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REFINEMENTS OF THE FAN-TODD'S INEQUALITIES

K. K. CHONG*

Abstract

Refinements to inequalities on inner product spaces are presented. In this respect, inequalities dealt with in this paper are: Cauchy's inequality, Bessel's inequality, Fan-Todd's inequality and Fan-Todd's determinantal inequality. In each case, a strictly increasing function is put forward, which lies between the smaller and the larger quantities of each inequality. As a result, an improved condition for equality of the Fan-Todd's determinantal inequality is deduced.

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§1. Introduction

In recent years, refinements or interpolations have played an important role on several types of inequalities with new results deduced as a consequence. Please refer to the papers [2, 8, 9, 12], etc. The aim of this paper is to furnish refinements of the Cauchy's and Bessel's inequalties as shown in Section 2, and also refinements of the Fan-Todd's inequality and the Fan-Todd's determinantal inequality in Sections 3 and 4, with an improved condition for equality derived.

First of all, we give some basic terms and definitions. An inner product space on a complex vector space X is a function that associates a complex number $\langle u, v\rangle$ with each pair of vectors u and v in X, in such a way that the following axioms are satisfied for all vectors u, v and w in X and all scalars λ:

(1) $\langle u, v\rangle = \overline{\langle v, u\rangle}$;
(2) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$;
(3) $\langle \lambda u, v\rangle = \lambda\langle u, v\rangle$;
(4) $\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0$ if and only if $v = 0$.

Here, $\overline{\langle v, u\rangle}$ denotes the complex conjugate of $\langle v, u\rangle$. A complex vector space with an inner product is called a complex inner product space. Let $\|u\| = \sqrt{\langle u, u\rangle}$ denote the norm of u. The content of the paper will be organized as follows: In Section 2, refinements of the

Manuscript received May 22, 2000.
*Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong, China.
E-mail: makkchon@inet.polyu.edu.hk

Cauchy's and Bessel's inequalities will be presented. In Section 3, refinements of the Fan-Todd's inequality will be put forward. Finally, in Section 4, refinements of the Fan-Todd's determinantal inequality will be presented, with the condition for equality improved.

§2. Refinements of the Cauchy's Inequality

The well-known Cauchy's inequality states as follows:

Theorem 2.1. *For any two vectors a and b in an inner product space X, we have*

$$|\langle a,b\rangle| \le \|a\|\|b\|. \tag{2.1}$$

The equality holds if and only if a and b are linearly dependent.

We can have a refinement of the Cauchy's inequality as follows:

Theorem 2.2. *Let a and b be two non-zero vectors in an inner product space (real or complex) X, such that $|\langle a,b\rangle| < \|a\|\|b\|$.*

For any $t\in[0,1]$, we define

$$a_t = [(1-t)\langle a,b\rangle b]/\|b\|^2 + ta \tag{2.2}$$

and

$$F(t) = \|a_t\|. \tag{2.3}$$

Then we have

(1) $\|a_t\| < \|a\|$ *for* $t\in[0,1)$ *and* $|\langle a,b\rangle| < \|a_t\|\|b\|$ *for* $t\in(0,1]$;

(2) *For* $0\le s<t\le 1$, *we have* $F(s)<F(t)$;

(3) $F(t)$ *is a strictly increasing function for* $t\in[0,1]$, *with* $F(0)=|\langle a,b\rangle|/\|b\|$ *and* $F(1)=\|a\|$ *i.e. we have the refinement* $|\langle a,b\rangle|/\|b\| < F(t) < \|a\|$ *for* $t\in(0,1)$.

Proof. (1) For $t\in[0,1)$, by (2.2), we have

$$\|a_t\| = \|[(1-t)\langle a,b\rangle b]/\|b\|^2 + ta\| \tag{2.4}$$

$$\le [(1-t)|\langle a,b\rangle|]/\|b\| + t\|a\| < \|a\|. \tag{2.5}$$

Hence, $\|a_t\|<\|a\|$ for $t\in[0,1)$.

For $t\in(0,1]$,

$$\langle a_t,b\rangle = (1-t)\langle a,b\rangle + t\langle a,b\rangle = \langle a,b\rangle. \tag{2.6}$$

As $|\langle a,b\rangle| < \|a_t\|\|b\|$ for $t\neq 0$, we have $|\langle a,b\rangle| < \|a_t\|\|b\|$ for $t\in(0,1]$. The proof of part (1) is complete.

