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ON THE GENERALIZED GLAISHER-HONG'S CONGRUENCES

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Abstract

Recently Hong Shaofang[6] has investigated the sums $\sum_{j=1}^{p-1}(np+j)^{-r}$ (with an odd prime number $p \geq 5$ and $n, r \in \mathbf{N}$) by Washington's p-adic expansion of these sums as a power series in n where the coefficients are values of p-adic L-fuctions[12]. Herethe author shows how a more general sums $\sum_{j=1}^{p^l-1}(np^l+j)^{-r}, l \in \mathbf{N}$, may be studied by elementary methods.

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§1. Notations and Introduction

Listed below are some general notations which will be used throughout this note:
p—a prime number greater than 3,
$c, l, m, r, s \in \mathbf{N}$,
$\binom{n}{k} = n!/(k!(n-k)!)$—the binomial coefficient,
B_k—the kth Bernoulli number in the "even suffix" notation, e.g., $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0, \cdots$,
$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}$—the Bernoulli polynomial.

As is known , the Bernoulli numbers are defined by the symbolic recurrence relation $B_{n+1} = (B+1)^{n+1}, n = 1, 2, \cdots, B_0 = 1$, which in the expanded form becomes

$$B_n = (n+1)^{-1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k.$$

Further, it is easily proved that $B_{2k+1} = 0$ for $k > 0$. And at last, we shall use the well-known Staudt-Clausen theorem for denominators and the Staudt theorem for numerators of B_k (see, [11, 7, 8]).

We shall also consider the sums

$$P(n, m, r) = \sum_{j=1,\ (j,m)=1}^{nm} j^{-r} \quad \text{and} \quad G(n, m, r) = \sum_{j=1,\ (j,m)=1}^{m-1} (nm+j)^{-r}.$$

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First of all we note that the sum $G(0,m,r) = W(m,r)$ is so named Wolstenholme-Leudesdorf's sums (see [2, 5], also [1, 12, 9]). Note that the last sums are connected by the relation $G(n,p,r) = P(n+1,p,r) - P(n,p,r)$. Using the relation and the Wasington's p-adic expantion of $P(n,p,r)$ as a power series in n with the coefficients which are values of p-adic L-functions (see [12]), S. F. Hong has generalized the Glaisher's results[3,4].

Theorem 1.1.[6] *Let p be an odd prime and let $n \geq 0, r \geq 1$ be integers.*

(i) *If $r \geq 1$ is odd and suppose $p \geq r+4$, then*

$$G(n,p,r) \equiv -\frac{(2n+1)r(r+1)}{2(r+2)} B_{p-r-2}p^2 (\operatorname{mod} p^3).$$

(ii) $G(n,p,p-2) \equiv -(2n+1)p (\operatorname{mod} p^2)$.

(iii) *If $r \geq 2$ is even and suppose $p \geq r+3$, then*

$$G(n,p,r) \equiv \frac{r}{r+1} B_{p-r-1} (\operatorname{mod} p^2).$$

Remark 1.1. The case $n = 0$ corresponds to the cited Glaisher's results[3,4].

As we shall show the developed elementary methods using division properties of Bernoulli numbers allow us to generalize the Glaisher-Hong's theorem without the help of p-adic L-functions.

Theorem 1.2. (i) *For $(2,r) = 1$ and $r+4 \leq p$ we have*

$$G(n,p^l,r) \equiv -p^{2l} \frac{r(r+1)(2r+1)}{2(p^{l-1}+r+1)} B_{\varphi(p^l)-r-1} (\operatorname{mod} p^{3l}). \tag{1.1}$$

(ii) $G(n,p^l,p-2) \equiv -\frac{p-2}{2} p^{2l}(2n+1) B_{(p-1)(p^{3l-1}-1)} (\operatorname{mod} p^{3l})$, *or*

$$G(n,p^l,p-2) \equiv -p^{2l-1}(2n+1) (\operatorname{mod} p^{2l}). \tag{1.2}$$

(iii) *For $2 \mid r$ and $r+3 \leq p$ we obtain*

$$G(n,p^l,r) \equiv \frac{rp^l}{p^{2l-2}+r} B_{\varphi(p^{2l-1})-r} (\operatorname{mod} p^{3l-1}). \tag{1.3}$$

It is obvious that in the case $l = 1$ Theorem 1.2 becomes Theorem 1.1.

