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PERTURBATIONS OF A KIND OF DEGENERATE QUADRATIC HAMILTONIAN SYSTEM WITH SADDLE-LOOP**

ZHAO Yulin*† WU Xiaoming* ZHU Siming*

Abstract

The authors investigate a kind of degenerate quadratic Hamiltonian systems with saddle-loop. Under quadratic perturbations, it is proved that the perturbed system has at most two limit cycles in the finite plane. The proof relies on a careful analysis of a related Abelian integral.

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§1. Introduction

Consider a Hamiltonian system with small perturbations

$$\begin{cases} \dot{x} = -\frac{\partial H}{\partial y} + \epsilon X(x,y,\epsilon), \\ \dot{y} = \frac{\partial H}{\partial x} + \epsilon Y(x,y,\epsilon), \end{cases} \tag{1.1$_\epsilon$}$$

where $X(x,y,\epsilon)$ and $Y(x,y,\epsilon)$ are polynomials of x, y with coefficients depending analytically on the small parameter ϵ, and the unperturbed system $(1.1)_0$ has at least one center surrounded by the compact component Γ_h of algebraic curve

$$H(x,y)=h, \quad \deg H(x,y)=m+1, \quad \max\{\deg X(x,y,\epsilon), \quad \deg Y(x,y,\epsilon)\}=m.$$

The number of limit cycles of $(1.1)_\epsilon$ which emerge from Γ_h is equal to the number of zeros of the displacement function

$$d(h,\epsilon)=\epsilon M_1(h)+\epsilon^2 M_2(h)+\cdots . \tag{1.2}$$

Therefore, the first nonvanishing Melnikov function $M_k(h)$ in (1.2) is important for the study of the limit cycles of $(1.1)_\epsilon$.

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*Department of Mathematics, Zhongshan University, Guangzhou 510275, China.
†**E-mail:** mcszyl@zsu.edu.cn

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This paper is concerned with the bifurcations of limit cycles in plane quadratic Hamiltonian systems under small quadratic perturbations, i.e.,

$$\deg H(x,y) = 3, \quad \max\{\deg X(x,y,\epsilon), \quad \deg Y(x,y,\epsilon)\} = 2.$$

Some results concerned with the cyclicity of the period annulus of generic Hamiltonian vector fields can be found in [2,4,5,6], etc. Contrary to the generic cases, in the degenerate cases the first order Melnikov function does not suffice to study the limit cycles of the perturbed system, since $M_1(h) \equiv 0$ does not yield necessarily that the perturbation is Hamiltonian (or even integrable). We must compute the higher order Melnikov functions $M_k(h), k \geq 2$. The degenerate cubic Hamiltonians with at least one center have the following normal forms[7]

$$H(x,y) = x[y^2 + Ax^2 - 3(A-1)x + 3(A-2)], \quad A \in \mathbf{R}, \tag{1.3}$$

$$H(x,y) = y^2 - x^3 + x. \tag{1.4}$$

The standard elliptic Hamiltonian (1.4) was extensively studied in many papers (see [11] and the references therein). For the Hamiltonians (1.3) with at least one center, I. D. Iliev classified it as follows:

(1) saddle-loop: $A \in (-\infty, -1) \cup (2, +\infty)$;
(2) hyperbolic segment: $A \in (-1, 0)$;
(3) elliptic segment: $A \in (0, 2)$;
(4) parabolic segment: $A = 0$;
(5) triangle: $A = -1$;
(6) non-Morsean point: $A = 2$.

The cyclicity of the period annulus of the non-generic quadratic Hamiltonian systems with $A = 0, -1, 2$ or $A \in (-1, 0)$ has been studied in the papers [7,8,13,17]. In what follows we consider the degenerate cubic Hamiltonians (1.3) with saddle-loop. Obviously, the unperturbed system $(1.1)_0$ has the center (1,0) and the saddle $(\frac{A-2}{A}, 0)$ for $A \in (2, +\infty)$. It is easy to compute that $H(x,y)$ has three critical values

$$h_c = H(1,0) = A - 3, \quad h_s = H\Big(\frac{A-2}{A}, 0\Big) = \frac{(A-2)^2(A+1)}{A^2}$$

