REFINEMENTS OF THE FAN-TODD’S INEQUALITIES

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Abstract

Reﬁnements to inequalities on inner product spaces are presented. In this respect, inequalities dealt with in this paper are: Cauchy’s inequality, Bessel’s inequality, Fan-Todd’s inequality and Fan-Todd’s determinantal inequality. In each case, a strictly increasing function is put forward, which lies between the smaller and the larger quantities of each inequality. As a result, an improved condition for equality of the Fan-Todd’s determinantal inequality is deduced.

Keywords Cauchy’s inequality, Bessel’s inequality, Fan-Todd’s inequality, Fan-Todd’s determinantal inequality, Reﬁnements of inequalities

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§1. Introduction

In recent years, reﬁnements or interpolations have played an important role on several types of inequalities with new results deduced as a consequence. Please refer to the papers [2, 8, 9, 12], etc. The aim of this paper is to furnish reﬁnements of the Cauchy’s and Bessel’s inequalities as shown in Section 2, and also reﬁnements of the Fan-Todd’s inequality and the Fan-Todd’s determinantal inequality in Sections 3 and 4, with an improved condition for equality derived.

First of all, we give some basic terms and deﬁnitions. An inner product space on a complex vector space $X$ is a function that associates a complex number $\langle u, v \rangle$ with each pair of vectors $u$ and $v$ in $X$, in such a way that the following axioms are satisﬁed for all vectors $u, v$ and $w$ in $X$ and all scalars $\lambda$:

1. $\langle u, v \rangle = \langle v, u \rangle$;
2. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$;
3. $\lambda \langle u, v \rangle = \langle u, \lambda v \rangle$;
4. $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ if and only if $v = 0$.

Here, $\overline{\langle v, u \rangle}$ denotes the complex conjugate of $\langle v, u \rangle$. A complex vector space with an inner product is called a complex inner product space. Let $\|u\| = \sqrt{\langle u, u \rangle}$ denote the norm of $u$. The content of the paper will be organized as follows: In Section 2, reﬁnements of the

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Cauchy’s and Bessel’s inequalities will be presented. In Section 3, refinements of the Fan-Todd’s inequality will be put forward. Finally, in Section 4, refinements of the Fan-Todd’s determinantal inequality will be presented, with the condition for equality improved.

§2. Refinements of the Cauchy’s Inequality

The well-known Cauchy’s inequality states as follows:

**Theorem 2.1.** For any two vectors $a$ and $b$ in an inner product space $X$, we have

$$|(a, b)| \leq \|a\|\|b\|.$$  \hspace{1cm} (2.1)

The equality holds if and only if $a$ and $b$ are linearly dependent.

We can have a refinement of the Cauchy’s inequality as follows:

**Theorem 2.2.** Let $a$ and $b$ be two non-zero vectors in an inner product space (real or complex) $X$, such that $|(a, b)| < \|a\|\|b\|$.

For any $t \in [0, 1]$, we define

$$a_t = [(1 - t)(a, b)b]/\|b\|^2 + ta$$  \hspace{1cm} (2.2)

and

$$F(t) = \|a_t\|.$$  \hspace{1cm} (2.3)

Then we have

(1) $\|a_t\| < \|a\|$ for $t \in [0, 1)$ and $|(a, b)| < \|a_t\|\|b\|$ for $t \in (0, 1]$;

(2) For $0 \leq s < t \leq 1$, we have $F(s) < F(t)$;

(3) $F(t)$ is a strictly increasing function for $t \in [0, 1]$, with $F(0) = |(a, b)|/\|b\|$ and $F(1) = \|a\|$ i.e. we have the refinement $|(a, b)|/\|b\| < F(t) < \|a\|$ for $t \in (0, 1)$.

**Proof.** (1) For $t \in [0, 1)$, by (2.2), we have

$$\|a_t\| = \|[1 - t](a, b)b]/\|b\|^2 + ta\|$$  \hspace{1cm} (2.4)

$$\leq \|[1 - t]|(a, b)|/\|b\| + t\|a\| < \|a\|.$$  \hspace{1cm} (2.5)

Hence, $\|a_t\| < \|a\|$ for $t \in [0, 1)$.

