Assume that U is not a half-space. Then for all $F \in \overline{U}$, there exist two matrices B, C and a scalar $\lambda \in [0,1]$ such that*

$$(B, C) \in (\partial U)^2, \quad F = \lambda B + (1-\lambda)C \quad \textit{and} \quad \mathrm{rank}(B - C) \leq 1.$$

Proof. By translating by $-F$, we can always assume that $0 \in U$ and we just need to decompose the zero matrix. Let us first recall the following facts concerning the gauge function of a convex set. Let $U \subset M_{mn}$ be an open, convex set such that $0 \in U$. For $F \in M_{mn}$, define

$$P_U(F) = \inf\{\alpha > 0, \quad \alpha^{-1}F \in U\}.$$

It is well known that

$$\forall t > 0, \quad \forall F \in M_{mn}, \quad P_U(tF) = tP_U(F)$$

(note that we do not assume $U = -U$),

$$\forall F, G \in M_{mn}, \quad P_U(F + G) \leq P_U(F) + P_U(G),$$

and

$$U = \{F \in M_{mn}; \;\; P_U(F) < 1\} \;\text{ and }\; \overline{U} = \{F \in M_{mn}; \; P_U(F) \leq 1\},$$

so that

$$\partial U = \{F \in M_{mn}; \; P_U(F) = 1\}.$$

The function P_U is a homogeneous function of degree one. We claim that there exists a rank-one matrix $a \otimes b$ such that $P(a \otimes b) > 0$ and $P(-a \otimes b) > 0$. This is equivalent to saying that there is a straight line $\mathbb{R}a \otimes b$ whose intersection with U is bounded. Indeed, since U is not a half-space, it is included in the intersection of two distinct half-spaces bounded by two affine hyperplanes H_1 and H_2. To prove the claim, it thus suffices to see that there is such a straight line that intersects both H_1 and H_2.

We argue by contradiction and assume that every straight line of the form $\mathbb{R}a \otimes b$ intersects at most one of the hyperplanes. Then, all rank-one matrices must belong to the union of the vector hyperplances $\widetilde{H}_1$ and $\widetilde{H}_2$ associated with H_1 and H_2. Two cases must be considered.

Assume first that H_1 and H_2 are parallel. Then $\widetilde{H}_1 = \widetilde{H}_2$ and the set of rank-one matrices is included in $\widetilde{H}_1$. Now this is impossible since the convex hull of this set is M_{mn}.

Assume next that H_1 and H_2 intersect on a set of codimension 2. Let l_1 be a nonzero linear form whose kernel is $\widetilde{H}_1$. Without loss of generality, consider a rank-one matrix such that $a \otimes b \in \widetilde{H}_1$ and $a \otimes b \notin \widetilde{H}_1 \cap \widetilde{H}_2$. Then, for $d \in \mathbb{R}^m$ arbitrary but small, we must have $l((a + d) \otimes d) \otimes b) = l(d \otimes b) = 0$. Therefore, by linearity, $l(d \otimes b) = 0$ for all $d \in \mathbb{R}^m$. Similarly, for all $e \in \mathbb{R}^n$, we have $l(a \otimes e) = 0$. Now, again for d and e small, we must have $l((a+d) \otimes (b+e)) = l(d \otimes e) = 0$, whence by linearity $l(d \otimes e) = 0$ for all $d \in \mathbb{R}^m$ and $e \in \mathbb{R}^n$. In other words, we find again that the set of rank-one matrices is included in $\widetilde{H}_1$, which is impossible as seen above.

We conclude the proof by applying Theorem 2.1 with $F = 0$, $\Psi = P_U$ and $\alpha = 1$.

Remark 2.4. Note that the theorem is evidently false if U is a half-space.

Theorem 2.4 does not apply as is to non open convex sets. It nonetheless has the following simple corollary.

Corollary 2.1. *Let U be any convex, strict subset of M_{mn}, E the minimal affine subspace of M_{mn} containing U and $\widetilde{E}$ the associated vector space. Assume that U is not a half-space of E and that the convex hull of the set of rank-one matrices in $\widetilde{E}$ is $\widetilde{E}$. Then for all $F \in \overline{U}$, there exist two matrices B, C and a scalar $\lambda \in [0, 1]$ such that*

$$(B, C) \in (\partial U)^2, \quad F = \lambda B + (1 - \lambda)C \;\; \textit{and} \;\; \operatorname{rank}(B - C) \leq 1,$$

where $\overline{U}$ and ∂U denote respectively the closure and boundary of U with respect to the topology of E.

Proof. Clear.

In the case when U is bounded, the assumptions can be slightly relaxed.

Corollary 2.2. *Let U be any bounded, convex, strict subset of M_{mn}, E the minimal affine subspace of M_{mn} containing U and $\widetilde{E}$ the associated vector space. Assume that $\tilde{E}$ contains at least one nonzero rank-one matrix. Then the conclusion of Corollary* 2.1 *hold true.*

Proof. We see from the proof of Theorem 2.4 that what we actually need is a rank-one straight line with two distinct points outside of $U - F$. If U is bounded, then any rank-one line will do.

