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VARIATIONAL PROPERTIES OF THE INTEGRATED MEAN CURVATURES OF TUBES IN SYMMETRIC SPACES***

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Abstract

Let P_t denote the tubular hypersurface of radius t around a given compatible submanifold in a symmetric space of arbitrary rank. The authors will obtain some relations between the integrated mean curvatures of P_t and their derivatives with respect to t. Moreover, the authors will emphasize the differences between the results obtained for rank one and arbitrary rank symmetric spaces.

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§1. Introduction

Let P be a submanifold of an n-dimensional riemannian manifold M. Expressions for the kth integrated mean curvatures, $M_k^P(t)$, (i. e. the integral of the kth mean curvature, $k = 0, 1, \cdots, n-1$), of a tubular hypersurface P_t of radius t about P, in terms of the Riemann curvature tensor of P, are calculated when M is Euclidean or a rank one symmetric space[5,13,7]. Moreover, when P is a closed convex hypersurface of the n-dimensional space of constant curvature λ^2, Santaló has obtained in [12] the following interesting relations between $M_k^P(t)$ and their derivatives with respect to t :

$$\frac{d}{dt}M_k^P(t) = -(n-(k+1))M_{k+1}^P(t) + \lambda^2\, k\, M_{k-1}^P(t). \tag{1.1}$$

In this paper we study the k-th integrated mean curvatures of tubular hypersurfaces about some compatible submanifolds in symmetric spaces $M = G/K$ of arbitrary rank. In particular we will extend (1.1) for geodesic balls and tubes around these submanifolds.

As it was noted in [6] and [9], where the authors obtained expressions for $M_0^P(t)$ (that is, the area of the tubular hypersurface P_t) by using the restricted roots of M, the theory of

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tubes in rank one symmetric spaces is simpler than this general theory in symmetric spaces because of the transitive action of the isotropy group of rank one symmetric spaces. Along this paper we will emphasize the differences between the relations of $M_k^P(t)$ obtained in rank one and arbitrary rank symmetric spaces.

The paper is structured as follows. In Section 2 we consider integrated mean curvatures of geodesic balls in symmetric spaces. In Sections 3 and 4 we extend the results in Section 2 for tubes around some compatible submanifolds in a symmetric space. Finally, in Section 5, we relate the theorems obtained in the preceding sections, for rank one symmetric spaces, with known comparison theorems for the mean curvature of tubular hypersurfaces (see, for instance, [4]), and with variational problems of functions of the mean curvatures for hypersurfaces and submanifolds in space forms[8,10]..

§2. Geodesic Balls in Symmetric Spaces

Let M be a compact symmetric space of dimension n and rank q. Later on we will extend the results to noncompact symmetric spaces, having taken into account the duality between compact and noncompact symmetric spaces.

Let $0 = \alpha_1(u) = \cdots = \alpha_q(u) < \alpha_{q+1}(u) \leq \cdots \leq \alpha_n(u)$, where $u \in \mathbf{h}$, maximal abelian subspace of $\mathbf{m}$, an arrangement of the set of positive restricted roots of M. The tangent space at any point of M is canonically identified with $\mathbf{m}$.

Let P_t be a geodesic sphere of radius t around a point p of M. Then, the eigenvalues of the shape operator $S(t)$ of P_t (principal curvatures of P_t) with respect to the vector u are given by[9]

$$k_i(t) = -\frac{1}{t}, \qquad i = 2, \cdots, q, \tag{2.1}$$

$$k_j(t) = -\frac{\alpha_j(u)}{\tan(t\alpha_j(u))}, \quad j = q+1, \cdots, n-1. \tag{2.2}$$

Definition 2.1. *The kth integrated mean curvature, $M_k^p(t)$, of P_t $(k = 0, 1, \cdots n-1)$, is defined by*

$$M_k^p(t) = \binom{n-1}{k}^{-1} \int_{P_t} \{k_{j1}, k_{j2}, \cdots, k_{jk}\} d\sigma, \tag{2.3}$$

where $\{k_{j1}, k_{j2}, \cdots, k_{jk}\}$ denotes the kth elementary symmetric function of the principal curvatures, and $d\sigma$ the area element of P_t.