and $h_0 = H(0, \pm\sqrt{3(A-2)}i) = 0$, where h_0 corresponds to the complex critical points $(0, \pm\sqrt{3(A-2)}i)$. If $A = 3$, then $h_c = h_0 = 0$, i.e., $H(x,y)$ has two critical values. For this reason, we conjecture that $A = 3$ might be the bifurcation point of the case (1). Therefore, in this paper we investigate the number of limit cycles for small quadratic perturbations of quadratic Hamiltonian systems which have the first integral (1.3) with $A = 3$, i.e.,

$$\begin{cases} \dot{x} = 2xy + \epsilon\Big(\sum\limits_{i+j\leq 2} a_{ij}(\epsilon)x^i y^j\Big), \\ \dot{y} = -3 + 12x - 9x^2 - y^2 + \epsilon\Big(\sum\limits_{i+j\leq 2} b_{ij}(\epsilon)x^i y^j\Big), \end{cases} \tag*{$(1.5)_\epsilon$}$$

where $a_{ij}(\epsilon)$ and $b_{ij}(\epsilon)$ depend analytically on the small parameter ϵ. The first integral of the unperturbed system $(1.5)_0$ is

$$H(x,y) = x(y^2 + 3(x-1)^2) = h. \tag{1.6}$$

It is easy to see that the period annulus Γ_h of $(1.5)_0$ is an oval lying on the real algebraic curves (1.6), $h \in (0, \frac{4}{9})$. Γ_0 and $\Gamma_{4/9}$ correspond to the center (1,0) and the homoclinic loop of $(1.5)_0$ respectively. The critical point $(\frac{1}{3}, 0)$ is a saddle of the unperturbed system $(1.5)_0$ (cf. Fig. 1).

Fig.1 Phase Portrait of System $(1.5)_0$

To study the lowest upper bound of the number of limit cycles for system $(1.5)_\epsilon$ in the finite plane, we divide the limit cycles of $(1.5)_\epsilon$ into three types:

(1) the limit cycles which emerge from the period annulus Γ_h of $(1.5)_0$, $h \in (0, \frac{4}{9})$;

(2) the limit cycles that tend to the saddle-loop $\Gamma_{4/9}$ as $\epsilon \to 0$;

(3) the limit cycles that tend to the center $(1, 0)$ of the system $(1.5)_0$ as $\epsilon \to 0$.

Recently, Iliev proved the following conclusion (as a consequence of [9, Theorem 3]):

Lemma 1.1. *For small ϵ, the number of limit cycles in $(1.5)_\epsilon$ that emerge from the period annulus of $(1.5)_0$ is equal to the number of zeros in the interval $(0, \frac{4}{9})$ of the function*

$$J(h) = \alpha J_{-1}(h) + \beta J_0(h) + \gamma J_1(h), \tag{1.7}$$

where α, β and γ are arbitrary real constants,

$$J_i(h) = \oint_{\Gamma_h} x^i y dx, \quad i = -1, 0, 1, \tag{1.8}$$

and Γ_h is the compact component of algebraic curve $H(x, y) = h$, defined by (1.6).

Using Lemma 1.1, we know that the number of limit cycles which emerge from Γ_h is equal to the number of zeros of the Abelian integrals $J(h)$. Therefore, we will investigate the first type limit cycles by estimating the number of zeros of $J(h)$.

To study the limit cycles that tend to $\Gamma_{4/9}$, the asymptotic behavious of $J(h)$ near $h = \frac{4}{9}$ must be given. Using the results proved in the papers [7] and [12], $J_{-1}(h), J_0(h)$ and $J_1(h)$ have the following expansions near $h = \frac{4}{9}$:

$$\begin{aligned} J_{-1}(h) &= J_{-1}\Big(\frac{4}{9}\Big) + 3\Big(\frac{4}{9} - h\Big) \ln\Big(\frac{4}{9} - h\Big) + \cdots, \\ J_0(h) &= J_0\Big(\frac{4}{9}\Big) + \Big(\frac{4}{9} - h\Big) \ln\Big(\frac{4}{9} - h\Big) + \cdots, \\ J_1(h) &= J_1\Big(\frac{4}{9}\Big) + \frac{1}{3}\Big(\frac{4}{9} - h\Big) \ln\Big(\frac{4}{9} - h\Big) + \cdots, \end{aligned} \tag{1.9}$$