For $t \in (0, 1]$,

$$\langle a_t, b \rangle = (1 - t)(a, b) + t(a, b) = \langle a, b \rangle.$$  \hspace{1cm} (2.6)

As $|(a, b)| < \|a_t\|\|b\|$ for $t \neq 0$, we have $|(a, b)| < \|a_t\|\|b\|$ for $t \in (0, 1]$. The proof of part (1) is complete.

(2) Suppose $0 < s < t < 1$. We have to set up the following identity first,

$$a_s = [(1 - s/t)(a, b)b]/\|b\|^2 + (s/t)a_t.$$  \hspace{1cm} (2.7)

The last equation can be verified as follows:

$$[(1 - s/t)(a, b)b]/\|b\|^2 + (s/t)a_t$$

$$= (1 - s/t)(a, b)b]/\|b\|^2 + s/t[(1 - t)(a, b)b]/\|b\|^2 + ta]$$

$$= [(1 - s/t) + s/t(1 - t)](a, b)b]/\|b\|^2 + sa$$

$$= [(1 - s)(a, b)b]/\|b\|^2 + sa = a_s.$$  \hspace{1cm} (2.8)

By (2.7) and part (1), we have $\|a_s\| < \|a_t\|$. Hence we have $F(s) < F(t)$ for $s < t$. The case for $s = 0$ and $t = 1$ can be shown easily. Hence the proof of part (2) is complete.

(3) From part (2), we have immediately the result that $F(t)$ is a strictly increasing function for $t \in [0, 1]$. Obviously, $F(0) = |(a, b)|/\|b\|$ and $F(1) = \|a\|$.
Remark 2.1. As the $\ell_2$ and $L_2$ spaces are inner product spaces, the above refinements can be applied to the Hölder’s inequalities in $\ell_2$ and $L_2$ spaces respectively.

In analysis (please refer to [5]), Bessel’s inequality states as follows:

**Theorem 2.3.** Let $X$ be an inner product space (real or complex) and $a \in X$. Let $e_1, e_2, \cdots, e_n$ be any finite collection of distinct elements of an orthonormal set $S$ in $X$. Then

$$\sum_{i=1}^{n} |\langle a, e_i \rangle|^2 \leq \|a\|^2. \quad (2.9)$$

A refinement of the Bessel’s inequality can be presented as follows:

**Theorem 2.4.** Let $X$ be an inner product space (real or complex) and $a$ be a nonzero vector in $X$. Let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal set in $X$, such that

$$\sum_{i=1}^{n} |\langle a, e_i \rangle|^2 < \|a\|^2 \quad (2.10)$$

and $\langle a, e_i \rangle$ are not all zero. Let

$$p = \langle a, e_1 \rangle e_1 + \cdots + \langle a, e_n \rangle e_n. \quad (2.11)$$

For any real number $t \in [0, 1]$, we define

$$a_t = [(1 - t)\langle a, p \rangle] / \|p\|^2 + ta \quad (2.12)$$

and

$$F(t) = \|a_t\|. \quad (2.13)$$

Then we have the following:

1. $\|a_t\| < \|a\|$ for $t \in (0, 1)$, and $\|p\|^2 = \sum_{i=1}^{n} |\langle a, e_i \rangle|^2 < \|a_t\|^2$ for $t \in (0, 1]$;
2. $F(s) < F(t)$ for $0 \leq s < t \leq 1$;
3. $F(t)$ is a strictly increasing function for $t \in [0, 1]$ with $F(0) = \sqrt{\sum_{i=1}^{n} |\langle a, e_i \rangle|^2}$ and $F(1) = \|a\|$, i.e. we have the refinement $\|p\| < F(t) < \|a\|$ for $t \in (0, 1)$.

**Proof.** (1)

$$\|\langle a, p \rangle\| = |\langle a, \langle a, e_1 \rangle e_1 + \cdots + \langle a, e_n \rangle e_n \rangle|$$

$$= |\langle a, e_1 \rangle \langle a, e_1 \rangle + \cdots + \langle a, e_n \rangle \langle a, e_n \rangle|$$

$$= \sum_{i=1}^{n} |\langle a, e_i \rangle|^2$$

$$= \langle p, p \rangle = \|p\|^2. \quad (2.14)$$

Hence we have $|\langle a, p \rangle| < \|a\| \|p\|$. By Theorem 2.2(1), we have

$$|\langle a, p \rangle| = \|p\|^2 < \|a_t\| \|p\|. \quad (2.15)$$

The last inequality implies that $\|p\|^2 < \|a_t\|^2$.