The $(n-1)$-dimensional volume of P_t is[9]

$$M_0^p(t) = A_p(t) = ct^{q-1} \int_C \prod_{j=q+1}^{n} \sin(t\alpha_j(u)) du, \tag{2.4}$$

where c is a known constant and $C = S^{q-1}(1) \cap D$, where D is a Weyl chamber and $S^{q-1}(1)$ the unit sphere in $\mathbf{h}$.

Proposition 2.1. *The kth integrated mean curvature of P_t can be expressed as:*

$$M_k^p(t) = \int_C f_k(t, u) du, \qquad k = 0, 1, \cdots, n-1,$$

where

$$f_0(t,u) = ct^{q-1}\prod_{j=q+1}^{n}\sin(t\alpha_j(u)),$$

$$f_k(t,u) = \binom{n-1}{k}^{-1}(-1)^k ct^{q-1}\Big[\sum_{l=0}^{k} B_l(t,u)\Big]\prod_{j=q+1}^{n}\sin(t\alpha_j(u)), \tag{2.5}$$

$$B_l(t,u) = \binom{q-1}{k-l}\frac{1}{t^{k-l}}\sum\frac{\alpha_{j_1}(u)\cdots\alpha_{j_l}(u)}{\tan(t\alpha_{j_1}(u))\cdots\tan(t\alpha_{j_l})}, \quad l=0,\cdots,k,$$

where the sum is extended to $q+1\le j_1<\cdots<j_l\le n$.

Note that

$$B_0(t,u) = \binom{q-1}{k}\frac{1}{t^k}.$$

Proof. Immediate from (2.1), (2.2), (2.3) and (2.4).

Theorem 2.1. *Let* $V_p(t)$ *denote the* n*-dimensional volume of the geodesic ball* P_t*; then*

(i) $\frac{d}{dt}V_p(t) = A_p(t),$ (2.6)

(ii) $\frac{d}{dt}A_p(t) = -(n-1)M_1^p(t),$ (2.7)

(iii) $\frac{d}{dt}f_k(t,u) = -(n-(k+1))f_{k+1}(t,u) + \sum_{m=0}^{k-1}\frac{\binom{n-1}{m}}{\binom{n-1}{k}}f_m(t,u)\sum_{j=q+1}^{n}\frac{\alpha_j^{k-m+1}(u)}{\tan^{k-m-1}(t\alpha_j(u))}.$ (2.8)

Proof. (i) is proved in [4] and (ii) is immediate comparing the derivative of (2.4), with respect to t, and (2.5) for $k=1$. In order to prove (iii) we will consider for simplicity the following notation

$$\alpha_j = \alpha_j(u), \quad \tan_j = \tan(t\alpha_j(u)), \quad S = \prod_{j=q+1}^{n}\sin(t\alpha_j(u)).$$

Since

$$\frac{d}{dt}S = \Big(\sum_{j=q+1}^{n}\frac{\alpha_j}{\tan_j}\Big)S,$$

from (2.5) we obtain

$$\binom{n-1}{k}\frac{(-1)^k}{c}\frac{d}{dt}f_k(t,u) = St^{q-1}G(t,u), \tag{2.9}$$

where

$$G(t,u) = \frac{q-1}{t}\sum_{l=0}^{k}B_l(t,u) + \sum_{l=0}^{k}\frac{d}{dt}B_l(t,u) + \Big(\sum_{l=0}^{k}B_l(t,u)\Big)\Big(\sum_{j=q+1}^{n}\frac{\alpha_j}{\tan_j}\Big).$$