which yields

$$J(h) = \alpha J_{-1}\Big(\frac{4}{9}\Big) + \beta J_0\Big(\frac{4}{9}\Big) + \gamma J_1\Big(\frac{4}{9}\Big) + \Big(3\alpha + \beta + \frac{1}{3}\gamma\Big)\Big(\frac{4}{9} - h\Big) \ln\Big(\frac{4}{9} - h\Big) + c_2\Big(\frac{4}{9} - h\Big) + \cdots, \tag{1.10}$$

where $c_2 = -J'(\frac{4}{9})$ when $3\alpha + \beta + \frac{1}{3}\gamma = 0$. Using Roussarie's[12] and Iliev's[7] results again, we have

Lemma 1.2. (i) *If $\alpha J_{-1}(\frac{4}{9}) + \beta J_0(\frac{4}{9}) + \gamma J_1(\frac{4}{9}) = 0$, $3\alpha + \beta + \frac{1}{3}\gamma \neq 0$, then the system $(1.5)_\epsilon$ has at most one limit cycle that tends to the saddle-loop of $(1.5)_0$ as $\epsilon \to 0$.*

(ii) *If* $\alpha J_{-1}(\frac{4}{9})+\beta J_0(\frac{4}{9})+\gamma J_1(\frac{4}{9})=3\alpha+\beta+\frac{1}{3}\gamma=0$, $c_2\neq 0$, *then the system* $(1.5)_\epsilon$ *has at most two limit cycles that tend to* $\Gamma_{4/9}$ *as* $\epsilon\to 0$.

(iii)[7] *The condition* $\alpha J_{-1}(\frac{4}{9})+\beta J_0(\frac{4}{9})+\gamma J_1(\frac{4}{9})=3\alpha+\beta+\frac{1}{3}\gamma=c_2=0$ *implies* $\alpha=\beta=\gamma=0$.

Using the same arguments as in the paper [11], we get

Lemma 1.3. *If*

$$J(h)=b_{m+1}h^{m+1}+o(h^{m+1}),\quad b_{m+1}\neq 0, 0<h\ll 1,$$

then the system $(1.5)_\epsilon$ *has at most* m *limit cycles that tend to* $(1,0)$ *as* $\epsilon\to 0$.

Applying Lemmas 1.1–1.3, we will prove the following main result of this paper:

Theorem 1.1. *For small* ϵ, *the maximum number of limit cycles of* $(1.5)_\epsilon$ *in the finite plane is equal to two.*

§2. Picard-Fuchs Equation and Basic Relations

In this section, we will derive the relations satisfied by $J_{-1}(h), J_0(h)$ and $J_1(h)$. This is crucial for our analysis.

Lemma 2.1.[7] *The vector-function* $\mathbf{V}=\mathrm{col}(J_{-1}(h),J_0(h),J_1(h))$ *satisfies the following system*

$$(\mathbf{B}h+\mathbf{C})\mathbf{V}'=\mathbf{D}\mathbf{V},\tag{2.1}$$

which is equivalent to the Picard-Fuchs equation

$$h(9h-4)\begin{pmatrix}J'_{-1}\\ J'_0\\ J'_1\end{pmatrix}=\begin{pmatrix}3h & 14 & -18\\ h & 6h+2 & -6\\ \frac{1}{3}h & -h+2 & 9h-6\end{pmatrix}\begin{pmatrix}J_{-1}\\ J_0\\ J_1\end{pmatrix},\tag{2.2}$$

where

$$\mathbf{B}=\begin{pmatrix}3&0&0\\-1&3&0\\1&-4&3\end{pmatrix},\quad \mathbf{C}=\begin{pmatrix}0&-6&6\\0&0&0\\0&0&0\end{pmatrix},\quad \mathbf{D}=\begin{pmatrix}1&0&0\\0&2&0\\0&-3&3\end{pmatrix}.$$

Lemma 2.2. *The following equalities hold*:

$$J''_1(h)=\frac{1}{3}J''_0(h),\quad J''(h)=\alpha J''_{-1}(h)+\Big(\beta+\frac{1}{3}\gamma\Big)J''_0(h),\quad h\in\Big(0,\frac{4}{9}\Big).\tag{2.3}$$

Proof. Differentiating both sides of (2.1), we have

$$(\mathbf{B}h+\mathbf{C})\mathbf{V}''=(\mathbf{D}-\mathbf{B})\mathbf{V}',\tag{2.4}$$

where

$$\mathbf{D}-\mathbf{B}=\begin{pmatrix}-2&0&0\\1&-1&0\\-1&1&0\end{pmatrix}.$$

Eliminating J'_0, J'_{-1} from the second and the third equations of (2.4), one gets

$$-hJ''_0+3hJ''_1=0,$$

which implies $J''_1=\frac{1}{3}J''_0$, $h\in(0,\frac{4}{9})$. The second equality of (2.3) follows from (1.7).