The remaining parts of the proof are similar to the proof of Theorem 2.2 with $b$ replaced by $p$, and the proof is omitted here.

§3. **Refinements of the Fan-Todd’s Inequality**

A. M. Ostrowski presented the following result (please refer to [4] or [5]):
**Theorem 3.1.** Let \( a = (a_1, \cdots, a_n) \) and \( b = (b_1, \cdots, b_n) \) be two sequences of non-proportional real numbers such that \( \sum_{i=1}^{n} a_i x_i = 0 \), and \( \sum_{i=1}^{n} b_i x_i = 1 \).

Let \( A = \sum_{i=1}^{n} a_i^2 \), \( B = \sum_{i=1}^{n} b_i^2 \), \( C = \sum_{i=1}^{n} a_i b_i \). Then we have \( \sum_{i=1}^{n} a_i^2 \geq \frac{A}{AB-C^2} \) with equality if and only if \( x_i = \frac{Ab_i - Cb_j}{Ab - C} \), \( 1 \leq i \leq n \).

Fan and Todd in [4] presented the following theorem:

**Theorem 3.2.** Let \( a = (a_1, \cdots, a_n) \) and \( b = (b_1, \cdots, b_n) \) with \( n \geq 2 \) be two sequences of real numbers such that \( a_i b_j \neq a_j b_i \) for \( i \neq j \). Then

\[
\frac{\sum_{i=1}^{n} a_i^2}{\left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{i=1}^{n} b_i^2 \right) - \sum_{i=1}^{n} a_i b_i^2} \leq \left( \frac{n}{2} \right)^{-2} \sum_{i=1}^{n} \left( \sum_{j=1 \atop j \neq i}^{n} \alpha a_j b_i - a_i b_j \right)^2.
\] (3.1)

Here, \( \binom{n}{2} \) denotes the number of combinations of \( n \) distinct objects chosen 2 at a time.

M. Bjelica in [6, pp.445–448] put forward the following refinement of Fan-Todd’s inequality:

**Theorem 3.3.** Let \( a = (a_1, \cdots, a_n) \) and \( b = (b_1, \cdots, b_n) \) with \( n \geq 2 \) be two sequences of real numbers such that \( a_i b_j \neq a_j b_i \) for \( i \neq j \). If \( |\alpha| \leq 1 \), then

\[
\frac{A}{AB-C^2} \leq \left( \frac{n}{2} \right)^{-2} \sum_{i=1}^{n} \left[ \sum_{j=1 \atop j \neq i}^{n} \alpha a_j b_i - a_i b_j \right] + (1 - \alpha) \frac{Ab_i - C a_i}{AB-C^2} \]

\[
\leq \left( \frac{n}{2} \right)^{-2} \sum_{i=1}^{n} \left( \sum_{j=1 \atop j \neq i}^{n} a_j b_i - a_i b_j \right)^2.
\] (3.2)

Z. M. Mitrovic in [7] established the following theorem:

**Theorem 3.4.** Let \( a \) and \( b \) be two linearly independent vectors in a complex inner product space \( V \) and let \( x \) be a vector in \( V \) such that \( (x, a) = \alpha \) and \( (x, b) = \beta \). Then

\[
G(a, b) ||x||^2 \geq ||\alpha b - \beta a||^2
\] (3.3)

with equality if and only if \( x = \frac{a, \beta a - \alpha b - (b, \beta a - \alpha b) a}{G(a,b)} \), where \( G(a,b) \) denotes the Gram determinant of vectors \( a \) and \( b \), i.e.

\[
G(a, b) = \left\| \frac{\langle a, a \rangle}{b, a} \right\| = \left\| \frac{\langle a, b \rangle}{b, b} \right\|.
\] (3.4)

The proofs of the above-mentioned four theorems can be found in [4–7]. It is natural to find some similar refinements for Theorem 3.4 in the complex inner product space. In fact, the following theorem is the answer to this problem.

