Chin. Ann. of Math.
23B:1(2002),67-74.

THE $\overline{\partial}$-PROBLEM FOR HOLOMORPHIC (0,2)-FORMS ON PSEUDOCONVEX DOMAINS IN SEPARABLE HILBERT SPACES AND D.F.N. SPACES***

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Abstract

This paper shows that the $\overline{\partial}$-problem for holomorphic $(0,2)$-forms on Hilbert spaces is solvable on pseudoconvex open subsets. By using this result, the authors investigate the existence of the solution of the $\overline{\partial}$-equation for holomorphic $(0,2)$-forms on pseudoconvex domains in D.F.N. spaces.

Keywords $\overline{\partial}$-problem, Pseudoconvex domain, Nuclear operator, D.F.N. space
2000 MR Subject Classification 32W05, 46E50
Chinese Library Classification O175.25, O177.91   **Document Code** A
Article ID 0252-9599(2002)01-0067-08

§1. Introduction

L. Hörmander[3] solved the $\overline{\partial}$-problem by using the L^2-estimates for partial differential operators in $\mathbf{C}^n$. J. Kajiwara[4] studied infinite dimensional generalizations of the potential kernel. Concerning the $\overline{\partial}$-problem in infinite dimensional spaces, P. Raboin[11] investigated the $\overline{\partial}$-equation for $C^\infty(0,1)$-forms in arbitrary pseudoconvex open subsets of separable Hilbert spaces without growth condition. J. F. Colombeau and B. Perrot[1] showed that a C^∞ solution u of $\overline{\partial}u=\omega$ can be obtained when ω is a closed C^∞ differential $(0,1)$-form on a arbitrary pseudoconvex domain of a D.F.N. space. On the other hand, S. Dineen[2] showed that the $\overline{\partial}$-problem is not solvable, for any domain in a locally convex space which does not admit a continuous norm. M. Nishihara[8,9] studied on special infinite dimensional spaces, correlating the Levi problem with the $\overline{\partial}$-problem in infinite dimensional space. R. L. Soraggi[13] proved the existence of a C^∞ solution u of $\overline{\partial}u=\omega$ which is of uniform bounded type on E for a holomorphic $(0,2)$-form ω on a D.F.N. space E. In this paper, we show the existence of the solution of the $\overline{\partial}$-equation for a holomorphic $(0,2)$-form f on a pseudoconvex domain Ω in a D.F.N. space E, using the results in [5,6,13,14] and following the argument of J. F. Colombeau and B. Perrot[1].

Manuscript received July 25, 2000.
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***The first author was supported by KOSEF postdoctoral fellowship 1998 and the second author was supported by the Brain Korea 21 Project, 1999 .

§2. The $\bar{\partial}$-Problem on Separable Hilbert Spaces

Let E and F be complex locally convex spaces. Let $\mathcal{L}_{\mathbf{R}}(E;F)$ and $\mathcal{L}_{\mathbf{C}}(\bar{E};F)$ be the vector spaces of continuous $\mathbf{R}$-linear and antilinear mappings from E to F, respectively.

Definition 2.1. *Let p and q be positive integers. $\Lambda^{(p,q)}(E)$ denotes the skew-symmetric subspace of a vector space $\mathcal{L}(^{p+q}\bar{E})$ of continuous p-$\mathbf{C}$-linear and q-antilinear forms on E. Let Ω be an open subset of E. We denote by $C^{\infty}_{(p,q)}(\Omega)$ the linear space of all $C^{\infty}(p,q)$-forms on Ω, equipped with the topology of uniform convergence on the compact subsets of E for the differential form and each derivative.*