§3. Relaxation Results

Let us start with a definition.

Definition 3.1. *We say that a function f on M_{mn} is convex outside a convex set $U \subset M_{mn}$ if there exists a convex function $\hat{f}$ defined on M_{mn} that coincides with f outside of U.*

This definition is consistent with the usual definition of a convex function on a nonconvex set when the convex hull of $M_{mn} \setminus U$ is equal to M_{mn}, i.e., when U is not a half-space. It is just saying that $f_{|M_{mn}\setminus U}$ is convex.

Let us generalize a result contained in [4] (see also [7]).

Theorem 3.1. *Let U be as in Corollary* 2.2. *Assume that f is a function of M_{mn} such that*

$$\forall F \in \partial U, \quad -\infty < \alpha = \inf_{M_{mn}} f = f(F), \tag{3.1}$$

and

$$f \text{ is convex outside } U. \tag{3.2}$$

Then we have

$$Cf(F) = Pf(F) = Qf(F) = Rf(F) = \begin{cases} f(F) & \text{if } F \notin U, \\ \alpha & \text{otherwise.} \end{cases}$$

Proof. Let us define a function $\breve{f}$ on M_{mn} by

$$\breve{f}(F) = \begin{cases} f(F) & \text{if } F \notin U, \\ \alpha & \text{otherwise.} \end{cases}$$

First of all, let us show that $\breve{f}$ is convex. Let F_1, F_2 be two matrices and $\lambda \in [0, 1]$.

If $\lambda F_1 + (1-\lambda)F_2 \in U$, since $\breve{f} \geq \alpha$, we have

$$\alpha = \breve{f}(\lambda F_1 + (1-\lambda)F_2) \leq \lambda \breve{f}(F_1) + (1-\lambda)\breve{f}(F_2).$$

Let us then consider the case when $\lambda F_1 + (1-\lambda)F_2 \notin U$. Let $\hat{f}$ be a convex function on M_{mn} that extends $f_{|M_{mn}\setminus U}$ to U. Therefore, by convexity, $\hat{f} \leq \alpha$ on U so that $\breve{f} \leq \hat{f}$ on M_{mn}. Consequently,

$$\begin{aligned} \breve{f}(\lambda F_1 + (1-\lambda)F_2) &= \hat{f}(\lambda F_1 + (1-\lambda)F_2) \\ &\leq \lambda \hat{f}(F_1) + (1-\lambda)\hat{f}(F_2) \\ &\leq \lambda \breve{f}(F_1) + (1-\lambda)\breve{f}(F_2). \end{aligned}$$

It clearly follows from the previous considerations that

$$Cf = \breve{f}.$$

Due to the inequality $Cf \le Pf \le Qf \le Rf \le f$, we thus see that

$$Cf = Pf = Qf = Rf = \breve{f} \quad \text{outside of} \quad U.$$

Let us now take $F \in \bar{U}$ (as before, the closure and boundary of U are taken relative to the affine space E). By Corollary 2.2, we can find two matrices A and B in ∂U, with $\mathrm{rank}(B - A) \le 1$, and F is a convex combination of A and B. Since Rf is rank-one-convex, it follows that

$$\alpha = Cf(F) \le Rf(F) = Rf(\lambda A + (1-\lambda)B) \le \lambda Rf(A) + (1-\lambda)Rf(B) = \alpha.$$

Therefore

$$Cf = Pf = Qf = Rf = \breve{f} \text{ in } \bar{U}$$

as well and the proof is complete.

Remark 3.1. This is a classical result when U is a Euclidean ball. See [4,7] for the case when U is a possibly degenerate ellipsoid.

Remark 3.2. An easy way of constructing functions f that are convex outside a convex is to take a convex function $g\colon \mathbb{R} \to \mathbb{R}$ that has a minimum at $t = 1$. Then it is not difficult to check that $f(F) = g(P_U(F))$ is convex outside of U and that Theorem 3.1 applies.

The following is also a generalization to homogeneous functions of a result in [7] concerning quadratic forms.

Theorem 3.2. *Let $\Psi\colon M_{mn} \to \mathbb{R}$ be a continuous homogeneous function of degree $p > 0$ such that there exist two rank-one matrices $a \otimes b$ and $c \otimes d$ satisfying $\Psi(a \otimes b) > 0$, $\Psi(-a \otimes b) > 0$, $\Psi(c \otimes d) < 0$ and $\Psi(-c \otimes d) < 0$. Let $\Phi\colon \mathbb{R} \to \mathbb{R}$ be bounded below and consider the function W defined on M_{mn} by*

$$W(F) = \Phi(\Psi(F)).$$

Then, for all F in M_{mn},

$$CW(F) = PW(F) = RW(F) = QW(F) = \inf_{t \in \mathbb{R}} \Phi(t).$$

Proof. Let $\mu = \inf_{t \in \mathbb{R}} \Phi(t)$. For all $\epsilon > 0$, there exists $\alpha_\epsilon \in \mathbb{R}$ such that $\mu \le \Phi(\alpha_\epsilon) \le \mu + \epsilon$. Let us consider a matrix F in M_{mn} such that $\Psi(F) \ne \alpha_\epsilon$. By Theorem 2.2, there exist $\lambda_\epsilon \in [0,1]$ and $B_\epsilon, C_\epsilon \in M_{mn}$ such that

$$F = \lambda_\epsilon B_\epsilon + (1-\lambda_\epsilon)C_\epsilon, \quad \mathrm{rank}\,\{B_\epsilon - C_\epsilon\} \le 1 \ \text{ and } \ \Psi(B_\epsilon) = \Psi(C_\epsilon) = \alpha_\epsilon.$$