**Theorem 3.5.** Let \( a \) and \( b \) be two linearly independent vectors in a complex inner product space \( V \) and let \( x \) be a vector in \( V \) such that \( (x, a) = \alpha \) and \( (x, b) = \beta \). Let

\[
y = \frac{\langle a, \beta a - \alpha b \rangle b - \langle b, \beta a - \alpha b \rangle a}{G(a,b)}.
\] (3.5)

Let \( \mathbb{D} = \{ t \in \mathbb{C} : |t| \leq 1 \} \) be the closed unit disk in the complex plane \( \mathbb{C} \). For any \( t \in \mathbb{D} \), we define

\[
F(t) = ||tx + (1-t)y||^2.
\]
Suppose \( x \neq y. \) Then \( F(t) \) depends only on the modulus of \( t \) and is a strictly increasing function of \( |t| \), with \( F(0) = \|y\|^2 \) and \( F(t) = \|x\|^2 \) for any \( t \in \partial \mathbb{D} \), the boundary of \( \mathbb{D} \), i.e. we have the refinement for \( t \in \mathbb{D} \):

\[
\|y\|^2 \leq F(t) \leq \|x\|^2.
\]  

**Proof.** It is straightforward to verify that \( \langle y, a \rangle = \alpha \) and \( \langle y, b \rangle = \beta. \) In fact, 

\[
G(a, b)\langle y, a \rangle = (a, \overline{ba} - \overline{va})\langle b, a \rangle - (b, \overline{ba} - \overline{va})\langle a, a \rangle
\]

\[
= |\beta||a|^2 - \alpha \|a\|^2| - \alpha \|b\|^2| + |\alpha||b|^2| - \alpha \|a\|^2| + \alpha \|b\|^2| - \alpha \|a\|^2| - |\alpha||b|^2|. \]

Hence, we have \( \langle y, a \rangle = \alpha. \) Also,  

\[
G(a, b)\langle y, b \rangle = (a, \overline{ba} - \overline{va})\langle b, b \rangle - (b, \overline{ba} - \overline{va})\langle a, b \rangle
\]

\[
= |\beta||a|^2 - \alpha \|a\|^2| - \alpha \|b\|^2| + |\alpha||b|^2| - \alpha \|a\|^2| - |\alpha||b|^2|. \]

Hence, we have \( \langle y, b \rangle = \beta. \) As a result, we have

\[
\langle y, y \rangle = \frac{1}{G(a, b)}[(a, \overline{ba} - \overline{va})\langle b, y \rangle - (b, \overline{ba} - \overline{va})\langle a, y \rangle]\]

\[
= \|\overline{ba} - \overline{va}\|^2/G(a, b). \]

Similarly, we have

\[
\langle y, x \rangle = \|\overline{ba} - \overline{vb}\|^2/G(a, b) = \|y\|^2. \]

Hence

\[
\langle x, y \rangle = \overline{\langle y, x \rangle} = \|y\|^2. \]  

\[
F(t) = \|tx + (1 - t)y\|^2 = \langle tx + (1 - t)y, tx + (1 - t)y \rangle
\]

\[
= t\|x\|^2 + t(1 - t)\langle x, y \rangle + (1 - t)\langle y, x \rangle + (1 - t)(1 - t)\|y\|^2
\]

\[
= t\|x\|^2 + (1 - t)\|y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 + \|y\|^2. \]  

By Theorem 3.4 and \( x \neq y, \|x\|^2 - \|y\|^2 > 0. \) Hence, \( F(t) \) is a strictly increasing function of \( |t| \) on \( \mathbb{D} \), depending only on \( |t| \), with \( F(0) = \|y\|^2 \) and \( F(t) = \|x\|^2 \) for any \( t \in \partial \mathbb{D}. \)

**§4. Refinement of the Fan-Todd’s Determinantal Inequality**

In [4], Fan and Todd presented the following celebrated theorem:

**Theorem 4.1.** Let \( n \) and \( m \) be two integers such that \( 2 \leq m \leq n \). Let \( a_i = \{a_{i1}, a_{i2}, \ldots, a_{in}\} \) be columns in the unitary \( n \)-space \( U^n \) such that every \( m \times m \) submatrix of the \( m \times n \) matrix

\[
\begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\]