Let $u \in C^1(\Omega;F)$ and let $u' : \Omega \longrightarrow \mathcal{L}_{\mathbf{R}}(E;F)$ be its derivative. For $x, y \in \Omega$, we define an operator $[\overline{\partial}] : C^1(\Omega;F) \longrightarrow C(\Omega;\mathcal{L}(\bar{E};F))$ as follows:

$$[\overline{\partial}]u(x)(y) = \frac{1}{2}[u'(x)(y) + iu'(x)(iy)].$$

Let $\omega : \Omega \longrightarrow \Lambda^{(p,q)}(E)$ be a $C^{\infty}(p,q)$-form. We define, for each $x \in \Omega$ and $y_1, \cdots, y_{p+q+1} \in E$,

$$\begin{aligned}&(\overline{\partial}\omega)(x)(y_1, \cdots, y_{p+q+1}) \\ &= \frac{1}{p+q+1}\sum_{k=1}^{p+q+1}(-1)^{k+1}[\overline{\partial}]\omega(x)(y_k)(y_1, \cdots, \hat{y_k}, \cdots, y_{p+q+1}),\end{aligned}$$

where $\hat{y_k}$ indicates that y_k is omitted.

Then we know the fact that for $\omega \in C^{\infty}_{(p,q)}(\Omega)$ and $x \in \Omega$, $(\overline{\partial}\omega)(x) \in \Lambda^{(p,q+1)}(E)$ and $[\overline{\partial}]\omega(x) \in \mathcal{L}_{\mathbf{C}}(\bar{E}, \Lambda^{(p,q)}(E))$. Thus $(\overline{\partial}\omega)(x)$ is the skew-symmetric component of $[\overline{\partial}]\omega(x)$. Hence for a $C^{\infty}(p,q)$ form u on E, we write $[\overline{\partial}]u = \overline{\partial}u + G(u)$ where $G(u)(x)$ is the symmetric component of $[\overline{\partial}]u(x)$.

R. L. Soraggi noted that for a $(0,1)$-form u the antilinear component $[\overline{\partial}]u$ of u' consists of $\overline{\partial}u$ and a symmetric part $G(u)$. This leads to some problems when considering integral representations of cylindrical solutions, since they involve $[\overline{\partial}]$ but not $\overline{\partial}$.

This is the reason why we impose the holomorphicity assumption and restrict ourselves to this case for $(0,2)$ forms. For further details we refer to [13] and [14]. No similar result is known for a $(0,q)$ form, $q \geq 3$, with holomorphic coefficients.

Definition 2.2.[7] *Let E and F be complex Banach spaces. Given $\omega : \Omega \to \mathcal{L}(\bar{E};F)$, we say that $u : \Omega \to F$ is a weak solution of $[\overline{\partial}]u = \omega$ if for every fixed $z \in \Omega$ and $x \in E$, the mapping $g : \lambda \to u(z + \lambda x)$ is continuous on a disc $\Delta = \Delta(0,r) \subset \mathbf{C}$ and in the sense of distributions, i.e. for all fixed $z, x \in E$, the function g satisfies*

$$\int_{\Delta}\frac{\partial\psi}{\partial\bar{\lambda}}(\lambda)g(\lambda)d\lambda = -\int_{\Delta}\psi(\lambda)\omega(z + \lambda x)(x)d\lambda$$

for all $\psi \in C^{\infty}_{o}(\Delta)$.

Note that if $z, x \in E$ and $\lambda \in \mathbf{C}$,

$$\frac{\partial g(\lambda)}{\partial\bar{\lambda}} = \frac{\partial}{\partial\bar{\lambda}}u(z + \lambda x) = [\overline{\partial}]u(z + \lambda x)(x).$$

As the first step to solve the $\overline{\partial}$-problem on a D. F. N. space, we show the existence of the solution of the $\overline{\partial}$-equation for a holomorphic $(0,2)$-form on a pseudoconvex domain of a separable Hilbert space. Let H be a separable Hilbert space and T be a nuclear injective self-adjoint operator on H (therefore T has a dense range). We denote by $H_T \subset H$ the range of T, equipped with the scalar product $(Tx,Ty)_{H_T}=(x,y)_H$ for $x,y\in H$. Let G be a separable Hilbert space such that H is contained in G with injective nuclear map. Then from [10], there is an orthonormal basis $\{e_j, j\geq 1\}$ of H made of eigenvectors for T, i.e. $T(e_j)=\lambda_j e_j$ with $\lambda_j\neq 0$ and $\lambda^2=\sum_{j=1}^{\infty}\lambda_j^2<\infty$ for $\lambda_j\in\mathbf{C}$. For $n\geq 1$, let