(2) Suppose $0<s<t<1$. We have to set up the following identity first,

$$a_s = [(1-s/t)\langle a_t,b\rangle b]/\|b\|^2 + (s/t)a_t. \tag{2.7}$$

The last equation can be verified as follows:

$$\begin{aligned}
&[(1-s/t)\langle a,b\rangle b]/\|b\|^2 + (s/t)a_t\\
&= (1-s/t)[\langle a,b\rangle b/\|b\|^2 + s/t[(1-t)\langle a,b\rangle b/\|b\|^2 + ta]\\
&= [(1-s/t)+s/t(1-t)]\langle a,b\rangle b/\|b\|^2 + sa\\
&= [(1-s)\langle a,b\rangle b]/\|b\|^2 + sa = a_s.
\end{aligned} \tag{2.8}$$

By (2.7) and part (1), we have $\|a_s\|<\|a_t\|$. Hence we have $F(s)<F(t)$ for $s<t$. The case for $s=0$ and $t=1$ can be shown easily. Hence the proof of part (2) is complete.

(3) From part (2), we have immediately the result that $F(t)$ is a strictly increasing function for $t\in[0,1]$. Obviously, $F(0)=|\langle a,b\rangle|/\|b\|$ and $F(1)=\|a\|$.

Remark 2.1. As the ℓ_2 and L_2 spaces are inner product spaces, the above refinements can be applied to the Hölder's inequalities in ℓ_2 and L_2 spaces respectively.

In analysis (please refer to [5]), Bessel's inequality states as follows:

Theorem 2.3. *Let X be an inner product space (real or complex) and $a \in X$. Let $e_1, e_2, \cdots, e_n$ be any finite collection of distinct elements of an orthonormal set S in X. Then*

$$\sum_{i=1}^{n} |\langle a, e_i\rangle|^2 \le \|a\|^2. \tag{2.9}$$

A refinement of the Bessel's inequality can be presented as follows:

Theorem 2.4. *Let X be an inner product space (real or complex) and a be a nonzero vector in X. Let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal set in X, such that*

$$\sum_{i=1}^{n} |\langle a, e_i\rangle|^2 < \|a\|^2 \tag{2.10}$$

and $\langle a, e_i\rangle$ are not all zero. Let

$$p = \langle a, e_1\rangle e_1 + \cdots + \langle a, e_n\rangle e_n. \tag{2.11}$$

For any real number $t \in [0, 1]$, we define

$$a_t = [(1-t)\langle a, p\rangle p]/\|p\|^2 + ta \tag{2.12}$$

and

$$F(t) = \|a_t\|. \tag{2.13}$$

Then we have the following:

(1) $\|a_t\| < \|a\|$ *for* $t \in [0, 1)$, *and* $\|p\|^2 = \sum\limits_{i=1}^{n} |\langle a, e_i\rangle|^2 < \|a_t\|^2$ *for* $t \in (0, 1]$;

(2) $F(s) < F(t)$ *for* $0 \le s < t \le 1$;

(3) $F(t)$ *is a strictly increasing function for* $t \in [0, 1]$ *with* $F(0) = \sqrt{\sum\limits_{i=1}^{n} |\langle a, e_i\rangle|^2}$ *and* $F(1) = \|a\|$, *i.e. we have the refinement* $\|p\| < F(t) < \|a\|$ *for* $t \in (0, 1)$.

Proof. (1)

$$\begin{aligned} |\langle a, p\rangle| &= |\langle a, \langle a, e_1\rangle e_1 + \cdots + \langle a, e_n\rangle e_n\rangle| \\ &= |\langle a, e_1\rangle \overline{\langle a, e_1\rangle} + \cdots + \langle a, e_n\rangle \overline{\langle a, e_n\rangle}| \\ &= \sum_{i=1}^{n} |\langle a, e_i\rangle|^2 \\ &= \langle p, p\rangle = \|p\|^2. \end{aligned} \tag{2.14}$$

Hence we have $|\langle a, p\rangle| < \|a\|\|p\|$. By Theorem 2.2(1), we have

$$|\langle a, p\rangle| = \|p\|^2 < \|a_t\|\|p\|. \tag{2.15}$$

The last inequality implies that $\|p\|^2 < \|a_t\|^2$.

The remaining parts of the proof are similar to the proof of Theorem 2.2 with b replaced by p, and the proof is omitted here.