(4.1)

is nonsingular. Let \( G(a_1, a_2, \ldots, a_{m-1}) \) denote the Gram determinant of the \( m - 1 \) columns \( a_1, a_2, \ldots, a_{m-1}; \) and let \( G(a_1, a_2, \ldots, a_n) \) denote the Gram determinant of the \( m \) columns \( a_1, a_2, \ldots, a_m. \) Let \( M(j_1, j_2, \ldots, j_{m-1}) \) denote the determinant of order \( m - 1 \) formed by the first \( m - 1 \) rows of (4.1) and the columns of (4.1) with indices \( j_1, j_2, \ldots, j_{m-1} \) taken
in this order. Let \( N(j_1, j_2, \cdots , j_{m-1}, j_m) \) denote the determinant of order \( m \) formed by the columns of (4.1) with indices \( j_1, j_2, \cdots , j_{m-1}, j_m \) taken in this order. Then

\[
G(a_1, \cdots , a_{m-1}) \leq \left( \frac{n}{m} \right) ^{-2} \sum _{j_m=1}^{n} \left| \sum _{j_2<j_3<\cdots <j_{m-1}}^{} M(j_1, j_2, \cdots , j_{m-1}) \right| ^{2} . \tag{4.2}
\]

Here, the Gram determinant is given by

\[
G(a_1, a_2, \cdots , a_m) = \begin{vmatrix} 
\langle a_1, a_1 \rangle & \langle a_1, a_2 \rangle & \cdots & \langle a_1, a_m \rangle \\
\langle a_2, a_1 \rangle & \langle a_2, a_2 \rangle & \cdots & \langle a_2, a_m \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle a_m, a_1 \rangle & \langle a_m, a_2 \rangle & \cdots & \langle a_m, a_m \rangle 
\end{vmatrix} . \tag{4.3}
\]

The proof of Theorem 4.1 can be found in [4]. In [1], Beesack presented the following theorem:

**Theorem 4.2.** Let \( a_1, a_2, \cdots , a_m (m \geq 1) \) be linearly independent vectors in a Hilbert space \( H \) and let \( \alpha_1, \alpha_2, \cdots , \alpha_m \) be given scalars. If \( x \in H \) satisfies

\[
\langle x, a_i \rangle = \alpha_i, \quad i = 1, 2, \cdots , m, \tag{4.4}
\]

then

\[
G^2 \| x \|^2 \geq \sum _{i=1}^{m} \gamma _i a_i^2 , \tag{4.5}
\]

where \( G = G(a_1, a_2, \cdots , a_m) \) is the Gram determinant of \( a_1, a_2, \cdots , a_m \), and \( \gamma _i \) is the determinant obtained from \( G \) by replacing the elements of the \( i \)-th row of \( G \) by \( \langle \alpha_1, \alpha_2, \cdots , \alpha_m \rangle \).

Moreover, equality holds in (4.5) if and only if \( Gx = \sum _{i=1}^{m} \gamma _i a_i \).

The proof of Theorem 4.2 can be found in [1].

**Remark 4.1.** The \( \gamma _i \)'s in Theorem 4.2 are the unique solution of the following system of equations:

\[
\langle a_1, a_1 \rangle \gamma _1 + \cdots + \langle a_m, a_1 \rangle \gamma _m = G \alpha _1 , \\
\vdots \\
\langle a_1, a_m \rangle \gamma _1 + \cdots + \langle a_m, a_m \rangle \gamma _m = G \alpha _m .
\]

Therefore, we have

\[
\sum _{j=1}^{m} \langle a_j, a_i \rangle \gamma _j = G \alpha _i, \quad i = 1, 2, \cdots , m , \tag{4.6}
\]

or

\[
\sum _{j=1}^{m} \overline{\gamma _j} \langle a_i, a_j \rangle = G \overline{\alpha _i}, \quad i = 1, 2, \cdots , m . \tag{4.7}
\]

Here \( \overline{\alpha} \) denotes the complex conjugate of \( \alpha \).

The following theorem is a generalization of Theorem 4.1 and Theorem 4.2, in the form of refinements of inequalities.