$$T_n:\mathbf{C}^n\longrightarrow\bigoplus_{j=1}^{n}\mathbf{C}\cdot e_j=H_n\subset H\subset G$$

be defined by

$$T_n(z_1,\cdots,z_n)=\sum_{j=1}^{n}z_j\lambda_j e_j,$$

and we define the orthogonal projection from H onto H_n by $P_n(y)=\sum_{j=1}^{n}y_je_j$ for $y=(y_1,\cdots,y_n,\cdots)\in H$.

For a holomorphic $(0,2)$-form f on a pseudoconvex domain in a separable Hilbert space, we can obtain a C^∞ solution of the $\overline{\partial}$-equation defined on $\mathbf{C}^n$, projecting f onto $\mathbf{C}^n$ and using the Hörmander's L^2-estimates, where a symmetric part of $[\overline{\partial}]$ is non-identically zero. Then we can construct a good cylindrical solution g_n of which the symmetric component, corresponding to the first solution, is identically zero as follows.

Theorem 2.1.[6] *Let Ω be a pseudoconvex open subset of G and let $f:\Omega\longrightarrow\Lambda^{(0,2)}(G)$ be a holomorphic $(0,2)$-form. Then there exists a $C^\infty(0,1)$-form $g_n:\Omega_n\longrightarrow\Lambda^{(0,1)}(\mathbf{C}^n)$ such that the symmetric part $G(g_n)$ is identically zero and $[\overline{\partial}]g_n=\overline{\partial}g_n=f_n$ for a holomorphic $(0,2)$-form f_n projected onto $\Omega_n=(T_n)^{-1}(\Omega\cap H_n)$.*

By using the solution obtained in Theorem 2.1, we can solve the $\overline{\partial}$-problem on a pseudoconvex domain in a Hilbert space.

Theorem 2.2. *Let Ω be a pseudoconvex open subset of G and let $\omega:\Omega\longrightarrow\Lambda^{(0,2)}(G)$ be a holomorphic $(0,2)$-form which is bounded on the bounded subsets of Ω. Then there exists $u:\Omega\cap H_T\longrightarrow\Lambda^{(0,1)}(G)$ such that u is a $C^\infty(0,1)$-form, bounded on the bounded subsets of $\Omega\cap H_T$ and $\overline{\partial}u=\omega$ on $\Omega\cap H_T$.*

Proof. In terms of the orthogonal projection P_n from H onto H_n, put $S_n=P_n^{-1}(\Omega\cap H_n)$. Let $n\geq 2$. We define, for $1\leq i,j\leq n$ and $t\in\Omega$, $\dot{\omega}_{ij}:\Omega\longrightarrow\mathbf{C}$ by $\dot{\omega}_{ij}(t)=\omega(t)(e_i,e_j)$ and

$$\omega_n(x)(y^1,y^2)=\sum_{i<j}^{n}\dot{\omega}_{ij}(P_nx)\bar{y}_i^1\bar{y}_j^2=\omega(P_nx)[P_ny^1,P_ny^2] \tag{2.1}$$

for $x=(x_1,\cdots,x_n)\in S_n$ and $y^i=(y_1^i,\cdots,y_n^i,\cdots)\in H, i=1,2$, that is,

$$\omega_n(x)=\sum_{i<j}^{n}\dot{\omega}_{ij}(P_nx)d\bar{x}_i\wedge d\bar{x}_j=\omega(P_nx)(P_n,P_n).$$