In order to rewrite $G(t,u)$ we have that, for $0\le l\le k$,

$$\frac{d}{dt}B_l(t,u) = -\binom{q-1}{k-l}\frac{k-l}{t^{k-l+1}}\sum\frac{\alpha_{j_1}\cdots\alpha_{j_l}}{\tan_{j_1}\cdots\tan_{j_l}} \tag{2.10}$$

$$-\binom{q-1}{k-l}\frac{1}{t^{k-l}}\sum\frac{\alpha_{j_1}\cdots\alpha_{j_l}}{\tan_{j_1}\cdots\tan_{j_1}}\left(\frac{\alpha_{j_1}}{\tan_{j_1}}+\cdots+\frac{\alpha_{j_l}}{\tan_{j_l}}\right) \tag{2.11}$$

$$-\binom{q-1}{k-l}\frac{1}{t^{k-l}}\sum\frac{\alpha_{j_1}\cdots\alpha_{j_l}}{\tan_{j_1}\cdots\tan_{j_l}}\left(\alpha_{j_1}\tan_{j_1}+\cdots+\alpha_{j_l}\tan_{j_l}\right). \tag{2.12}$$

Note that, for $k=0$, we have

$$\frac{d}{dt}B_0(t,u) = -\binom{q-1}{k}\frac{k}{t^{k+1}}.$$

Now, given l such that $0 \le l \le k$, the expression $B_l(t,u)\left(\sum_{j=q+1}^{n}\frac{\alpha_j}{\tan_j}\right)$ + (2.11) is

$$\begin{aligned}
&\binom{q-1}{k-l}\frac{1}{t^{k-l}}\Big[\Big(\sum\frac{\alpha_{j_1}\cdots\alpha_{j_l}}{\tan_{j_1}\cdots\tan_{j_l}}\Big)\Big(\sum_{j=q+1}^{n}\frac{\alpha_j}{\tan_j}\Big)\\
&-\sum\frac{\alpha_{j_1}\cdots\alpha_{j_l}}{\tan_{j_1}\cdots\tan_{j_l}}\Big(\frac{\alpha_{j_1}}{\tan_{j_1}}+\cdots+\frac{\alpha_{j_l}}{\tan_{j_l}}\Big)\Big]\\
&=\binom{q-1}{k-l}\frac{l+1}{t^{k-l}}\sum_{j1<\cdots<j_{l+1}}\frac{\alpha_{j_1}\cdots\alpha_{j_{l+1}}}{\tan_{j_1}\cdots\tan_{j_{l+1}}}.
\end{aligned}\tag{2.13}$$

Moreover, the sum

$$\frac{q-1}{t}B_l(t,u)+(2.10)+(2.13),$$

when, in (2.13), the corresponding term to $l-1$ is considered, can be expressed as

$$\begin{aligned}
&\Big(\frac{1}{t^{k-l+1}}\sum\frac{\alpha_{j_1}\cdots\alpha_{j_l}}{\tan_{j_1}\cdots\tan_{j_l}}\Big)\Big[\binom{q-1}{k-l}((q-1)-(k-l))+\binom{q-1}{k-l+1}l\Big]\\
&=\Big(\frac{1}{t^{k-l+1}}\sum\frac{\alpha_{j_1}\cdots\alpha_{j_l}}{\tan_{j_1}\cdots\tan_{j_l}}\Big)\binom{q-1}{k-l+1}(k+1).
\end{aligned}\tag{2.14}$$

On the other hand, (2.12) can be written as

$$\begin{aligned}
&-\binom{q-1}{k-l}\frac{1}{t^{k-l}}\sum\frac{\alpha_{j_1}\cdots\alpha_{j_l}}{\tan_{j_1}\cdots\tan_{j_l}}(\alpha_{j_1}\tan_{j_1}+\cdots+\alpha_{j_l}\tan_{j_l})\\
&=-\binom{q-1}{k-l}\frac{1}{t^{k-l}}\Big[\Big(\sum\frac{\alpha_{j_1}\cdots\alpha_{j_{l-1}}}{\tan_{j_1}\cdots\tan_{j_{l-1}}}\Big)\Big(\sum_{j=q+1}^{n}\alpha_j^2\Big)\\
&-\Big(\sum\frac{\alpha_{j_1}\cdots\alpha_{j_{l-2}}}{\tan_{j_1}\cdots\tan_{j_{l-2}}}\Big)\Big(\sum_{j=q+1}^{n}\frac{\alpha_j^3}{\tan_j}\Big)+\cdots\\
&+(-1)^l\Big(\sum_{j=q+1}^{n}\frac{\alpha_j}{\tan_j}\Big)\Big(\sum_{j=q+1}^{n}\frac{\alpha_j^l}{\tan_j^{l-2}}\Big)+(-1)^{l+1}\Big(\sum_{j=q+1}^{n}\frac{\alpha_j^{l+1}}{\tan_j^{l-1}}\Big)\Big].
\end{aligned}\tag{2.15}$$