**Theorem 4.3.** Let \( a_1, a_2, \cdots , a_m (m \geq 2) \) be linearly independent vectors in a complex inner product space \( X \) and let \( \alpha_1, \alpha_2, \cdots , \alpha_m \) be given scalars. Let \( x \in X \) satisfy

\[
\langle x, a_i \rangle = \alpha_i , \quad i = 1, 2, \cdots , m . \tag{4.8}
\]
Let \( y \in X \) be defined by
\[
Gy = \sum_{i=1}^{m} \gamma_i a_i,
\]
where \( G \) and \( \gamma_i \) have the same meanings as in Theorem 4.2. Suppose \( x \neq y \). For \( t \in \mathbb{D} \), the closed unit disk in \( \mathbb{C} \), we define \( Q(t) \) as follows:
\[
Q(t) = \|tx + (1 - t)y\|^2.
\]
Then \( Q(t) \) depends only on the modulus of \( t \) and is a strictly increasing function of \( |t| \) for \( t \in \mathbb{D} \), with \( Q(0) = \|y\|^2 \) and \( Q(t) = \|x\|^2 \) for \( t \in \partial \mathbb{D} \) the boundary of \( \mathbb{D} \). That is, for \( t \in \mathbb{D} \), with \( t \neq 0 \) and \( |t| \neq 1 \), we have
\[
\|y\|^2 < Q(t) < \|x\|^2.
\]

**Proof.** By (4.8) and (4.9), we have
\[
G\langle y, x \rangle = \sum_{i=1}^{m} \gamma_i(a_i, x) = \sum_{i=1}^{m} \gamma_i a_i,
\]
\[
G\langle y, y \rangle = \left\langle \sum_{i=1}^{m} \gamma_i a_i, \frac{1}{G} \sum_{j=1}^{m} \gamma_j a_j \right\rangle
= \frac{1}{G} \sum_{i=1}^{m} \sum_{j=1}^{m} \gamma_i \gamma_j (a_i, a_j)
= \sum_{i=1}^{m} \gamma_i a_i.
\]
Hence
\[
\langle y, x \rangle = \langle y, y \rangle.
\]
Also, we have
\[
\langle x, y \rangle = \overline{\langle y, x \rangle} = \langle y, y \rangle.
\]

For any \( t \in \mathbb{D} \), we have
\[
Q(t) = \|tx + (1 - t)y\|^2
= \langle tx + (1 - t)y, tx + (1 - t)y \rangle
= |t|^2 \|x\|^2 + (1 - |t|) \|y\|^2
= |t|^2 \|x\|^2 + (1 - |t|) \|y\|^2 + |t|^2 \|y\|^2.
\]

By Theorem 4.2 and \( x \neq y \), we have \( \|x\|^2 - \|y\|^2 > 0 \). Hence, \( Q(t) \) is a strictly increasing function of \( |t| \), depending only on the modulus of \( t \) with \( Q(0) = \|y\|^2 \) and \( Q(t) = \|x\|^2 \) for \( t \in \partial \mathbb{D} \). This completes the proof of the theorem.

**Corollary 4.1.** Let \( n \) and \( m \) be two integers such that \( 2 \leq m \leq n \). Let \( a_i = (a_{i1}, a_{i2}, \cdots, a_{im}) \), \( i = 1, 2, \cdots, m \), be \( m \) vectors in \( U^n \), the unitary \( n \)-space, such that every \( m \times m \)
If a vector \( x \) of the system of equations in (4.4) is a linear combination of \( y \) then the minimum \( \langle \cdot \rangle \) is a determinantal inequality.

4.2, providing us with a necessary and sufficient condition for equality of the Fan-Todd’s lemma was put forward by Fan and Todd in [4].

The minimum value of \( \| y \|^2 = \frac{G(a_1, \ldots, a_m)}{G(a_1, \ldots, a_m)} \leq \left( \frac{n}{m} \right)^{-2} \sum_{j=1}^{n} \sum_{j_1 < j_2 < \cdots < j_{m-1} \neq k} \frac{M(j_1, \ldots, j_{m-1})}{N(j_1, j_2, \ldots, j_m)} \right)^2.

Furthermore, equality holds if and only if \( x = y \).