Then, from the solution g_n in Theorem 2.1, let us define a cylindrical solution $u_n:S_n\longrightarrow\Lambda^{(0,1)}(H)$ for the holomorphic (0,2)-form ω_n on S_n. Since for $z=(z_1,\cdots,z_n)\in\mathbf{C}^n$, $y=(y_1,\cdots,y_n,\cdots)\in H$ and $0\neq\lambda_j\in\mathbf{C}$, from the definitions of T_n and P_n, we get the following composition

$$T_n^{-1}\circ P_n(y)=T_n^{-1}\Big(\sum_{j=1}^{n}y_je_j\Big)=\Big(\frac{y_1}{\lambda_1},\cdots,\frac{y_n}{\lambda_n}\Big),$$

we can define u_n from g_n in Theorem 2.1 by

$$u_n(x)(y) = g_n(T_n^{-1} \circ P_n(x))(T_n^{-1} \circ P_n(y)) = g_n\Big(\frac{x_1}{\lambda_1}, \cdots, \frac{x_n}{\lambda_n}\Big)\Big(\frac{y_1}{\lambda_1}, \cdots, \frac{y_n}{\lambda_n}\Big)$$

for $x = \sum\limits_{j=1}^{n} x_j e_j \in S_n$ and $y = \sum\limits_{j=1}^{\infty} y_j e_j \in H$. Since g_n is C^∞, u_n is a $C^\infty(0,1)$-form on S_n and

$$u_n(x) = \sum_{j=1}^{n} \frac{1}{\lambda_j} a_j\Big(\frac{x_1}{\lambda_1}, \cdots, \frac{x_n}{\lambda_n}\Big) d\bar{x}_j \quad \text{if} \quad g_n(z) = \sum_{j=1}^{n} a_j(z) d\bar{z}_j.$$

In Theorem 2.1, we could define a holomorphic $(0,2)$-form f_n on the pseudoconvex open set Ω_n in $\mathbf{C}^n$, as defining ω_n,

$$f_n(z) = \sum_{i<j}^{n} \omega[T_n(z)](T_n e_j, T_n e_i) d\bar{z}_i \wedge d\bar{z}_j.$$

Then we can write for $z \in \mathbf{C}^n$ and $\eta_1, \eta_2 \in \mathbf{C}$,

$$f_n(z)(\eta_1, \eta_2) = \omega[T_n(z)](T_n(\eta_1), T_n(\eta_2)).$$

Thus we obtain by the definition of u_n and f_n and Theorem 2.1,

$$\begin{aligned} \overline{\partial} u_n(x)(y) &= \overline{\partial} g_n\Big(\frac{x_1}{\lambda_1}, \cdots, \frac{x_n}{\lambda_n}\Big)\Big(\frac{y_1}{\lambda_1}, \cdots, \frac{y_n}{\lambda_n}\Big) \\ &= f_n\Big(\frac{x_1}{\lambda_1}, \cdots, \frac{x_n}{\lambda_n}\Big)\Big(\frac{y_1}{\lambda_1}, \cdots, \frac{y_n}{\lambda_n}\Big) \\ &= \omega\Big(T_n\Big(\frac{x_1}{\lambda_1}, \cdots, \frac{x_n}{\lambda_n}\Big)\Big)\Big(T_n\Big(\frac{y_1}{\lambda_1}, \cdots, \frac{y_n}{\lambda_n}\Big)\Big) \\ &= \omega(P_n(x))(P_n(y)) = \omega_n(x)(y). \end{aligned}$$

Now, we look for estimates for u_n. The measures μ and μ_T denote the Gauss measure on H and the image by T, respectively. Then the following fact was proved in [12]: if $z_0 \in H_T$ then the translated measure $\mu_T(B - z_0)$ for each Borel set B of H is equivalent to μ_T with a density

$$\frac{d\mu_{T,z_0}}{d\mu_T}(x) = \rho_T(z_0, x) \quad \text{for } x \in H_T, \tag{2.2}$$

where

$$\rho_T(z_0, x) = \exp\Big[-\frac{1}{2}|z_0|^2_{H_T} + \mathrm{Re}\langle T^{-1}z_0; T^{-1}x\rangle\Big]. \tag{2.3}$$