§3. Refinements of the Fan-Todd's Inequality

A. M. Ostrowski presented the following result (please refer to [4] or [5]):

Theorem 3.1. *Let $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$ be two sequences of non-proportional real numbers such that $\sum\limits_{i=1}^{n} a_i x_i = 0$, and $\sum\limits_{i=1}^{n} b_i x_i = 1$.*

Let $A = \sum\limits_{i=1}^{n} a_i^2$, $B = \sum\limits_{i=1}^{n} b_i^2$, $C = \sum\limits_{i=1}^{n} a_i b_i$. Then we have $\sum\limits_{i=1}^{n} x_i^2 \geq \frac{A}{AB-C^2}$ with equality if and only if $x_i = \frac{Ab_i - Ca_i}{AB-C^2}$, $1 \leq i \leq n$.

Fan and Todd in [4] presented the following theorem:

Theorem 3.2. *Let $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$ with $n \geq 2$ be two sequences of real numbers such that $a_i b_j \neq a_j b_i$ for $i \neq j$. Then*

$$\frac{\sum\limits_{i=1}^{n} a_i^2}{\Big(\sum\limits_{i=1}^{n} a_i^2\Big)\Big(\sum\limits_{i=1}^{n} b_i^2\Big) - \Big(\sum\limits_{i=1}^{n} a_i b_i\Big)^2} \leq \binom{n}{2}^{-2} \sum_{i=1}^{n} \Big(\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{a_j}{a_j b_i - a_i b_j}\Big)^2. \tag{3.1}$$

Here, $\binom{n}{2}$ denotes the number of combinations of n distinct objects chosen 2 at a time.

M. Bjelica in [6, pp.445–448] put forward the following refinement of Fan-Todd's inequality:

Theorem 3.3. *Let $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$ with $n \geq 2$ be two sequences of real numbers such that $a_i b_j \neq a_j b_i$ for $i \neq j$. If $|\alpha| \leq 1$, then*

$$\begin{aligned} \frac{A}{AB - C^2} &\leq \binom{n}{2}^{-2} \sum_{i=1}^{n} \Big[\sum_{\substack{j=1 \\ j\neq i}}^{n} \alpha \frac{a_j}{a_j b_i - a_i b_j} + (1-\alpha)\frac{Ab_i - Ca_i}{AB - C^2}\Big]^2 \\ &\leq \binom{n}{2}^{-2} \sum_{i=1}^{n} \Big(\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{a_j}{a_j b_i - a_i b_j}\Big)^2. \end{aligned} \tag{3.2}$$

Z. M. Mitrovic in [7] established the following theorem:

Theorem 3.4 *Let a and b be two linearly independent vectors in a complex inner product space V and let x be a vector in V such that $\langle x, a\rangle = \alpha$ and $\langle x, b\rangle = \beta$. Then*

$$G(a, b)\|x\|^2 \geq \|\overline{\alpha} b - \overline{\beta} a\|^2 \tag{3.3}$$

with equality if and only if $x = \frac{\langle a, \overline{\beta} a - \overline{\alpha} b\rangle b - \langle b, \overline{\beta} a - \overline{\alpha} b\rangle a}{G(a,b)}$, where $G(a, b)$ denotes the Gram determinant of vectors a and b, i.e.

$$G(a, b) = \left\| \begin{matrix} \langle a, a\rangle & \langle a, b\rangle \\ \langle b, a\rangle & \langle b, b\rangle \end{matrix} \right\|. \tag{3.4}$$

The proofs of the above-mentioned four theorems can be found in [4–7]. It is natural to find some similar refinements for Theorem 3.4 in the complex inner product space. In fact, the following theorem is the answer to this problem.

Theorem 3.5. *Let a and b be two linearly independent vectors in a complex inner product space V and let x be a vector in V such that $\langle x, a\rangle = \alpha$ and $\langle x, b\rangle = \beta$. Let*

$$y = \frac{\langle a, \overline{\beta} a - \overline{\alpha} b\rangle b - \langle b, \overline{\beta} a - \overline{\alpha} b\rangle a}{G(a, b)}. \tag{3.5}$$