Now, having (2.14) and (2.15) in mind, $G(t,u)$ can be written as

$$\begin{aligned}
G(t,u)=&(k+1)\sum_{l=0}^{k+1}\binom{q-1}{k-l+1}\frac{1}{t^{k-l+1}}\sum\frac{\alpha_{j_1}\cdots\alpha_{j_l}}{\tan_{j_1}\cdots\tan_{j_l}}\\
&-\sum_{l=1}^{k}\binom{q-1}{k-l}\frac{1}{t^{k-l}}\Big[\Big(\sum\frac{\alpha_{j_1}\cdots\alpha_{j_{l-1}}}{\tan_{j_1}\cdots\tan_{j_{l-1}}}\Big)\Big(\sum_{j=q+1}^{n}\alpha_j^2\Big)\\
&-\Big(\sum\frac{\alpha_{j_1}\cdots\alpha_{j_{l-2}}}{\tan_{j_1}\cdots\tan_{j_{l-2}}}\Big)\Big(\sum_{j=q+1}^{n}\frac{\alpha_j^3}{\tan_j}\Big)+\cdots\\
&+(-1)^l\Big(\sum_{j=q+1}^{n}\frac{\alpha_j}{\tan_j}\Big)\Big(\sum_{j=q+1}^{n}\frac{\alpha_j^l}{\tan_j^{l-2}}\Big)+(-1)^{l+1}\Big(\sum_{j=q+1}^{n}\frac{\alpha_j^{l+1}}{\tan_j^{l-1}}\Big)\Big].
\end{aligned}\tag{2.16}$$

Finally, comparing (2.9) (with the above expression for $G(t,u)$) and the definitions of $f_k(t,u)$, $k=0,1,\cdots,n-1$, in (2.5), we obtain the desired result.

Remark 2.1. If M is a noncompact symmetric space, Proposition 2.1 and Theorem 2.1 are still valid changing trigonometric functions to hyperbolic functions (see [9] for details).

For rank one symmetric spaces (space forms), the set of positive restricted roots of M are independent of the choice of the vector u; therefore, we can substitute functions $f_k(t,u)$ for $M_k^P(t)$, in (2.8). Moreover, if M is a space of constant curvature λ^2, equation (2.8) has the form (1.1).

§3. Tubes About Totally Geodesic Submanifolds in Symmetric Spaces

Let $P=U/L$ be a totally geodesic submanifold in M, of dimension p, and $\mathbf{u}=\mathbf{p}+\mathbf{l}$ the canonical decomposition of $\mathbf{u}$. We assume that the orthogonal complement $\mathbf{p}^\perp$ of $\mathbf{p}$ in $\mathbf{m}$ is a Lie triple system; then, $P^\perp=\mathrm{Exp}(\mathbf{p}^\perp)$ is a totally geodesic submanifold of M and a Riemannian global symmetric space $P^\perp=U'/L'$. We suppose that $\mathrm{rank}(P^\perp)=r\le q$ and $\mathbf{a}\subset\mathbf{h}$ is a maximal Abelian subspace of $\mathbf{p}^\perp$. Then, if α_j, $1\le j\le n-p$, denote the positive restricted root system of $\mathbf{p}^\perp$, and β_i, $1\le i\le p$, that of $\mathbf{p}$, the principal curvature functions of the tube P_t of radius t around P are[9]

$$\begin{aligned} &k_a(t)=-\frac{1}{t},\quad a=2,\cdots,r,\\ &k_j(t)=-\frac{\alpha_j}{\tan_j},\quad j=r+1,\cdots,n-p,\\ &k_b(t)=0,\quad b=1,\cdots,q-r,\\ &k_i(t)=\beta_i\tan_i,\quad i=q-r+1,\cdots,p.\end{aligned} \tag{3.1}$$

Note that we have used for simplicity the notation in the preceding section and $\tan_i=\tan(\beta_i)$.