Since RW is rank-one-convex, it follows that

$$\begin{aligned} \mu \le CW(F) &\le PW(F) \le QW(F) \le RW(F) \\ &\le \lambda_\epsilon RW(B_\epsilon) + (1-\lambda_\epsilon)RW(C_\epsilon) \\ &\le \lambda_\epsilon W(B_\epsilon) + (1-\lambda_\epsilon)W(C_\epsilon) = \Phi(\alpha_\epsilon) \le \mu + \epsilon. \end{aligned}$$

Now, if F is such that $\Psi(F) = \alpha_\epsilon$, then we trivially have

$$\mu \le CW(F) \le RW(F) \le W(F) \le \mu + \epsilon.$$

Therefore, letting ϵ go to zero, we obtain

$$CW(F) = PW(F) = QW(F) = RW(F) = \mu,$$

which concludes the proof.

Remark 3.3. It follows immediately from the previous theorem that for a function of the form $W(F) = \Phi(\Psi(F))$ satisfying the hypotheses of the theorem,

$$W \text{ convex } \Leftrightarrow W \text{ polyconvex} \Leftrightarrow W \text{ quasiconvex } \Leftrightarrow W \text{ rank-one-convex } \Leftrightarrow \Phi \text{ constant.}$$

§4. Invariance of Envelopes

In this section, we give a fairly general result on the various convex envelopes of a function that has invariance properties with respect to some group action.

Let us be given a group G and an action of this group on M_{mn} in the form of a homomorphism from G into $\mathrm{GL}\,(M_{mn})$. We denote by $g \cdot F$ the action of an element g of G on a matrix F. We say that the group action is gradient-compatible if there exists a corresponding linear action of G on $W_c^{1,\infty}(\mathbb{R}^n;\mathbb{R}^m)$, still denoted by $g\,\cdot$, such that for all $\varphi \in W_0^{1,\infty}(D;\mathbb{R}^m)$ and all bounded domains D of $\mathbb{R}^n$, we have $g \cdot \varphi \in W_0^{1,\infty}(\Theta_g(D);\mathbb{R}^m)$ with

$$g \cdot \nabla\varphi(x) = \nabla(g \cdot \varphi)(\Theta_g(x)), \tag{4.1}$$

where Θ_g is an affine isomorphism in $\mathbb{R}^n$. From now on, the assumption of gradient-compatibility will always be made, even when it is not necessary.

Let $W: M_{mn} \to \mathbb{R}$ be a given function. For fixed g in G, we denote by W_g the function on M_{mn} defined by $W_g(F) = W(g\cdot F)$. Let $\chi: G \to \mathbb{R}_+^*$ be a group homomorphism. We will say that a function W is (G,χ)-equivariant if

$$\forall F \in M_{mn}, \quad \forall g \in G, \qquad W_g(F) = \chi(g)W(F). \tag{4.2}$$

Examples 4.1. Examples of such group actions and equivariances that are relevant in applications are

(1) Homogeneity: $G = \mathbb{R}_+^*$, for $g = t$, $g \cdot F = tF$, $g \cdot \varphi(x) = \varphi(tx)$, $\Theta_g(x) = t^{-1}x$ and $\chi(g) = t^p$ for some $p > 0$.

(2) Positive homogeneity: $G = \{-1, 1\} \times \mathbb{R}_+^*$, where $\{-1, 1\}$ is the multiplicative incarnation of $\mathbb{Z}/2\mathbb{Z}$, for $g = (s,t)$, $g \cdot F = stF$, $g \cdot \varphi(x) = \varphi(stx)$, $\Theta_g(x) = (st)^{-1}x$ and $\chi(g) = t^p$ for some $p > 0$.

(3) Material frame indifference: $G = \mathrm{SO}\,(m)$, for $g = Q$, $g \cdot F = QF$, $g \cdot \varphi(x) = Q\varphi(x)$, $\Theta_g(x) = x$ and $\chi(g) = 1$.

(4) Isotropy: $G = \mathrm{SO}\,(n)$, for $g = R$, $g \cdot F = FR$, $g \cdot \varphi(x) = \varphi(Rx)$, $\Theta_g(x) = R^T x$ and $\chi(g) = 1$. The same action with $G \subset \mathrm{SO}\,(n)$ corresponds to material symmetry.

Let us start with a property of gradient-compatible group actions.