Then we have for a continuous plurisubharmonic function φ, defining a plurisubharmonic weight $\tilde{\varphi}_n(z) = \varphi \circ T_n(z) + \frac{1}{2}||z||^2_{\mathbf{C}^n}$ in $\mathbf{C}^n$,

$$\begin{aligned} ||u_n(z)||^2_{L^2(S_n,\bar{H},\mu_T)} &= \int_{S_n} ||u_n(z)||^2_{\Lambda^{(0,1)}(H)} e^{-\varphi\circ P_n(z)} d\mu_T \\ &= \int_{\Omega_n} ||g_n(z)||^2_{\Lambda^{(0,1)}(\mathbf{C}^n)} e^{-\tilde{\varphi}_n(z)} \frac{d\sigma_{2n}}{(2\pi)^n} \\ &\le 4 \int_{\Omega_n} ||f_n(z)||^2_{\Lambda^{(0,2)}(\mathbf{C}^n)} e^{-\tilde{\varphi}_n(z)} \frac{d\sigma_{2n}}{(2\pi)^n} \\ &\le 8M, \end{aligned} \tag{2.4}$$

where $M = \lambda^4$ for $\lambda = (\lambda_1, \lambda_2)$ and $d\sigma_{2n}$ is the Lebesgue measure on $\mathbf{C}^n$. Hence, the cylindrical solution $u_n : S_n \longrightarrow \mathcal{L}(\bar{H})$ also has L^2-estimate. Therefore, we get a $C^\infty_{(0,1)}$-

solution u_n on S_n such that $\overline{\partial} u_n = \omega_n$ for a holomorphic $(0,2)$-form ω_n defined on S_n and u_n has L^2-estimate in (2.4).

From now on we construct a continuous weak solution u on $\Omega \cap H_T$. Let us observe the following : if $x \in \Omega \cap H$ and if $\delta(x, (\Omega \cap H)^c)$ denotes the distance in H between x and the complement of $\Omega \cap H$ in H, then

$$P_n\Big[B\Big(x, \frac{1}{2}\delta(x, (\Omega \cap H)^c)\Big)\Big] \subset \Omega$$

for n large enough, if $B(x, r)$ denotes the closed ball in H of center x and radius r. We take a dense sequence $\{x_n\}_{n=1}^{\infty}$ in Ω and we denote $B_{x_n} = B\Big(x_n, \frac{1}{2}\delta(x_n, (\Omega \cap H)^c)\Big)$. By (2.4), $\{u_n \exp(-\frac{1}{2}\varphi \circ P_n), n \geq 2\}$ is bounded in the space $L^2(H, \bar{H}_T, \mu_T)$ of square μ_T-Bochner integrable mappings from H into $\overline{H}_T$ endowed with the Hilbert structures given by

$$\langle f, g \rangle = \int_H \Big(g(x); f(x)\Big)_{H_T} d\mu_T,$$

where f and $g \in L^2(H, \overline{H}_T, \mu_T)$. Hence there exists a subsequence of the sequence

$$\{u_n e^{\frac{1}{2}(-\varphi \circ P_n)}, n \geq 2\},$$

which we still denote by $\{u_n e^{\frac{1}{2}(-\varphi \circ P_n)}, n \geq 2\}$, which is defined on B_{x_n} for n large enough and which, for every $n \in \mathbf{N}$, is weakly convergent in

$$L^2(B_{x_n}, \overline{H}_T, \mu_T) \quad \text{to } g_{B_{x_n}} \in L^2(B_{x_n}, \overline{H}_T, \mu_T).$$

We set

$$u_{B_{x_n}}(x) = g_{B_{x_n}}(x) e^{\frac{1}{2}\varphi(x)}.$$

Now, if $z_0 \in \Omega \cap H_T$, let $\varepsilon > 0$ be small enough so that $B(z_0, \varepsilon)$ is contained in some ball B_{x_n}. We denote by B the above ball B_{x_n} and by B_ε the ball $B(0, \varepsilon)$. Let $z_0 \in \Omega \cap H_T, e \in H_T$ and $x \in B_\varepsilon$. We define the following C^∞ function on the open unit disc $\Delta(0,1)$ of $\mathbf{C}$:

$$\begin{aligned} \Delta = \Delta(0,1) &\longrightarrow \mathbf{C} \\ \lambda &\longmapsto (u_n(z_0 + \lambda x); e)_{H_T}. \end{aligned}$$