Proposition 3.1. *The kth integrated mean curvature of P_t can be expressed as:*

$$M_k^P(t)=\int_C f_k(t,u)du,\quad k=0,1,\cdots,n-1,$$

$C=S^{r-1}\cap D$ (D is a Weyl chamber of the linear action $\mathrm{Ad}:L'\times\mathbf{p}^\perp\longrightarrow\mathbf{p}^\perp$ and S^{r-1} is the unit sphere in $\mathbf{a}$), and

$$f_0(t,u)=c\mathrm{Vol}(P)t^{r-1}\prod_{j=r+1}^{n-p}\sin(t\alpha_j(u))\prod_{i=q-r+1}^{p}\cos(t\beta_i(u)),$$

$$\begin{aligned} f_k(t,u)=&\frac{c(-1)^k\mathrm{Vol}(P)}{\binom{n-1}{k}}\Big[\sum_{l=0}^{k}A_l(t,u)\Big]t^{r-1}\\ &\cdot\prod_{j=r+1}^{n-p}\sin(t\alpha_j(u))\prod_{i=q-r+1}^{p}\cos(t\beta_i(u))\end{aligned} \tag{3.2}$$

for $k=1,\cdots,n-1$; where c is a known constant and

$$A_l(t,u)=\binom{r-1}{k-l}\frac{1}{t^{k-l}}\sum_{s=0}^{l}(-1)^{l-s}A_{s,l}(t,u),\quad 0\le l\le k,$$

where

$$A_{s,l}(t,u)=\sum\frac{\alpha_{j_1}\cdots\alpha_{j_s}}{\tan_{j_1}\cdots\tan_{j_s}}\beta_{i_{s+1}}\cdots\beta_{i_l}\tan_{i_{s+1}}\cdots\tan_{i_l}, \tag{3.3}$$

and the sum is extended to

$$r+1 \le j_1 < \cdots < j_s \le n-p \quad \textit{and} \quad q-r+1 \le i_{s+1} < \cdots < i_l \le p.$$

Note that

$$A_0(t,u) = \binom{r-1}{k}\frac{1}{t^k}.$$

Proof. Immediate from (2.3), (2.17) and the expression of $A_P(t) = M_0^P(t)$ given in Theorem 3.2 of [9].

Remark 3.1. From (3.1) we have that $f_k(t,u)$ defined in (3.2) is zero when $k > n-q+r$.

Theorem 3.1. *Let $V_P(t)$ denote the n-dimensional volume of the tube P_t; then*

(i) $\frac{d}{dt}V_P(t) = A_P(t)$, (3.4)

(ii) $\frac{d}{dt}A_P(t) = -(n-1)M_1^P(t)$, (3.5)

(iii)

$$\frac{d}{dt}f_k(t,u) = -(n-(k+1))f_{k+1}(t,u) + \sum_{m=0}^{k-1}\frac{\binom{n-1}{m}}{\binom{n-1}{k}}f_m(t,u)$$
$$\cdot\Big(\sum_{j=r+1}^{n-p}\frac{\alpha_j^{k-m+1}}{\tan_j^{k-m-1}} + (-1)^{k-m+1}\sum_{i=q-r+1}^{p}(\beta_i^{k-m+1})(\tan_i^{k-m-1})\Big). \tag{3.6}$$

Proof. (i) is proved in [4] and (ii) is immediate. In order to prove (iii) we will consider the following notation

$$S = \prod_{j=r+1}^{n-p}\sin(t\alpha_j(u)) \text{ and } T = \prod_{i=q-r+1}^{p}\cos(t\beta_i(u)).$$