Let $\mathbb{D} = \{t \in \mathbb{C} : |t| \leq 1\}$ be the closed unit disk in the complex plane $\mathbb{C}$. For any $t \in \mathbb{D}$, we define

$$F(t) = \|tx + (1-t)y\|^2.$$

Suppose $x \neq y$. Then $F(t)$ depends only on the modulus of t and is a strictly increasing function of $|t|$, with $F(0) = \|y\|^2$ and $F(t) = \|x\|^2$ for any $t \in \partial\mathbb{D}$, the boundary of $\mathbb{D}$, i.e. we have the refinement for $t \in \mathbb{D}$:

$$\|y\|^2 \leq F(t) \leq \|x\|^2. \tag{3.6}$$

Proof. It is straight forward to verify that $\langle y, a\rangle = \alpha$ and $\langle y, b\rangle = \beta$. In fact,

$$\begin{aligned} G(a,b)\langle y,a\rangle &= \langle a, \overline{\beta}a - \overline{\alpha}b\rangle\langle b,a\rangle - \langle b, \overline{\beta}a - \overline{\alpha}b\rangle\langle a,a\rangle \\ &= [\beta\|a\|^2 - \alpha\langle a,b\rangle]\langle b,a\rangle - [\beta\langle b,a\rangle - \alpha\|b\|^2]\|a\|^2 \\ &= \alpha[\|a\|^2\|b\|^2 - |\langle a,b\rangle|^2]. \end{aligned} \tag{3.7}$$

Hence, we have $\langle y, a\rangle = \alpha$. Also,

$$\begin{aligned} G(a,b)\langle y,b\rangle &= \langle a, \overline{\beta}a - \overline{\alpha}b\rangle\langle b,b\rangle - \langle b, \overline{\beta}a - \overline{\alpha}b\rangle\langle a,b\rangle \\ &= [\beta\|a\|^2 - \alpha\langle a,b\rangle]\|b\|^2 - [\beta\langle b,a\rangle - \alpha\|b\|^2]\langle a,b\rangle \\ &= \beta[\|a\|^2\|b\|^2 - |\langle a,b\rangle|^2]. \end{aligned} \tag{3.8}$$

Hence, we have $\langle y, b\rangle = \beta$. As a result, we have

$$\begin{aligned} \langle y,y\rangle &= \frac{1}{G(a,b)}[\langle a, \overline{\beta}a - \overline{\alpha}b\rangle\langle b,y\rangle - \langle b, \overline{\beta}a - \overline{\alpha}b\rangle\langle a,y\rangle] \\ &= \|\overline{\beta}a - \overline{\alpha}b\|^2/G(a,b). \end{aligned} \tag{3.9}$$

Similarly, we have

$$\langle y,x\rangle = \|\overline{\beta}a - \overline{\alpha}b\|^2/G(a,b) = \|y\|^2. \tag{3.10}$$

Hence

$$\langle x,y\rangle = \overline{\langle y,x\rangle} = \|y\|^2. \tag{3.11}$$

$$\begin{aligned} F(t) &= \|tx + (1-t)y\|^2 = \langle tx + (1-t)y, tx + (1-t)y\rangle \\ &= t\bar{t}\|x\|^2 + t(1-\bar{t})\langle x,y\rangle + (1-t)\bar{t}\langle y,x\rangle + (1-t)(1-\bar{t})\|y\|^2 \\ &= t\bar{t}\|x\|^2 + (1-t\bar{t})\|y\|^2 = |t|^2(\|x\|^2 - \|y\|^2) + \|y\|^2. \end{aligned} \tag{3.12}$$

By Theorem 3.4 and $x \neq y$, $\|x\|^2 - \|y\|^2 > 0$. Hence, $F(t)$ is a strictly increasing function of $|t|$ on $\mathbb{D}$, depending only on $|t|$, with $F(0) = \|y\|^2$ and $F(t) = \|x\|^2$ for any $t \in \partial\mathbb{D}$.

§4. Refinement of the Fan-Todd's Determinantal Inequality

In [4], Fan and Todd presented the following celebrated theorem:

Theorem 4.1. *Let n and m be two integers such that $2 \leq m \leq n$. Let $a_i = \{a_{i1}, a_{i2}, \cdots, a_{in}\}$ $(1 \leq i \leq m)$ be m vectors in the unitary n-space U^n such that every $m \times m$ submatrix of the $m \times n$ matrix*