Lemma 4.1. *Let $g\,\cdot$ be a gradient-compatible action. We have that for any matrix A of rank less than one,* $\mathrm{rank}(g \cdot A) \le 1$.

Proof. For any φ in $\mathcal{D}(\mathbb{R}^n;\mathbb{R}^m)$, the Fourier transform of its gradient is an $m \times n$ at most rank-one matrix at all points ξ in Fourier space. Any rank-one matrix may be obtained in this way.

Let thus A be such that $\mathrm{rank}A \le 1$, and $\varphi \in \mathcal{D}(\mathbb{R}^n;\mathbb{R}^m)$ and $\xi \in \mathbb{R}^n$ such that

$$\widetilde{\nabla\varphi}(\xi) = A.$$

It is clear that for any L^2 matrix-valued function F on $\mathbb{R}^n$, we have

$$\widetilde{g\cdot F}(\xi)=\int_{\mathbb{R}^n} g\cdot(e^{-i(x|\xi)}F(x))\,dx = g\cdot\int_{\mathbb{R}^n} e^{-i(x|\xi)}F(x)\,dx = g\cdot\widetilde{F}(\xi)$$

(approximate F by a dominated sequence of simple functions). In particular,

$$g\cdot A=(\widetilde{g\cdot\nabla\varphi})(\xi).$$

For all g in G, we write the associated affine isomorphism as

$$\Theta_g(x)=A_gx+b_g,$$

where A_g is an $n\times n$ nonsingular matrix and b_g a vector in $\mathbb{R}^n$. Now

$$\begin{aligned}(\widetilde{g\cdot\nabla\varphi})(\xi)&=\int_{\mathbb{R}^n} e^{-i(x|\xi)}g\cdot\nabla\varphi(x)\,dx\\ &=\int_{\mathbb{R}^n} e^{-i(x|\xi)}\nabla(g\cdot\varphi(\Theta_g(x)))\,dx\\ &=\frac{1}{\det A_g}\int_{\mathbb{R}^n} e^{-i((A_g^{-1}y-A_g^{-1}b_g)|\xi)}\nabla(g\cdot\varphi(y))\,dy\\ &=\frac{e^{i((A_g^{-1}b_g)|\xi)}}{\det A_g}\int_{\mathbb{R}^n} e^{-i(A_g^{-1}y|\xi)}\nabla(g\cdot\varphi(y))\,dy\\ &=\frac{e^{i((A_g^{-1}b_g)|\xi)}}{\det A_g}(\widetilde{\nabla(g\cdot\varphi)})(A_g^{-T}\xi).\end{aligned}$$

This computation is justified by the fact that $g\cdot\varphi$ is in $W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^m)$ with compact support, by gradient-compatibility of the group action, in fact it is in $\mathcal{D}(\mathbb{R}^n;\mathbb{R}^m)$. Consequently, $(\widetilde{g\cdot\nabla\varphi})(\xi)$ is a scalar multiple of the Fourier transform of a gradient and is thus of rank at most one.

Theorem 4.1. *If a function W on M_{mn} is convex (resp. polyconvex, quasiconvex, rank-one-convex), then for all g in G, W_g is convex (resp. polyconvex, quasiconvex, rank-one-convex).*

Proof. Let us start with the simplest case, i.e., W convex. Take two matrices F_1 and F_2 and $\lambda,\mu\geq 0$ such that $\lambda+\mu=1$. Then, since G acts linearly on M_{mn},

$$W_g(\lambda F_1+\mu F_2)=W(g\cdot(\lambda F_1+\mu F_2))=W(\lambda g\cdot F_1+\mu g\cdot F_2)\leq\lambda W_g(F_1)+\mu W_g(F_2).$$

We turn to the case when W is quasiconvex. Let D be a domain in $\mathbb{R}^n$ and $F\in M_{mn}$. We want to prove that W_g is quasiconvex at F. Let us thus consider an arbitrary function $\varphi\in W_0^{1,\infty}(D;\mathbb{R}^m)$. Since the action of G is gradient-compatible, it follows that

$$\begin{aligned}\int_D W_g(F+\nabla\varphi(x))\,dx&=\int_D W(g\cdot F+g\cdot\nabla\varphi(x))\,dx\\ &=\int_D W(g\cdot F+\nabla(g\cdot\varphi)(\Theta_g(x)))\,dx\\ &=\frac{1}{\det A_g}\int_{\Theta_g(D)} W(g\cdot F+\nabla(g\cdot\varphi)(y))\,dy\\ &\geq\frac{|\Theta_g(D)|}{\det A_g}W(g\cdot F)=|D|W_g(F),\end{aligned}$$

where A_g is the matrix of Θ_g as in the previous lemma, and the claim is proved.