By the Cauchy integral formula for C^∞ functions on the disc $\bar{\Delta}(0,1)$, we obtain for $\lambda \in \Delta$,

$$\begin{aligned} (u_n(z_0 + \lambda x); e) &= \frac{1}{2\pi i} \int_{|\alpha|=1} (u_n(z_0 + \alpha x); e) \frac{d\alpha}{\alpha - \lambda} \\ &\quad + \frac{1}{2\pi i} \iint_{\bar{\Delta}} \frac{\partial}{\partial \bar{\alpha}} (u_n(z_0 + \alpha x); e) \frac{d\alpha \wedge d\bar{\alpha}}{\alpha - \lambda} \\ &= \frac{1}{2\pi i} \int_{|\alpha|=1} (u_n(z_0 + \alpha x); e) \frac{d\alpha}{\alpha - \lambda} \\ &\quad + \frac{1}{2\pi i} \iint_{\bar{\Delta}} [\overline{\partial}] u_n(z_0 + \alpha x)(x, e) \frac{d\alpha \wedge d\bar{\alpha}}{\alpha - \lambda} \\ &= \frac{1}{2\pi i} \int_{|\alpha|=1} (u_n(z_0 + \alpha x); e) \frac{d\alpha}{\alpha - \lambda} \\ &\quad + \frac{1}{2\pi i} \iint_{\bar{\Delta}} \omega(P_n(z_0 + \alpha x))(P_n x, P_n e) \frac{d\alpha \wedge d\bar{\alpha}}{\alpha - \lambda}. \end{aligned}$$

Letting $\lambda = 0$, by (2.1) we have

$$(u_n(z_0); e) = \frac{1}{2\pi}\int_0^{2\pi} (u_n(z_0 + e^{i\theta}x); e)d\theta + 2\int_0^1 \Big[\frac{1}{2\pi}\int_0^{2\pi} \omega[P_n(z_0 + re^{i\theta}x)](P_n x, P_n e)d\theta\Big]dr. \tag{2.5}$$

Now, we integrate (2.5) in $x \in B_\varepsilon$ with respect to the measure μ_T and apply Fubini's theorem to the second integral. By applying the rotation invariance of some integrals with respect to μ_T, we have

$$\mu_T(B_\varepsilon)(u_n(z_0); e) = \int_{z_0+B_\varepsilon} (u_n(x); e)\rho_T(z_0, x)d\mu_T + 2\int_0^1\int_{B_\varepsilon} \omega\Big[P_n(z_0 + rx)\Big](P_n x, P_n e)d\mu_T dr.$$

Since for $x \in \Omega \cap H_T$, $e \in H_T$ and ρ_T in (2.2) and (2.3)

$$\begin{aligned}&\int_{z_0+B_\varepsilon} \Big([u_B(x) - u_n(x)]; e\Big)_{H_T} \rho_T(z_0, x)d\mu_T\\ =& \int_{z_0+B_\varepsilon} \Big(u_n(x); \rho_T(z_0, x)e\Big)\Big[e^{-\frac{1}{2}[\varphi\circ P_n(x) - \varphi(x)]} - 1\Big]d\mu_T\\ &- \int_{z_0+B_\varepsilon} \Big(u_n(x)e^{-\frac{1}{2}\varphi\circ P_n(x)} - u_B(x)e^{-\frac{1}{2}\varphi(x)}; e\cdot \rho_T(z_0, x)e^{\frac{1}{2}\varphi(x)}\Big)d\mu_T\end{aligned}$$

and the first and second parts in the integration tend to zero as $n \to \infty$, we have

$$\lim_{n\to\infty} \mu_T(B_\varepsilon)(u_n(z_0); e) = \int_{z_0+B_\varepsilon} (u_B(x); e)\rho_T(z_0, x)d\mu_T + 2\int_0^1\int_{B_\varepsilon} \omega(z_0 + rx)(x, e)d\mu_T dr.$$