Remark 1.2. (1) The proof of Theorem 1.2 (see §2) is true for $n = 0$ too, if we put that the empty sum $P(0,m,r)$ is equal to zero.

(2) The congruences (1.1)–(1.3) generalize both the classical results and the cited papers of D. Boyd, L. C. Washington, S. F. Hong and the author. Some corollaries of our main result concern partial sums of the harmonic and allied series.

§2. Proof of Theorem 1.2

In order to prove Theorem 1.2 we shall use the following short remark.

Lemma 2.1. *Let $t = \varphi(p^{cl}) - r$ with $cl \leq \varphi(p^{cl}) - r$ and $c, l, r \in \mathbf{N}$. Then*

$$G(n,p^l,r) \equiv \{B_{t+1}((n+1)p^l) - B_{t+1}(np^l)\}/(t+1) (\operatorname{mod} p^{cl}). \tag{2.1}$$

Indeed, $i^t \equiv i^{-r} (\operatorname{mod} p^{cl})$ for $(i,p) = 1$. Noting that $i^t \equiv 0 (\operatorname{mod} p^{cl})$ with $p|i$, from Bernoulli formula

$$\sum_{i=0}^{x-1} i^s = \{B_{s+1}(x) - B_{s+1}\}/(s+1), \quad s, x \in \mathbf{N},$$

we conclude

$$P(n,p^l,r) \equiv \sum_{i=1}^{np^l-1} i^t \equiv \{B_{t+1}(np^l) - B_{t+1}\}/(t+1) (\operatorname{mod} p^{cl}). \tag{2.2}$$

To obtain the congruence (2.1) it remains to observe that

$$G(n,p^l,r)=P(n+1,p^l,r)-P(n,p^l,r).$$

Remark 2.1. It seems that the proposed lemma has an independent interest. If we fix the number l in the congruence (2.2) and note that the left part of the congruence (2.1) is independent of c, then we obtain p-adic approximation of $P(n,p^l,r)$ provided that $c\to\infty$. Some applications of the congruence (2.2) and its variants (e.g., analogs of Bernoulli formula for negative values of degrees of naturals) are contained in the notes [9, 10].

Proof of Theorem 1.2. It is obvious that the congruence (2.1) implies

$$G(n,p^l,r)\equiv\sum_{k=1}^{t+1}\frac{1}{k}\binom{t}{k-1}p^{lk}B_{t+1-k}\{(n+1)^k-n^k\}(\bmod p^{cl})$$

or

$$G(n,p^l,r)\equiv p^lB_t+\frac{t}{2}p^{2l}B_{t-1}(2n+1)+\frac{t(t-1)}{6}p^{3l}B_{t-2}\{3n^2+3n+1\}+\cdots(\bmod p^{cl}). \quad (2.3)$$

To obtain the generalization of Washington-Hong's results I restrict myself to the case $c=3$. Since $\mathrm{ord}_p\{p^lB_{t+1-k}p^{l(k-4)}p^{3l}/k\}\geq 3l$ for $l\geq 1$ and $k\geq 4$, in the considered case we have

$$G(n,p^l,r)\equiv p^lB_t+\frac{t}{2}p^{2l}B_{t-1}(2n+1)+\frac{t(t-1)}{6}p^{3l}B_{t-2}\{3n^2+3n+1\}(\bmod p^{cl}). \quad (2.4)$$

To finish the proof we consider the following cases:

(i) Let $(2,r)=1$. Then $(2,t)=1$ and we find that

$$G(n,p^l,r)\equiv\frac{t}{2}p^{2l}B_{t-1}(2n+1)(\bmod p^{3l}) \quad (2.5)$$

with $t=\varphi(p^{3l})-r$ and $3l\leq\varphi(p^{3l})-r$. So, by the binary Kummer-Staudt's congruence for Bernoulli numbers we conclude that

$$B_{t-1}\equiv\frac{\varphi(p^{3l})-r-1}{\varphi p^l)-r-1}B_{\varphi(p^l)-r-1}(\bmod p^l),\quad r+2\leq p-2.$$

Therefore, the congruence (2.5) implies (1.1).