Consider next the case when W is polyconvex. Let $\tau(m,n)$ be the number of all minors of an $m \times n$ matrix and $M(F) \in \mathbb{R}^{\tau(m,n)}$ be the vector of all minors of F following a given ordering. Since W is polyconvex, there exists a convex function $\widehat{W}$ on $\mathbb{R}^{\tau(m,n)}$ such that for all F,

$$W(F) = \widehat{W}(M(F)).$$

Let $M_{p,ij}(F)$ be a given minor of order p of F. We consider the function $Z\colon F \mapsto M_{p,ij}(F)$. This function is a null Lagrangian, i.e., a quasiaffine mapping: both Z and $-Z$ are quasiconvex. From the previous step, we deduce that Z_g is also a null Lagrangian. It is known that null Lagrangians are affine combinations of minors (see [2]). Moreover, since Z is a p-linear form on a subset of the column vectors, it follows that Z_g is actually a linear combination of the minors of order p of F, with coefficients depending only on g.

The above considerations show that there exists a linear mapping $T_g\colon \mathbb{R}^{\tau(m,n)} \to \mathbb{R}^{\tau(m,n)}$ such that

$$M(g \cdot F) = T_g(M(F)).$$

Consequently,

$$W_g(F) = \widehat{W}(M(g \cdot F)) = \widehat{W} \circ T_g(M(F)),$$

and the function $\hat{W} \circ T_g$ is convex. Hence, W_g is polyconvex.

Finally, let us assume that W is rank-one-convex. Let us take two matrices F_1 and F_2 such that $\mathrm{rank}(F_1 - F_2) \leq 1$ and a scalar $\lambda \in [0,1]$. We have

$$\begin{aligned} W_g(\lambda F_1 + (1-\lambda)F_2) &= W(\lambda g \cdot F_1 + (1-\lambda) g \cdot F_2) \\ &\leq \lambda W(g \cdot F_1) + (1-\lambda) W(g \cdot F_2) \\ &= \lambda W_g(F_1) + (1-\lambda) W_g(F_2) \end{aligned}$$

since $\mathrm{rank}(g \cdot F_1 - g \cdot F_2) = \mathrm{rank}(g \cdot (F_1 - F_2)) \leq 1$ by Lemma 4.1.

Remark 4.1. In the above proof, the group structure of G does not play any role. The result applies if G is just a set and its action on M_{mn} is replaced by any mapping from G into $\mathcal{L}(M_{mn})$ that is gradient-compatible.

We now consider the various envelopes.

Theorem 4.2. *Let $W : M_{mn} \to \mathbb{R}$ be (G,χ)-equivariant. Then CW, PW, QW and RW are (G,χ)-equivariant.*

Proof. Let us only treat the case of the convex envelope. The three other cases are identical.

Let us thus be given a (G,χ)-equivariant function W. By definition, we have

$$CW = \sup\{Z; Z \text{ convex and } Z \leq W\}.$$

For any such Z, we have

$$Z_g(F) = Z(g \cdot F) \leq W(g \cdot F) = \chi(g) W(F) \text{ and } Z_g \text{ is convex}.$$

Since $\chi(g) \in \mathbb{R}_+^*$, it follows that for all $g \in G$,

$$\chi(g)^{-1} Z_g(F) \leq CW(F).$$

Taking the supremum of this inequality over all competing functions Z, we obtain that for all $g \in G$ and all $F \in M_{mn}$,

$$(CW)_g(F) \leq \chi(g) CW(F).$$

We now use the fact that G is a group and χ is a homomorphism. Applying the previous inequality to g^{-1} and $g \cdot F$, we obtain

$$CW(F) = (CW)_{g^{-1}}(g \cdot F) \leq \chi(g^{-1})CW(g \cdot F) = \chi(g)^{-1}(CW)_g(F).$$

Consequently

$$(CW)_g(F) = \chi(g)CW(F),$$

and CW is (G, χ)-equivariant.

Example 4.2. If W is homogeneous of degree p, then for all $t > 0$,

$$CW(tF) = t^p CW(F), PW(tF) = t^p PW(F), QW(tF) = t^p QW(F), RW(tF) = t^p RW(F).$$

The same equalities hold for all t, but with absolute values, if W is positively homogeneous. If W is frame-indifferent (resp. isotropic), then CW, PW, QW and RW are frame-indifferent (resp. isotropic). See [14] for more specific results concerning $\mathrm{SO}\,(m)$ and $\mathrm{SO}\,(n)$ left- and right-invariance.

§5. A Few Applications

We begin by recalling a result of [12].

Theorem 5.1.[12] *Let $W\colon M_{22} \to \mathbb{R}$ be continuous, positively homogeneous of degree one, frame indifferent and isotropic. Then*

$$W \text{ rank-one-convex} \Longleftrightarrow W \text{ convex}.$$

The following is an immediate consequence of the previous theorem and the invariance results of the previous section.

Theorem 5.2. *Let $W\colon M_{22} \to \mathbb{R}$ be continuous, positively homogeneous of degree one, frame indifferent and isotropic, then*

$$CW = PW = QW = RW.$$

Proof. By Theorem 4.2, all envelopes share the same invariances as W, i.e., they are positively homogeneous of degree one, frame indifferent and isotropic. Therefore, the rank-one-convex envelope is convex, hence the result.