Hence $\{u_n(z_0); n \geq 2\} \subset \mathcal{L}(\bar{H}_T)$ for each fixed $z_0 \in \Omega \cap H_T$, and $\{u_n(z_0)(e),\ n \geq 2\}$ is a convergent sequence in $\mathbf{C}$ for every $e \in H_T$. By applying an extended version of Banach theorem we have, for all $z_0 \in \Omega \cap H_T$,

$$u_n(z_0) \to u(z_0) \in \mathcal{L}(\bar{H}_T).$$

Then it follows from [13] that u is bounded on the balls of $\Omega \cap H_T$ and u is a weak solution to the $\overline{\partial}$-problem. By [7] the solution u is C^∞ on $\Omega \cap H_T$ and so satisfies all conditions of Theorem 2.2. This completes the proof.

§3. The $\overline{\partial}$-Problem on D.F.N. Spaces

We apply the results about the $\overline{\partial}$-problem on Hilbert spaces to show the existence of the solution for the $\overline{\partial}$-equation on D.F.N. spaces.

Lemma 3.1. *Let $H_0 \subset H_1 \subset H_2$ be separable, complex Hilbert spaces with nuclear injections. Let Ω be a pseudoconvex open subset of H_2 and let $\omega : \Omega \longrightarrow \Lambda^{(0,2)}(H_2)$ be a holomorphic (0,2)-form on Ω. Then there exists a $C^\infty(0,1)$-form $u : \Omega \cap H_0 \longrightarrow \Lambda^{(0,1)}(H_0)$ such that $\overline{\partial}u = \omega$ on $\Omega \cap H_0$.*

Proof. By using Theorem 2.2 and following an argument of J. F. Colombeau and B. Perrot[1], we can prove this lemma.

Proposition 3.1.[1] *Let E and F be two separable Hilbert spaces with a compact inclusion mapping from F to E. Let Ω be a pseudoconvex open subset of E with $\Omega \cap F \neq \emptyset$. Then the restriction mapping $\mathbf{H}(\Omega) \longrightarrow \mathbf{H}(\Omega \cap F)$ has dense range.*

Lemma 3.2. *Let E and F be two separable Hilbert spaces with a compact inclusion mapping from F to E. Let Ω be a pseudoconvex open subset of E with $\Omega \cap F \neq \emptyset$ and K be a compact subset of $\Omega \cap F$. If $\varepsilon > 0$ is given, for any holomorphic $(0,1)$-form h in $\Omega \cap F$, then there is a holomorphic $(0,1)$-form $\tilde{h}$ in Ω such that $||\tilde{h} - h||_K \leq \varepsilon$.*

Proof. For holomorphic functions $h_j : \Omega \cap F \longrightarrow \mathbf{C}$, we can define $h : \Omega \cap F \longrightarrow \Lambda^{(0,1)}(F)$ by $h(z) = \sum_{j=1}^{n} h_j(z) d\bar{z}_j$ for $z \in \Omega \cap F$. Then, by Proposition 3.1, for any $\varepsilon > 0$ there exist $\tilde{h}_j \in \mathbf{H}(\Omega)$ such that $|\tilde{h}_j - h_j|_K \leq \varepsilon$. Hence we obtain a holomorphic $(0,1)$-form $\tilde{h} : \Omega \longrightarrow \Lambda^{(0,1)}(E)$ such that $\tilde{h}(z) = \sum_{j=1}^{n} \tilde{h}_j(z) d\bar{z}_j$ for $z \in \Omega$. Then we have $||\tilde{h} - h||_K \leq \varepsilon$.