Then we have

$$\frac{d}{dt}S = \Big(\sum_{j=r+1}^{n-p}\frac{\alpha_j}{\tan_j}\Big)S, \quad \frac{d}{dt}T = -\Big(\sum_{i=q-r-1}^{p}\beta_i\tan_i\Big)T,$$

and

$$\frac{d}{dt}A_l(t,u) = -\binom{r-1}{k-l}\frac{k-l}{t^{k-l+1}}\sum_{s=0}^{l}(-1)^{l-s}A_{s,l}(t,u) \tag{3.7}$$
$$+\binom{r-1}{k-l}\frac{1}{t^{k-l}}\sum_{s=0}^{l}(-1)^{l-s}A_{s,l}(t,u) \tag{3.8}$$
$$\cdot\Big[\Big(\beta_{i_{s+1}}\tan_{i_{s+1}} + \cdots\beta_{i_l}\tan_{i_l} - \frac{\alpha_{j_1}}{\tan_{j_1}} - \cdots - \frac{\alpha_{j_s}}{\tan_{j_s}}\Big) \tag{3.9}$$
$$+\Big(\frac{\beta_{i_{s+1}}}{\tan_{i_{s+1}}} + \cdots\frac{\beta_{i_l}}{\tan_{i_l}} - \alpha_{j_1}\tan_{j_1} - \cdots - \alpha_{j_s}\tan_{j_s}\Big)\Big]. \tag{3.10}$$

From (3.2) we have

$$\frac{(-1)^k}{c\text{Vol}(P)}\binom{n-1}{k}\frac{d}{dt}f_k(t,u) = STt^{r-1}G(t,u), \tag{3.11}$$

where

$$G(t,u) = \sum_{l=0}^{k}\frac{d}{dt}A_l(t,u) + \Big(\sum_{l=0}^{k}A_l(t,u)\Big)\Big[\frac{r-1}{t} + \sum_{j=r+1}^{n-p}\frac{\alpha_j}{\tan_j} - \sum_{i=q-r+1}^{p}\beta_i\tan_i\Big]. \tag{3.12}$$

Now, similarly to (2.13), for a given l (and assuming in (3.9) its product with (3.8)), we obtain

$$\begin{aligned} &A_l(t,u)\Big[\sum_{j=r+1}^{n-p}\frac{\alpha_j}{\tan_j}-\sum_{i=q-r+1}^{p}\beta_i\tan_i\Big]+(3.9)\\ &=\binom{r-1}{k-l}\frac{l+1}{t^{k-l}}\sum_{s=0}^{l+1}(-1)^{l-s+1}A_{s,l+1}. \end{aligned}\tag{3.13}$$

Therefore, considering in the above expression the corresponding term to $l-1$, in order to get the corresponding factor to $\frac{1}{t^{k-l+1}}$, we have

$$\frac{r-1}{t}A_l(t,u)+(3.7)+(3.13)=\binom{r-1}{k-l+1}\frac{k+1}{t^{k-l+1}}\sum_{s=0}^{l}(-1)^{l-s}A_{s,l}.\tag{3.14}$$

On the other hand, (3.10) (considered with its product by (3.8)) can be written as

$$\begin{aligned} &-\binom{r-1}{k-l}\frac{1}{t^{k-l}}\Big[\Big(\sum_{j=r+1}^{n-p}\alpha_j^2+\sum_{i=q-r+1}^{p}\beta_i^2\Big)\sum_{s=0}^{l-1}(-1)^{l-s-1}A_{s,l-1}\\ &-\Big(\sum_{j=r+1}^{n-p}\frac{\alpha_j^3}{\tan_j}-\sum_{i=q-r+1}^{p}\beta_i^3\tan_i\Big)\sum_{s=0}^{l-2}(-1)^{l-s-2}A_{s,l-2}+\cdots\\ &+(-1)^{l+1}\Big(\sum_{j=r+1}^{n-p}\frac{\alpha_j^{l+1}}{\tan_{l-1}}+(-1)^{l-1}\sum_{i=q-r+1}^{p}\beta_i^{l+1}\tan_i^{l-1}\Big)\Big]. \end{aligned}\tag{3.15}$$

Finally, comparing (3.2) with (3.11), having (3.14) and (3.15) in mind to develop $G(t,u)$, and, in particular, comparing f_{k+1} with (3.14) (note that, because of (3.13), l varies in (3.14) from 0 to $k+1$), we obtain the desired result.