(ii) Let $r=p-2$. In this case $B_t=B_{t-2}=0$ and the congruence (2.4) implies

$$G(n,p^l,p-2)\equiv-\frac{p-2}{2}p^{2l}(2n+1)B_{(p-1)(p^{3l-1}-1)}(\bmod p^{3l}).$$

Therefore, with $pB_{k(p-1)}\equiv -1(\bmod p), k\in\mathbf{N}$, we obtain

$$G(n,p^l,p-2)\equiv\frac{p-2}{2}p^{2l-1}(2n+1)(\bmod p^{2l})\quad\text{or}$$
$$G(n,p^l,p-2)\equiv -p^{2l-1}(2n+1)(\bmod p^{2l}).$$

(iii) At last, let $2\mid r$, so that $2\mid t$. The congruence (2.4) implies

$$G(n,p^l,r)\equiv p^lB_t+\frac{t(t-1)}{6}p^{3l}B_{t-2}\{3n^2+3n+1\}(\bmod p^{3l}).$$

If $r+2\leq p-1$, then $\mathrm{ord}_pB_{t-2}\geq -1$ and

$$G(n,p^l,r)\equiv p^lB_t(\bmod p^{3l-1}), r+3\leq p. \quad (2.6)$$

Further, because $t=\varphi(p^3)-r, r\leq p-3$, by the congruence

$$B_{\varphi(p^{3l})-r}\equiv\frac{p^{3l-1}(p-1)-p^{2l-2}(p-1)+p^{2l-2}(p-1)-r}{p^{2l-2}(p-1)-r}B_{\varphi(p^{2l})-r}(\bmod p^{2l-1})\quad\text{or}$$

$$B_{\varphi(p^{3l})-r}\equiv\frac{r}{p^{2l-2}+r}B_{\varphi(p^{2l-1})-r}(\bmod p^{2l-1}),$$

we see that the congruence (2.6) implies $G(n,p^l,r) \equiv p^l \frac{r}{p^{2l-2}+r} B_{\varphi(p^{2l-1})-r} (\mathrm{mod}\, p^{3l-1})$, $r+3 \le p$. The proof is complete.

§3. Corollaries

In the present section we shall give some simple consequences of Theorem 1.2. First, remark that the case $G(0,p^l,r) = W(p^l,r)$ was studied by the author ([9, Corollary 2]; see, also §1). Therefore, as before, it is sufficient to propose $n \ge 1$.

Corollary 3.1. (i) *If* $(2,r)=1, r+4 \le p$ *and* $\mathrm{ord}_p(2n+1) \le l$, *then we have*

$$\mathrm{ord}_p G(n,p^l,r) \ge \mathrm{ord}_p(2n+1) + 2l. \tag{3.1}$$

In particular, $\mathrm{ord}_p(2n+1) = 0 \Rightarrow \mathrm{ord}_p G(n,p^l,r) \ge 2l$.

(ii)

$$(p, 2n+1) = 1 \Rightarrow \mathrm{ord}_p G(n,p^l,p-2) = 2l-1, \tag{3.2}$$

$$p \mid (2n+1) \Rightarrow \mathrm{ord}_p G(n,p^l,p-2) \ge 2l. \tag{3.3}$$

(iii) *If* $2|r$ *and* $r+3 \le p$, *then*

$$\mathrm{ord}_p G(n,p^l,r) \ge l. \tag{3.4}$$

Remark 3.1. The corollary partially do not coincide with Hong's ones (in the case $l=1$) because the note [6] contains a mistake. In general, Hong's assertion $\mathrm{ord}_p B_{p-r-2} = 0$, $r \ge 1$ and $r+4 \le p$, is not correct. As is known, for the first irregular prime $p=37$ and $r=3$ we have $B_{32} \equiv 0 (\mathrm{mod}\, 37)$ (see, e.g., [11]). By the way, the mistake is contained in the proof of Corollary 4.2.

Proof of Corollary 3.1. From Theorem 1.2 it follows that in the case (i) it is enough to observe that by the Staudt-Clausen's theorem $\mathrm{ord}_p B_{\varphi p^l)-r-1} \ge 0$. Further, the implications (3.2) and (3.3) are the simple consequences of the congruence (1.2). And finally, in the case (iii) we know that $\mathrm{ord}_p r = 0$ and $\mathrm{ord}_p B_{\varphi(p^{2l-1})-r} \ge 0$, so that the congruence (1.4) implies (3.4).

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