Another result in [12] is as follows, slightly rewritten. Let W be a continuous, homogeneous function of degree one on M_{22}. Assume that there exists an automorphism S on M_{22} and a nonzero matrix $A \in M_{22}$ such that $\det A = \det(S(A)) = 0$ in such a way that the function $Z\colon M_{22} \to \mathbb{R}$, $Z(F) = W(S(F))$, is rotationally invariant.

Theorem 5.3. *Let W satisfy the above hypotheses, then*

$$W \text{ rank-one-convex} \Longleftrightarrow W \text{ convex} \Longleftrightarrow Z \text{ convex} \Longleftrightarrow Z \text{ rank-one-convex.}$$

In terms of envelopes, this theorem implies the following

Theorem 5.4. *Let W be as above, then $CW = PW = QW = RW$.*

Proof. We define two group actions on M_{22}.

For the first group action, we take $G = \mathrm{SO}(2)$ with the usual group structure and define

$$Q \cdot_1 F = S[QS^{-1}(F)].$$

Clearly,

$$Q' \cdot_1 (Q \cdot_1 F) = S[Q'S^{-1}(Q \cdot_1 F)] = S[Q'S^{-1}(S[QS^{-1}(F)])] = S[Q'QS^{-1}(F)] = (Q'Q) \cdot_1 F,$$

and

$$\mathrm{Id} \cdot_1 F = F,$$

so that this is a group action. Moreover

$$W(Q \cdot_1 F) = W(S[QS^{-1}(F)]) = Z(QS^{-1}(F)) = Z(S^{-1}(F)) = W(F)$$

so that W is invariant under this group action. Therefore, by Theorem 4.2, all the envelopes of W are also invariant under the group action. This means for example that, setting $Z_C(F) = CW(S(F))$,

$$Z_C(QF) = CW(S[Q(F)]) = CW(S[QS^{-1}(S[F])]) = CW(Q \cdot_1 S(F)) = CW(S(F)) = Z_C(F),$$

and Z_C is left-SO(2) invariant. The same holds true for the other three envelopes.

For the second action, we take $G = \mathrm{SO}(2)$ with the reverse group structure $((R', R) \mapsto RR')$ and define

$$R \cdot_2 F = S[S^{-1}(F)R].$$

Clearly,

$$R' \cdot_2 (R \cdot_2 F) = S[S^{-1}(R \cdot_2 F)R'] = S[S^{-1}(S[S^{-1}(F)R])R'] = S[S^{-1}(F)RR'] = (R'R) \cdot_2 F,$$

and

$$\mathrm{Id} \cdot_2 F = F,$$

so that this is again a group action. Moreover

$$W(R \cdot_2 F) = W(S[S^{-1}(F)R]) = Z(S^{-1}(F)R) = Z(S^{-1}(F)) = W(F)$$

so that W is also invariant under the second group action. Therefore, by Theorem 4.2, all the envelopes of W are invariant under the second group action. As before, this means that Z_C is right-SO(2) invariant, and the same holds true for the other three envelopes.

We have thus shown that all the envelopes satisfy the hypotheses of Theorem 5.3. Therefore, the rank-one-convex envelope is convex, hence the result.

Example 5.1. The previous results can be applied to the following examples taken from [12]. Let $W: M_{22} \to \mathbb{R}$ be positively homogeneous of degree one and having one of the following forms

$$W(F) = \Phi(\|F\|, \alpha \det F + \beta F^1 \cdot F^2), \quad \alpha\beta \neq 0,$$

or

$$W(F) = \Phi(\|F^1\|, \|F^2\|)$$

for some $\Phi: \mathbb{R}^2 \to \mathbb{R}$, where the various norms are the usual Euclidean norms on M_{22} and $\mathbb{R}^2$. Then all the envelopes coincide. For the first example, take

$$S(F) = \frac{1}{\alpha^2 + \beta^2} \begin{pmatrix} \sqrt{\alpha^2 + \beta^2} F_{11} & \beta F_{22} + \alpha F_{12} \\ \sqrt{\alpha^2 + \beta^2} F_{21} & -\beta F_{12} + \alpha F_{22} \end{pmatrix},$$

so that $W(S(F)) = \Phi\big(\frac{1}{\sqrt{\alpha^2+\beta^2}}\|F\|, \frac{1}{\sqrt{\alpha^2+\beta^2}} \det F\big)$ is rotationally invariant. The second example reduces to the first example by writing first

$$\Phi(\|F^1\|, \|F^2\|) = \Psi(\sqrt{\|F^1\|^2 + \|F^2\|^2}, \|F^1\|^2 - \|F^2\|^2) = \Psi(\|F\|, \|F^1\|^2 - \|F^2\|^2)$$

and then taking $S(F) = \frac{1}{\sqrt{2}}(F^1 + F^2 | F^1 - F^2)$.

Let us then give a series of miscellaneous examples of applications of the results of the previous sections. As a rule, if F is an $m\times n$ matrix, we denote by F^j its j-th column.