Remark 3.2. When $\operatorname{rank}(P^\perp)=1$, we can substitute in (3.6) functions $f_k(t,u)$ for $M_k^P(t)$ and we obtain a relation between the integrated mean curvatures of P_t. Example:

$$P=\frac{SO(n)}{SO(p)\times SO(n-p)}\subset\frac{SO(n+1)}{SO(p)\times SO(n-p+1)}=M.$$

For rank one symmetric spaces M (space forms) we can also substitute in (3.6) functions $f_k(t,u)$ for $M_k^P(t)$. Moreover, in order to get compatible submanifolds P of M, P is not necessary to be totally geodesic or a principal orbit; for instance, a complex submanifold of the complex projective space $\mathbb{C}P^n$ is compatible with $\mathbb{C}P^n$ and any submanifold in the sphere S^n is compatible with S^n (see [4]). On the other hand, if P is a submanifold of a space of constant curvature λ^2, equation (3.6) has the form (1.1).

§4. Tubes About Principal Orbits in Symmetric Spaces

Now, let P denote a principal orbit of the canonical isometric action $\rho: K\times M\longrightarrow M$ of the Lie group K on the symmetric space $M=G/K$. If we consider the canonical decomposition of $\mathbf{m}$,

$$\mathbf{m}=\mathbf{h}\oplus\sum_{\alpha\in\Delta}\mathbf{m}_\alpha,\tag{4.1}$$

where Δ denotes the set of positive restricted roots of M and $\mathbf{m}_\alpha$ the root subspace of α, it is known that the subspace $\mathbf{p}$ which is identified with T_pP is given by $\mathbf{p}=\sum_{\alpha\in\Delta}\mathbf{m}_\alpha$ (see

[14]). Then, in a similar way to (3.1) (see [9]),

$$k_a(t) = -\frac{1}{t}, \quad a = 2, \cdots, q,$$
$$k_i(t) = \frac{\alpha_i^2 \tan_i + k_i\alpha_i}{\alpha_i - k_i \tan_i}, \quad i = q+1, \cdots, n, \tag{4.2}$$

where k_i are finite (in Section 3, $k_i = 0$ because P was considered totally geodesic).

Therefore, proceeding as in [4] or [9], we have that

$$A_P(t) = \int_P \int_{S^{q-1}(1)} t^{q-1}\,\theta_u(t) du\, dP, \tag{4.3}$$

where

$$\theta_u(t) = \prod_{i=q+1}^{n} \Big(\cos(t\alpha_i(u)) - \frac{k_i}{\alpha_i}\sin(t\alpha_i(u))\Big). \tag{4.4}$$

Proposition 4.1. *The k-th integrated mean curvature of P_t can be expressed as*

$$M_k^p(t) = \int_P \int_{S^{q-1}(1)} f_k(t,u) du\, dP, \quad k = 0, 1, \cdots, n-1,$$

where

$$f_k(t,u) = \binom{n-1}{k}^{-1} t^{q-1}\Big[\sum_{l=0}^{k} A_l(t,u)\Big] \prod_{j=q+1}^{n} \Big(\cos(t\alpha_j) - \frac{k_j}{\alpha_j}\sin(t\alpha_j)\Big), \tag{4.5}$$

$$A_l(t,u) = \binom{q-1}{k-l} \frac{(-1)^{k-l}}{t^{k-l}} \sum \Big(\frac{\alpha_{j_1}^2 \tan_{j_1} + k_{j_1}\alpha_{j_1}}{\alpha_{j_1} - k_{j_1}\tan_{j_1}}\Big) \cdots \Big(\frac{\alpha_{j_l}^2 \tan_{j_l} + k_{j_l}\alpha_{j_1}}{\alpha_{j_l} - k_{j_l}\tan_{j_l}}\Big),$$

where the sum is extended to $q+1 \le j_1 < \cdots < j_l \le n$.

Proof. Immediate from (4.2), (4.3) and (4.4).