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{4.1}$$

is nonsingular. Let $G(a_1, a_2, \cdots, a_{m-1})$ denote the Gram determinant of the $m-1$ vectors $a_1, a_2, \cdots, a_{m-1}$; and let $G(a_1, a_2, \cdots, a_m)$ denote the Gram determinant of the m vectors $a_1, a_2, \cdots, a_m$. Let $M(j_1, j_2, \cdots, j_{m-1})$ denote the determinant of order $m-1$ formed by the first $m-1$ rows of (4.1) *and the columns of* (4.1) *with indices $j_1, j_2, \cdots, j_{m-1}$ taken* *in this order. Let* $N(j_1, j_2, \cdots, j_{m-1}, j_m)$ *denote the determinant of order* m *formed by the columns of* (4.1) *with indices* $j_1, j_2, \cdots, j_{m-1}, j_m$ *taken in this order. Then*

$$\frac{G(a_1, \cdots, a_{m-1})}{G(a_1, \cdots, a_m)} \leq \binom{n}{m}^{-2} \sum_{j_m=1}^{n} \Big| \sum_{\substack{j_1<j_2<\cdots<j_{m-1} \\ j_1,\cdots,j_{m-1}\neq j_m}} \frac{M(j_1, j_2, \cdots, j_{m-1})}{N(j_1, j_2, \cdots, j_m)} \Big|^2. \tag{4.2}$$

Here, the Gram determinant is given by

$$G(a_1, a_2, \cdots, a_m) = \begin{vmatrix} \langle a_1, a_1\rangle & \langle a_1, a_2\rangle & \cdots & \langle a_1, a_m\rangle \\ \langle a_2, a_1\rangle & \langle a_2, a_2\rangle & \cdots & \langle a_2, a_m\rangle \\ \vdots & \vdots & & \vdots \\ \langle a_m, a_1\rangle & \langle a_m, a_2\rangle & \ldots & \langle a_m, a_m\rangle \end{vmatrix}. \tag{4.3}$$

The proof of Theorem 4.1 can be found in [4].

In [1], Beesack presented the following theorem:

Theorem 4.2. *Let* $a_1, a_2, \cdots, a_m (m \geq 1)$ *be linearly independent vectors in a Hilbert space* H *and let* $\alpha_1, \alpha_2, \cdots, \alpha_m$ *be given scalars. If* $x \in H$ *satisfies*

$$\langle x, a_i\rangle = \alpha_i, \quad i = 1, 2, \cdots, m, \tag{4.4}$$

then

$$G^2\|x\|^2 \geq \Big\| \sum_{i=1}^{m} \gamma_i a_i \Big\|^2, \tag{4.5}$$

where $G = G(a_1, a_2, \cdots, a_m)$ *is the Gram determinant of* $a_1, a_2, \cdots, a_m$, *and* γ_i *is the determinant obtained from* G *by replacing the elements of the ith row of* G *by* $(\alpha_1, \alpha_2, \cdots, \alpha_m)$. *Moreover, equality holds in* (4.5) *if and only if* $Gx = \sum\limits_{i=1}^{m} \gamma_i a_i$.

The proof of Theorem 4.2 can be found in [1].

Remark 4.1. The γ_i's in Theorem 4.2 are the unique solution of the following system of equations:

$$\begin{aligned} \langle a_1, a_1\rangle\gamma_1 + \cdots + \langle a_m, a_1\rangle\gamma_m &= G\alpha_1, \\ &\cdots \\ \langle a_1, a_m\rangle\gamma_1 + \cdots + \langle a_m, a_m\rangle\gamma_m &= G\alpha_m. \end{aligned}$$

Therefore, we have

$$\sum_{j=1}^{m} \langle a_j, a_i\rangle\gamma_j = G\alpha_i, \quad i = 1, 2, \cdots, m, \tag{4.6}$$

or

$$\sum_{j=1}^{m} \overline{\gamma_j}\langle a_i, a_j\rangle = G\overline{\alpha_i}, \quad i = 1, 2, \cdots, m. \tag{4.7}$$

Here $\overline{\alpha}$ denotes the complex conjugate of α.

The following theorem is a generalization of Theorem 4.1 and Theorem 4.2, in the form of refinements of inequalities.