Theorem 3.1. *Let E be a D.F.N. space and Ω be a pseudoconvex domain in E. Let $f : \Omega \longrightarrow \Lambda^{(0,2)}(E)$ be a holomorphic $(0,2)$-form. Then there exists a $C^\infty(0,1)$-form g on Ω such that $\overline{\partial} g = f$.*

Proof. Since E is a nuclear Silva space, it is the inductive limit of an increasing sequence of Hilbert spaces E_n with a nuclear injection $E_n \to E_{n+1}$ for every n. Then there exists an increasing exhaustive sequence of compact subsets K_n of Ω, where we may assume that K_n is compact in E_n. We set $\Omega(n) = \Omega \cap E_n$.

Now we consider the restriction of f to $\Omega \cap E_{n+1}$. From Theorem 2.2, there exists a $C^\infty(0,1)$-form u_n on $\Omega(n) \subset E_n$ such that $\overline{\partial} u_n = f$ on $\Omega(n)$. In order to start an induction we set $g_2 = u_2$; then $u_3 - g_2$ is defined and is a $C^\infty(0,1)$-form on $\Omega(2)$. Since

$$\overline{\partial}(u_3 - g_2) = \overline{\partial} u_3 - \overline{\partial} g_2 = \overline{\partial} u_3 - \overline{\partial} u_2 = 0$$

on $\Omega(2)$, $u_3 - g_2$ is a holomorphic $(0,1)$-form on $\Omega(2)$. From Lemma 3.2, this holomorphic $(0,1)$-form may be approximated uniformly on K_2 by holomorphic $(0,1)$-forms on $\Omega \cap E_3$. Therefore, there is a holomorphic $(0,1)$-form h_2 in $\Omega \cap E_3$ such that

$$\sup_{x \in K_2} |u_3(x) - g_2(x) - h_2(x)| \leq \frac{1}{2^2}.$$

If we set $g_3 = u_3 - h_2$, we have

$$\begin{cases} g_3 \text{ is a } C^\infty(0,1)\text{-form on } \Omega(3) = \Omega \cap E_3, \\ \overline{\partial} g_3 = f \text{ on } \Omega(3) \text{ (since } \overline{\partial} h_2 = 0), \\ \sup_{x \in K_2} |g_3(x) - g_2(x)| \leq \frac{1}{2^2}. \end{cases}$$

By an induction we obtain a sequence (g_n) of $C^\infty(0,1)$-forms on $\Omega(n)$ such that

$$\begin{cases} \overline{\partial} g_n = f \text{ on } \Omega(n), \\ \sup_{x \in K_{n-1}} |g_n(x) - g_{n-1}(x)| \leq (\frac{1}{2})^{n-1}. \end{cases}$$

For every $x \in \Omega$, there is some n large enough such that $x \in K_n \subset \Omega(n)$. Thus $g_n(x)$ is defined for n large enough and

$$|g_n(x) - g_{n-1}(x)| \leq \left(\frac{1}{2}\right)^{n-1}$$

for n large enough. We set

$$g(x) = \lim_{n\to\infty} g_n(x).$$

We notice that $\overline{\partial}(g_{n+k} - g_n) = 0$ on $\Omega(n)$, thus $g_{n+k} - g_n$ is a holomorphic $(0,1)$-form in $\Omega(n)$. When $k \to \infty$, $(g_{n+k} - g_n)$ converges to holomorphic $(0,1)$-form $g - g_n$ in $\Omega(n)$ since every compact subset of $\Omega(n)$ is contained in K_l, for some l large enough. Therefore $g = (g-g_n)+g_n$ is a $C^\infty(0,1)$-form on $\Omega(n)$. Since this holds for any n, g is a $C^\infty(0,1)$-form on Ω. Furthermore, $g - g_n$ is a holomorphic $(0,1)$-form in $\Omega(n)$ and $\overline{\partial}g_n = f$ on $\Omega(n)$, hence $\overline{\partial}g = f$ on $\Omega(n)$ for any n, i.e., $\overline{\partial}g = f$ on Ω.

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