Theorem 4.1. *Let $V_p(t)$ denote the n-dimensional volume of the tube P_t; then*

(i) $\frac{d}{dt}V_p(t) = A_p(t)$, (4.6)

(ii) $\frac{d}{dt}A_p(t) = -(n-1)M_1^p(t)$, (4.7)

(iii)

$$\frac{d}{dt} f_k(t,u) = -(n-(k+1)) f_{k+1}(t,u) + \sum_{m=0}^{k-1} \frac{\binom{n-1}{m}}{\binom{n-1}{k}} f_m(t,u)(-1)^{k-m-1} \sum_{j=q+1}^{n} \alpha_j^2 \Big(\frac{\alpha_j^2 \tan_j + k_j\alpha_j}{\alpha_j - k_j\tan_j}\Big)^{k-m-1}. \tag{4.8}$$

Proof. (i) is proved in [4] and (ii) is immediate from (4.2)–(4.4). To prove (iii) we consider the notation

$$S = \prod_{j=q+1}^{n} \Big(\cos(t\alpha_j) - \frac{k_j}{\alpha_j}\sin(t\alpha_j)\Big).$$

Then

$$\frac{d}{dt}S = -\Big(\sum_{j=q+1}^{n} \frac{\alpha_j^2 \tan_j + k_j\alpha_j}{\alpha_j - k_j\tan_j}\Big) S.$$

Now, from (4.5) we have

$$\binom{n-1}{k} \frac{d}{dt} f_k(t,u) = S t^{q-1} G(t,u),$$

where

$$G(t,u) = \sum_{l=0}^{k} \frac{d}{dt} A_l(t,u) + \sum_{l=0}^{k} A_l(t,u) \Big[\frac{q-1}{t} - \sum_{j=q+1}^{n} \frac{\alpha_j^2 \tan_j + k_j \alpha_j}{\alpha_j - k_j \tan_j} \Big]. \tag{4.9}$$

Finally, having in mind that

$$\frac{d}{dt} \Big(\frac{\alpha_j^2 \tan_j + k_j \alpha_j}{\alpha_j - k_j \tan_j} \Big) = \alpha_j^2(u) + \Big(\frac{\alpha_j^2 \tan_j + k_j \alpha_j}{\alpha_j - k_j \tan_j} \Big)^2,$$

and proceeding as in Theorems 2.1 and 3.1, we obtain the result.

Remark 4.1. For noncompact symmetric spaces the results in Sections 3 and 4 are still valid considering hyperbolic functions instead of trigonometric functions. However, since P will be a noncompact submanifold, it is not possible to obtain a finite value of P or a finite integration over P (see [6]).

Moreover, as it was expected, (4.8) coincides with (3.6) when P is a totally geodesic principal orbit of M (for instance, $P = S^{m-1} \times S^{n-1} \subset S^m \times S^n = M$).

§5. Related Topics: Comparison Theorems and Variational Problems

From comparison results for $\mathrm{tr}(S(t))$ it is possible to obtain comparison theorems for different geometric invariants as the volume, the mean exit time or the first Dirichlet eigenvalue in rank one symmetric spaces (see, for instance, [4]). In this section we only want to note that formulas (2.7), (3.5) and (4.7), for rank one symmetric spaces, allow to obtain comparison results for the quotient $M_1^P(t)/A_P(t)$, if the corresponding comparison results for $\mathrm{tr}(S(t))$ are known.

Indeed, from (1.7), (3.5) and (4.7) we have

$$\frac{d}{dt} \log(A(t)) = -(n-1) \frac{M_1^P(t)}{A_P(t)}.$$

Moreover, from (4.3), when M is a rank one symmetric space, we obtain

$$\log(A(t)) = \log C_1 + (n-1) \log t + \log \theta_u(t),$$

where C_1 is a constant; so, from [4],

$$\frac{d}{dt} \log(A(t)) = \frac{n-1}{t} + \frac{\theta_u'(t)}{\theta_u(t)} = -\mathrm{tr}(S(t)).$$

Therefore, comparison results for $\mathrm{tr}(S(t))$ give comparison results for $M_1^P(t)/A_P(t)$.

On the other hand, when a one-parameter family $X_t : P \longrightarrow M$ of immersions of P into M is considered, related results to our theorems can be found in [10], when P is a hypersurface and M a space form, and in [8], when P is any submanifold of the space form M or P is a hypersurface of any riemannian manifold M.

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