**Proof.** Similar to the proof in [4, Theorem 1] or as in the proof of Theorem 4.4 below, we can show that \( \langle a_i, x \rangle = 0 \) if \( 1 \leq i \leq m - 1 \), and \( \langle a_m, x \rangle = 1 \). Hence, Theorem 4.3 is applicable with \( X = U^n \), \( \alpha_1 = \alpha_2 = \cdots = m-1 = 0 \), \( \alpha_m = 1 \), and as \( \gamma_m = \frac{G(a_1, \ldots, a_m)}{G(a_1, \ldots, a_m)} \)

\[ \| y \|^2 = \left( \frac{1}{G} \right) \sum_{i=1}^{m} \gamma_i a_i = \gamma_m a_m / G = \frac{G(a_1, \ldots, a_m)}{G(a_1, \ldots, a_m)}. \]

Hence, the Fan-Todd’s determinantal inequality is deduced as a consequence of \( \| y \|^2 \leq \| x \|^2 \) in (4.11). By Theorem 4.2, we have, equality holds if and only if \( x = y \).

**Remark 4.2.** It is clear that Theorem 4.3 is a generalization of Theorem 4.1 and Theorem 4.2, providing us with a necessary and sufficient condition for equality of the Fan-Todd’s determinantal inequality.

In an attempt to give a criterion on \( x \), for which \( \langle x, x \rangle \) will be the minimum, the following lemma was put forward by Fan and Todd in [4].

**Lemma 4.1.** Let \( a_1, a_2, \ldots, a_m \) be \( m \) linearly independent vectors in \( U^n \) \((2 \leq m \leq n)\). If a vector \( x \) in \( U^n \) varies under the conditions:

\[ \langle a_i, x \rangle = 0 \quad \text{if} \quad 1 \leq i \leq m - 1, \]

\[ \langle a_i, x \rangle = 1 \quad \text{if} \quad i = m, \]

then the minimum of \( \langle x, x \rangle \) is \( \frac{G(a_1, \ldots, a_m)}{G(a_1, \ldots, a_m)} \). Furthermore, this minimum value is attained if and only if \( x \) is a linear combination of \( a_1, a_2, \ldots, a_m \).

From Corollary 4.1, we have the improved result to Lemma 4.1, with a more explicit expression in the linear combination of \( a_1, a_2, \ldots, a_m \) as follows.

**Lemma 4.2.** With the same assumptions and notations as in Theorem 4.1 and Lemma 4.1, we have:

(i) The minimum of \( \langle x, x \rangle \) is \( \frac{G(a_1, \ldots, a_m)}{G(a_1, \ldots, a_m)} \).

(ii) The minimum value of \( \langle x, x \rangle \) is attained if and only if

\[ x = (1/G) \sum_{i=1}^{m} \gamma_i a_i, \]
where \( \gamma_i \) are the unique solution of the system of equations in (4.6):

\[
\sum_{j=1}^{m} (a_j, a_i) \gamma_j = G a_i \quad i = 1, 2, \cdots, m.
\]

In the next theorem, a deduction of the weighted Fan-Todd’s inequality will also be deduced as an application of our refinement Theorem 4.3. The original statement of Theorem 4.4 can be found in [4].

**Theorem 4.4.** In addition to the hypotheses of Theorem 4.1, let \( p_{j_1, j_2, \cdots, j_m} \) be complex numbers defined for every set of \( m \) distinct positive integers \( j_1, j_2, \cdots, j_m \leq n \) such that the following two conditions are fulfilled:

(i) \( p_{j_1, j_2, \cdots, j_m} \) is independent of the arrangement of \( j_1, j_2, \cdots, j_m \);

(ii) \( P = \sum_{1 \leq j_1 < j_2 < \cdots < j_m \leq n} p_{j_1, j_2, \cdots, j_m} \neq 0 \).