Example 5.2. Let $m=n=2$ and

$$W(F)=\Phi(\det F+F^1\cdot F^2)$$

where $x\cdot y$ denotes the standard scalar product in $\mathbb{R}^2$. Taking $\Psi(F)=(\det F+F^1\cdot F^2)$, we see that Ψ is a homogeneous function of degree 2, thus positively homogeneous. Furthermore $\Psi(a\otimes b)=b_1b_2\|a\|^2$ so that Ψ changes sign on rank-one matrices. Therefore

$$QW(F)=\inf_{t\in\mathbb{R}}\Phi(t),$$

by Theorem 3.2.

Example 5.3. A quite similar example is as follows. Let $m=n$ and

$$W(F)=\Phi(\det F+\|F^1\|^n-\|F^2\|^n).$$

Taking $\Psi(F)=(\det F+\|F^1\|^n-\|F^2\|^n)$, we see that Ψ is a homogeneous function of degree n. Moreover, the choices $a=F^1$, $b=e_1$ and $c=F^2$, $d=e_2$ provide rank-one matrices on which Ψ changes sign. Therefore

$$QW(F)=\inf_{t\in\mathbb{R}}\Phi(t),$$

by Theorem 3.2 again.

Let us complicate things a little.

Example 5.4. Let m and n be arbitrary. Consider a polyconvex function Z on M_{mn} that depends only on the minors of order greater than 2. Take

$$W(F)=\Phi\Big(\sum_{i=1}^{s}|F^i|^p-\sum_{i=s+1}^{n}|F^i|^p\Big)+Z(F)\qquad\text{for some } 2\le s\le n-2.$$

Assume that $\inf\limits_{t\in\mathbb{R}}\Phi(t)=0$. Then

$$QW(F)=Z(F).$$

For simplicity, we assume that the infimum of Φ is attained at $t=\alpha$. Let $\Psi(F)=\big(\sum\limits_{i=1}^{s}|F^i|^p-\sum\limits_{i=s+1}^{n}|F^i|^p\big)$. Let $F\neq 0$ be such that $\Psi(F)\neq\alpha$. Select j such that $F^j\neq 0$. Then, for all b in $\mathbb{R}^n$,

$$\Psi(F^j\otimes b)=\Big(\sum_{i=1}^{s}|b_i|^p-\sum_{i=s+1}^{n}|b_i|^p\Big)\|F^j\|^p.$$

Whatever be the value of j, since there is always at least one positive term and one negative term in the sum, we can always find b and d such that $b_j=d_j=0$ and $\Psi(F^j\otimes b)=\Psi(-F^j\otimes b)>0$ and $\Psi(F^j\otimes d)=\Psi(-F^j\otimes d)<0$. Therefore, by Theorem 2.3, we can find matrices A and B with $\operatorname{rank}(A-B)\le 1$ such that $\Psi(A)=\Psi(B)=\alpha$, $Z(A)=Z(B)=Z(F)$ and $F=\lambda A+(1-\lambda)B$. By the rank-one convexity of QW, it follows that $QW(F)\le Z(F)$. Now, Z is polyconvex, hence quasiconvex, therefore $Z(F)=QW(F)$. The equality still holds for $F=0$ by continuity.

Example 5.5. Consider two disjoint, nonempty subsets I and J of $\{1,\cdots,m\}\times$

$\{1,\cdots,n\}$ and let

$$\Psi(F)=\sum_{(i,j)\in I}|F_{ij}|^p-\sum_{(i,j)\in J}|F_{ij}|^p.$$

If $W(F)=\Phi(\Psi(F))$ with $\inf_{t\in\mathbb{R}}\Phi=\mu$, then $QW(F)=\mu$.

Example 5.6. Let us now consider an example that is a generalization of the James-Ericksen stored energy function. The James-Ericksen stored energy function was introduced as a model to study problems of phase transitions in crystals in 2D (see [10] for example). The generalization we propose here is as follows:

$$W(F)=\phi_1(F)+\phi_2(F)+\phi_3(F)$$

with

$$\begin{cases}\phi_1(F)=\kappa_1\Big|\sum\limits_{i,j=1}^{2}|F_{ij}|^p-2\Big|^q,\\ \phi_2(F)=\kappa_2|F^1\cdot F^2|^r,\\ \phi_3(F)=\kappa_3\Big|\Big|\dfrac{|F_{11}|^s+|F_{21}|^s-(|F_{12}|^s+|F_{22}|^s)}{2}\Big|^t-\epsilon^t\Big|^u.\end{cases}$$

The James-Ericksen stored energy function is obtained when all exponents are set equal to 2 (see [5] and [7] for relaxation results pertaining to the James-Ericksen energy).

In the general case, we have

$$Q\phi_1(F)=\kappa_1\Big(\Big(\sum_{i,j=1}^{2}|F_{ij}|^p-2\Big)_+\Big)^q,\qquad Q\phi_2(F)=0,\qquad Q\phi_3(F)=0.$$

These equalities follow from Theorem 3.1 for the first one and are similar to Examples 5.1 and 5.2 for the other two.

Example 5.7. Let us close this article by detailing an example taken from [9], Remark 2.1, concerning a multiple well problem again in connection with phase transitions in crystals. Let us be given a finite set of $m\times n$ matrices F_i, $1\le i\le k$ with $k\ge 2$ that are pairwise rank-one connected in the sense that $\mathrm{rank}(F_i-F_j)\le 1$ for all i and j. We denote by $\mathrm{co}\,\{F_i\}$ the convex hull of this set.