Theorem 4.3. *Let* $a_1, a_2, \cdots, a_m (m \geq 2)$ *be linearly independent vectors in a complex inner product space* X *and let* $\alpha_1, \alpha_2, \cdots, \alpha_m$ *be given scalars. Let* $x \in X$ *satisfy*

$$\langle x, a_i\rangle = \alpha_i, \quad i = 1, 2, \cdots, m. \tag{4.8}$$

Let $y \in X$ be defined by

$$Gy = \sum_{i=1}^{m} \gamma_i a_i, \tag{4.9}$$

where G and γ_i have the same meanings as in Theorem 4.2. Suppose $x \neq y$. For $t \in \mathbb{D}$, the closed unit disk in $\mathbb{C}$, we define $Q(t)$ as follows:

$$Q(t) = \|tx + (1-t)y\|^2. \tag{4.10}$$

Then $Q(t)$ depends only on the modulus of t and is a strictly increasing function of $|t|$ for $t \in \mathbb{D}$, with $Q(0) = \|y\|^2$ and $Q(t) = \|x\|^2$ for $t \in \partial\mathbb{D}$ the boundary of $\mathbb{D}$. That is, for $t \in \mathbb{D}$, with $t \neq 0$ and $|t| \neq 1$, we have

$$\|y\|^2 < Q(t) < \|x\|^2. \tag{4.11}$$

Proof. By (4.8) and (4.9), we have

$$G\langle y, x\rangle = \sum_{i=1}^{m} \gamma_i \langle a_i, x\rangle = \sum_{i=1}^{m} \gamma_i \overline{\alpha_i}, \tag{4.12}$$

$$\begin{aligned} G\langle y, y\rangle &= \Big\langle \sum_{i=1}^{m} \gamma_i a_i, (1/G) \sum_{j=1}^{m} \gamma_j a_j \Big\rangle \\ &= (1/G) \sum_{i=1}^{m} \sum_{j=1}^{m} \gamma_i \overline{\gamma_j} \langle a_i, a_j\rangle \\ &= \sum_{i=1}^{m} \gamma_i \overline{\alpha_i}. \end{aligned} \tag{4.13}$$

Hence

$$\langle y, x\rangle = \langle y, y\rangle. \tag{4.14}$$

Also, we have

$$\langle x, y\rangle = \overline{\langle y, x\rangle} = \langle y, y\rangle. \tag{4.15}$$

For any $t \in \mathbb{D}$, we have

$$\begin{aligned} Q(t) &= \|tx + (1-t)y\|^2 \\ &= \langle tx + (1-t)y, tx + (1-t)y\rangle \\ &= t\bar{t}\|x\|^2 + t(1-\bar{t})\langle x, y\rangle + (1-t)\bar{t}\langle y, x\rangle + (1-t)(1-\bar{t})\|y\|^2 \\ &= |t|^2\|x\|^2 + (1 - t\bar{t})\|y\|^2 \\ &= |t|^2(\|x\|^2 - \|y\|^2) + \|y\|^2. \end{aligned} \tag{4.16}$$

By Theorem 4.2 and $x \neq y$, we have $\|x\|^2 - \|y\|^2 > 0$. Hence, $Q(t)$ is a strictly increasing function of $|t|$, depending only on the modulus of t with $Q(0) = \|y\|^2$ and $Q(t) = \|x\|^2$ for $t \in \partial\mathbb{D}$. This completes the proof of the theorem.

Corollary 4.1. *Let n and m be two integers such that $2 \leq m \leq n$. Let $a_i = (a_{i1}, a_{i2}, \cdots, a_{in})$, $i = 1, 2, \cdots, m$, be m vectors in U^n, the unitary n-space, such that every $m \times m$*

submatrix of the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \tag{4.17}$$

is nonsingular. Let $x = (x_1, x_2, \cdots, x_n)$ *be the vector in* U^n *such that for* $k = 1, 2, \cdots, n$,

$$\overline{x_k} = \binom{n}{m}^{-1} \sum_{\substack{j_1<j_2<\cdots<j_{m-1}\\ j_1,\cdots,j_{m-1}\neq k}} \frac{M(j_1, \cdots, j_{m-1})}{N(j_1, \cdots, j_{m-1}, k)}. \tag{4.18}$$

Let y *be the vector in* U^n *defined as* $y = (1/G)\sum_{i=1}^{m} \gamma_i a_i$, *where* γ_i *are the unique solution of the system of equations in* (4.6). *Then we have*

$$\|y\|^2 = \frac{G(a_1, \cdots, a_{m-1})}{G(a_1, \cdots, a_m)} \le \binom{n}{m}^{-2} \sum_{j_m=1}^{n} \Big| \sum_{\substack{j_1<j_2<\cdots<j_{m-1}\\ j_1,\cdots,j_{m-1}\neq j_m}} \frac{M(j_1, j_2, \cdots, j_{m-1})}{N(j_1, j_2, \cdots, j_m)} \Big|^2.$$

Furthermore, equality holds if and only if $x = y$.