Then

\[
G(a_1, \cdots, a_{m-1}) \leq \frac{1}{|P|^2} \sum_{j_m=1}^{n} \sum_{j_1 < j_2 < \cdots < j_{m-1}} p_{j_1, j_2, \cdots, j_m} \frac{M(j_1, j_2, \cdots, j_{m-1})}{\gamma(j_1, j_2, \cdots, j_{m-1}, k)}. \tag{4.20}
\]

**Proof.** Define a vector \( x = (x_1, x_2, \cdots, x_n) \in U^n \) by

\[
x_k = \frac{1}{P} \sum_{j_1 < j_2 < \cdots < j_{m-1} \neq k} p_{j_1, j_2, \cdots, j_{m-1}} \frac{M(j_1, j_2, \cdots, j_{m-1})}{\gamma(j_1, j_2, \cdots, j_{m-1}, k)}. \tag{4.21}
\]

Let \( y \in U^n \) be defined by

\[
y = \frac{1}{G} \sum_{i=1}^{m} \gamma_i a_i. \tag{4.22}
\]

Following the proof of [4, Theorem 1], we show first that

\[
(a_i, x) = 0, \quad i = 1, 2, \cdots, m-1 \text{ and } (a_m, x) = 1, \tag{4.23}
\]

\[
(a_i, x) = \frac{1}{P} \sum_{k=1}^{n} a_{ik} \sum_{j_1 < \cdots < j_{m-1} \neq k} p_{j_1, j_2, \cdots, j_{m-1}} \frac{M(j_1, j_2, \cdots, j_{m-1})}{\gamma(j_1, j_2, \cdots, j_{m-1}, k)}. \tag{4.24}
\]

For any ordered \( m \)-tuple \( [h_1, h_2, \cdots, h_m] \) of integers such that

\[
1 \leq h_1 < h_2 < \cdots < h_m \leq n, \tag{4.25}
\]

the sum on the right side of (4.24) contains exactly \( m \) terms

\[
a_{ik} \frac{M(j_1, j_2, \cdots, j_{m-1})}{\gamma(j_1, j_2, \cdots, j_{m-1}, k)} \tag{4.26}
\]

\( (j_1 < j_2 < \cdots < j_{m-1}) \) such that \( [j_1, j_2, \cdots, j_{m-1}, k] \) is merely a rearrangement of \( [h_1, h_2, \cdots, h_m] \). The sum of these \( m \) terms is denoted by \( S_i(h_1, h_2, \cdots, h_m) \), which can be written as:
\[ S_i(h_1, h_2, \ldots, h_m) = \sum_{\nu=1}^{m} \frac{a_{ih_\nu} M(h_1, \ldots, h_{\nu-1}, h_{\nu+1}, \ldots, h_m)}{N(h_1, \ldots, h_{\nu-1}, h_{\nu+1}, \ldots, h_m, h_\nu)} \]  
\[ = \frac{\sum_{\nu=1}^{m} (-1)^{m+\nu} a_{ih_\nu} M(h_1, \ldots, h_{\nu-1}, h_{\nu+1}, \ldots, h_m)}{N(h_1, \ldots, h_{m-1}, h_m)} \]  
\[ = \begin{cases} 0 & \text{if } 1 \leq i \leq m-1, \\ 1 & \text{if } i = m. \end{cases} \]  
\[ \langle a_i, x \rangle = \frac{1}{P} \sum_{1 \leq h_1 < h_2 < \cdots < h_m \leq n} p_{h_1 h_2 \cdots h_m} S_i(h_1, h_2, \ldots, h_m) \]  
\[ = \begin{cases} 0 & \text{if } 1 \leq i \leq m-1, \\ 1 & \text{if } i = m. \end{cases} \]  
This completes the proof of (4.23). By Theorem 4.3, we have \( \|y\|^2 \leq \|x\|^2 \). As in Corollary 4.1, we have
\[ \|y\|^2 = (1/G) \sum_{i=1}^{m} \frac{G(a_1, a_2, \ldots, a_{m-1})}{G(a_1, \ldots, a_{m-1}, a_m)} \]
\[ \leq \frac{1}{|P|^2} \sum_{k=1}^{n} \left| \sum_{j_1 < j_2 < \cdots < j_{m-1} \neq k} p_{j_1 j_2 \cdots j_{m-1}} M(j_1, j_2, \ldots, j_{m-1})^2 \right| \frac{N(j_1, j_2, \ldots, j_{m-1})}{N(j_1, j_2, \ldots, j_{m-1})^2}. \]  
This completes the proof of Theorem 4.4. Finally, we would remark that we have a similar statement for equality to hold in (4.30) as in Corollary 4.1.

References