Proposition 5.1.[9] *Let $W\colon M_{mn}\to\mathbb{R}_+$ be such that*

$$\forall i=1,\cdots,k,\quad W(F_i)=0.$$

If $F\in\mathrm{co}\,\{F_i\}$, then

$$CW(F)=PW(F)=QW(F)=RW(F)=0. \tag{5.1}$$

Proof. First of all, it is clear that $CW(F)=0$. It is thus sufficient to show that $RW(F)=0$. To prove this, it suffices to show that any F in $\mathrm{co}\,\{F_i\}$ can be obtained as the result of a finite sequence of rank-one-convex combinations starting from the F_i, $i=1,\cdots,k$. The rank-on-convexity of RW combined with the fact that $RW(F_i)=0$ will then give the result.

We argue by induction on k. For $k=2$, any F in the segment $[F_1,F_2]$ is trivially a rank-one-convex combination of F_1 and F_2.

Let us assume as our induction hypothesis that the property holds true for any set of $k-1$ pairwise rank-one connected matrices. Let us set $U=\mathrm{co}\,\{F_i\}$ and E the minimal affine subspace containing U. Let us consider $F \in U$. The hypotheses of Corollary 2.2 are clearly satisfied. Thus, there exist $\lambda \in [0,1]$, A and B in ∂U with $\mathrm{rank}(A-B) \le 1$ such that $F = \lambda A + (1-\lambda)B$. Now, since A and B belong to ∂U, each of them is a convex combination of at most $k-1$ of the F_i. Indeed, U is equal to the adherence of its interior in E. The induction hypothesis yields the result.

Let us extend the previous result to a situation in which the wells are not necessarily directly rank-one connected. In the following, we thus do not assume that $\mathrm{rank}(F_i - F_j) \le 1$.

Prpoposition 5.2. *Let $W\colon M_{mn} \to \mathbb{R}_+$ be such that the following hypotheses are satisfied:*

(i) $W(F_i) = 0$ *for all* $i = 1, \cdots, k$,

(ii) *For all* $i = 1, \cdots, k$, *there exist* $\widehat{F}_i$ *such that* $\mathrm{rank}(\widehat{F}_i - \widehat{F}_j) \le 1$ *and* $\mathrm{rank}(F_i - \widehat{F}_i) \le 1$ *with* $\widehat{F}_i \in [F_i, \widehat{F}_{i+1}]$, *using the convention that* $\widehat{F}_{k+1} = \widehat{F}_1$.

Then, for all $F \in \mathrm{co}\,\{\widehat{F}_i\}$,

$$CW(F) = PW(F) = QW(F) = RW(F) = 0.$$

Proof. In view of Proposition 5.1, it is clearly enough to prove that $RW(\widehat{F}_i) = 0$ for all i. It follows from the hypothesis (ii) that for all i, there exists $\lambda_i \in [0,1]$ such that

$$\widehat{F}_i = \lambda_i F_i + (1-\lambda_i)\widehat{F}_{i+1}.$$

By rank-one-convexity, we deduce that

$$RW(\widehat{F}_i) \le (1-\lambda_i) RW(\widehat{F}_{i+1}),$$

so that

$$RW(\widehat{F}_1) \le \Big(\prod_{i=1}^{k}(1-\lambda_i)\Big) RW(\widehat{F}_1).$$

Therefore, either $RW(\widehat{F}_1) = 0$, or there exists i_0 such that $\lambda_{i_0} = 1$, in which case $\widehat{F}_{i_0} = F_{i_0}$ and $RW(\widehat{F}_{i_0}) = 0$. In both cases, we are done.

Remark 5.1. In the previous theorem, it is possible to replace the condition $\mathrm{rank}(\widehat{F}_i - \widehat{F}_j) \le 1$ by the weaker condition $\mathrm{rank}(\widehat{F}_i - \widehat{F}_{i+1}) \le 1$ if we assume in addition that all $\widehat{F}_i$ belong to one plane and that they form a convex polygon. An example of such a configuration was introduced in [17] for other purposes, with four diagonal 2×2 matrices situated at the vertices of a square (see also [8] for a discussion of this example).

Fig. 5.1 A Five Matrix Configuration

Remark 5.2. It follows from Proposition 5.1 that

$$\inf\Big\{\int_\Omega W(\nabla v(x))\,dx; v\in W^{1,\infty}(\Omega;\mathbb{R}^m), v(x)=Fx \text{ on } \partial\Omega\Big\}=0$$

if F is in the convex hull of the wells F_i. In [9], upper bounds for the infimum of the energy over various finite element spaces are given. These upper bounds are explicit powers of the mesh size, and easily imply the above equality.

Acknowledgements. The work of the second author was completed during a stay at the Texas Institute for Computational and Applied Mathematics, University of Texas at Austin, with the support of the TICAM Visiting Fellowship program.

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