Proof. Similar to the proof in [4, Theorem 1] or as in the proof of Theorem 4.4 below, we can show that $\langle a_i, x\rangle = 0$, $i = 1, 2, \cdots, m-1$, and $\langle a_m, x\rangle = 1$. Hence, Theorem 4.3 is applicable with $X = U^n$, $\alpha_1 = \alpha_2 = \cdots = \alpha_{m-1} = 0$, $\alpha_m = 1$, and as $\gamma_m = G(a_1, \cdots, a_{m-1})$

$$\|y\|^2 = (1/G)\sum_{i=1}^{m} \gamma_i \overline{\alpha_i} = \gamma_m \overline{\alpha_m}/G = \gamma_m/G = \frac{G(a_1, \cdots, a_{m-1})}{G(a_1, \cdots, a_m)}. \tag{4.19}$$

Hence, the Fan-Todd's determinantal inequality is deduced as a consequence of $\|y\|^2 \le \|x\|^2$ in (4.11). By Theorem 4.2, we have, equality holds if and only if $x = y$.

Remark 4.2. It is clear that Theorem 4.3 is a generalization of Theorem 4.1 and Theorem 4.2, providing us with a necessary and sufficient condition for equality of the Fan-Todd's determinantal inequality.

In an attempt to give a criterion on x, for which $\langle x, x\rangle$ will be the minimum, the following lemma was put forward by Fan and Todd in [4].

Lemma 4.1. *Let* $a_1, a_2, \cdots, a_m$ *be* m *linearly independent vectors in* U^n $(2 \le m \le n)$. *If a vector* x *in* U^n *varies under the conditions*:

$$\langle a_i, x\rangle = 0 \quad \text{if } 1 \le i \le m-1,$$
$$\langle a_i, x\rangle = 1 \quad \text{if } i = m,$$

then the minimum of $\langle x, x\rangle$ *is* $\frac{G(a_1,\cdots,a_{m-1})}{G(a_1,\cdots,a_m)}$. *Furthermore, this minimum value is attained if and only if* x *is a linear combination of* $a_1, a_2, \cdots, a_m$.

From Corollary 4.1, we have the improved result to Lemma 4.1, with a more explicit expression in the linear combination of $a_1, a_2, \cdots, a_m$ as follows.

Lemma 4.2. *With the same assumptions and notations as in Theorem* 4.1 *and Lemma* 4.1, *we have*

(i) *The minimum of* $\langle x, x\rangle$ *is* $\frac{G(a_1,\cdots,a_{m-1})}{G(a_1,\cdots,a_m)}$.

(ii) *The minimum value of* $\langle x, x\rangle$ *is attained if and only if*

$$x = (1/G)\sum_{i=1}^{m} \gamma_i a_i,$$

where γ_i are the unique solution of the system of equations in (4.6) :

$$\sum_{j=1}^{m}\langle a_j, a_i\rangle\gamma_j = G\alpha_i \quad i = 1, 2, \cdots, m.$$

In the next theorem, a deduction of the weighted Fan-Todd's inequality will also be deduced as an application of our refinement Theorem 4.3. The original statement of Theorem 4.4 can be found in [4].

Theorem 4.4. *In addition to the hypotheses of Theorem* 4.1, *let* $p_{j_1,j_2,\cdots,j_m}$ *be complex numbers defined for every set of* m *distinct positive integers* $j_1, j_2, \cdots, j_m \le n$ *such that the following two conditions are fulfilled*:

(i) $p_{j_1,j_2,\cdots,j_m}$ *is independent of the arrangement of* $j_1, j_2, \cdots, j_m$;

(ii) $P = \sum_{1\le j_1<j_2<\cdots<j_m\le n} p_{j_1,j_2,\cdots,j_m} \ne 0.$

Then

$$\frac{G(a_1,\cdots,a_{m-1})}{G(a_1,\cdots,a_m)} \le \frac{1}{|P|^2}\sum_{j_m=1}^{n}\Big|\sum_{\substack{j_1<j_2<\cdots<j_{m-1}\\ j_1,\cdots,j_{m-1}\ne j_m}}^{n} p_{j_1j_2\cdots j_{m-1}j_m}\frac{M(j_1,j_2,\cdots,j_{m-1})}{N(j_1,j_2,\cdots,j_m)}\Big|^2. \quad (4.20)$$

Proof. Define a vector $x = (x_1, x_2, \cdots, x_n) \in U^n$ by

$$\overline{x_k} = \frac{1}{P}\sum_{\substack{j_1<j_2<\cdots<j_{m-1}\\ j_1,\cdots,j_{m-1}\ne k}} p_{j_1,j_2,\cdots,j_{m-1}k}\frac{M(j_1,j_2,\cdots,j_{m-1})}{N(j_1,j_2,\cdots,j_{m-1},k)}. \quad (4.21)$$

Let $y \in U^n$ be defined by

$$y = (1/G)\sum_{i=1}^{m}\gamma_i a_i. \quad (4.22)$$

Following the proof of [4, Theorem 1], we show first that

$$\langle a_i, x\rangle = 0, \quad i = 1, 2, \cdots, m-1 \text{ and } \langle a_m, x\rangle = 1, \quad (4.23)$$

$$\langle a_i, x\rangle = \frac{1}{P}\sum_{k=1}^{n} a_{ik}\sum_{\substack{j_1<\cdots<j_{m-1}\\ j_1,\cdots,j_{m-1}\ne k}} p_{j_1j_2\cdots j_{m-1}k}\frac{M(j_1,j_2,\cdots,j_{m-1})}{N(j_1,\cdots,j_{m-1},k)}. \quad (4.24)$$

For any ordered m-tuple $[h_1, h_2, \cdots, h_m]$ of integers such that

$$1 \le h_1 < h_2 < \cdots < h_m \le n, \quad (4.25)$$

the sum on the right side of (4.24) contains exactly m terms

$$a_{ik}\cdot\frac{M(j_1,j_2,\cdots,j_{m-1})}{N(j_1,\cdots,j_{m-1},k)} \quad (4.26)$$

$(j_1 < j_2 < \cdots < j_{m-1})$ such that $[j_1, j_2, \cdots, j_{m-1}, k]$ is merely a rearrangement of $[h_1, h_2, \cdots, h_m]$. The sum of these m terms is denoted by $S_i(h_1, h_2, \cdots, h_m)$, which can be written as:

$$S_i(h_1, h_2, \cdots, h_m) = \sum_{\nu=1}^{m} a_{ih_\nu} \frac{M(h_1, \cdots, h_{\nu-1}, h_{\nu+1}, \cdots, h_m)}{N(h_1, \cdots, h_{\nu-1}, h_{\nu+1}, \cdots, h_m, h_\nu)} \tag{4.27}$$

$$= \frac{\sum_{\nu=1}^{m} (-1)^{m+\nu} a_{ih_\nu} \cdot M(h_1, \cdots, h_{\nu-1}, h_{\nu+1}, \cdots, h_m)}{N(h_1, \cdots, h_{m-1}, h_m)}$$

$$= \begin{cases} 0 & \text{if } 1 \le i \le m-1, \\ 1 & \text{if } i = m. \end{cases} \tag{4.28}$$

Then (4.24) becomes

$$\langle a_i, x\rangle = \frac{1}{P} \sum_{1 \le h_1 < h_2 < \cdots < h_m \le n} p_{h_1 h_2 \cdots, h_m} S_i(h_1, h_2, \cdots, h_m)$$

$$= \begin{cases} 0 & \text{if } 1 \le i \le m-1, \\ 1 & \text{if } i = m. \end{cases} \tag{4.29}$$

This completes the proof of (4.23). By Theorem 4.3, we have $\|y\|^2 \le \|x\|^2$. As in Corollary 4.1, we have

$$\|y\|^2 = (1/G) \sum_{i=1}^{m} \gamma_i \overline{\alpha_i} = \frac{G(a_1, a_2, \cdots, a_{m-1})}{G(a_1, \cdots, a_{m-1}, a_m)}$$

$$\le \frac{1}{|P|^2} \sum_{k=1}^{n} \Big| \sum_{\substack{j_1 < j_2 < \cdots < j_{m-1} \\ j_1, \cdots, j_{m-1} \ne k}} p_{j_1 j_2 \cdots j_{m-1} k} \frac{M(j_1, j_2, \cdots, j_{m-1})}{N(j_1, j_2, \cdots, j_{m-1} k)} \Big|^2. \tag{4.30}$$

This completes the proof of Theorem 4.4.

Finally, we would remark that we have a similar statement for equality to hold in (4.30) as in Corollary 4.1.

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