Lemma 2.3. *The Abelian integrals* J_{-1}, J_0 *satisfy the following equation*

$$h(9h-4)\begin{pmatrix}J'''_{-1}\\ J'''_0\end{pmatrix}=\begin{pmatrix}-15h+8 & -16\\ h & -12h\end{pmatrix}\begin{pmatrix}J''_{-1}\\ J''_0\end{pmatrix}.\tag{2.5}$$

Proof. Differentiating both sides of (2.4), we obtain

$$(\mathbf{B}h+\mathbf{C})\mathbf{V}'''=(\mathbf{D}-2\mathbf{B})\mathbf{V}''. \tag{2.6}$$

Substituting $J_1''=\frac{1}{3}J_0''$ into the first two equations of (2.6), one gets (2.5).

Lemma 2.4. (i) *The following expansions hold near* $h=0$:

$$\begin{aligned} J_{-1}(h)&=J_0'(0)\Big(h+\frac{1}{2}h^2+\frac{35}{72}h^3+\cdots\Big),\\ J_0(h)&=J_0'(0)\Big(h+\frac{1}{4}h^2+\frac{5}{24}h^3+\cdots\Big),\\ J_1(h)&=J_0'(0)\Big(h+\frac{1}{12}h^2+\frac{5}{72}h^3+\cdots\Big). \end{aligned} \tag{2.7}$$

(ii) $J_0'(h)>0,\ h\in[0,\frac{4}{9})$.

Proof. (i) Since $h=0$ corresponds to the Hamiltonian value of the center of $(1.5)_0$, we know that $J_i(h)$ $(i=-1,0,1)$ is analytic at $h=0$. Substituting $J_i(h)=\sum\limits_{k=1}^{\infty}a_k^ih^k$ into (2.2), one gets (2.7).

(ii) Using Green formula, we have

$$J_0(h)=\iint_{\mathrm{int}\Gamma_h}dxdy,$$

which implies $J_0(h)$ is an increasing function, i.e., $J'(h)>0, h\in(0,\frac{4}{9})$. The inequality $J'(0)>0$ follows from (2.7).

Lemma 2.5. $J_0''(h)>0,\ h\in(0,\frac{4}{9})$.

Proof. Substituting $J_1''=\frac{1}{3}J_0''$ into the first two equations of (2.4), we have

$$\begin{cases} 3hJ_{-1}''-4J_0''=-2J_{-1}',\\ -hJ_{-1}''+3hJ_0''=J_{-1}'-J_0'. \end{cases} \tag{2.8}$$

Solving J_{-1}'' from (2.8), one gets

$$hJ_{-1}''=(-6h+4)J_0''-2J_0'. \tag{2.9}$$

Substituting (2.9) into the second equation of (2.5), we obtain

$$h(9h-4)J_0'''=(-18h+4)J_0''-2J_0'. \tag{2.10}$$

Since $J_0'(h)>0$, we can define the ratio $v(h)=\frac{J_0''(h)}{J_0'(h)}$ in the interval $(0,\frac{4}{9})$. The equation (2.10) implies that $v(h)$ satisfies the following Ricatti equation

$$h(9h-4)v'=-h(9h-4)v^2+(-18h+4)v-2. \tag{2.11}$$

Consider the following system

$$\begin{cases} \dot{h}=h(9h-4),\\ \dot{v}=-h(9h-4)v^2+(-18h+4)v-2. \end{cases} \tag{2.12}$$

Obviously, the system (2.12) has two saddle points at $(0,\frac{1}{2})$ and $(\frac{4}{9},-\frac{1}{2})$ respectively. The straight lines $h=0$ and $h=\frac{4}{9}$ are invariant lines of (2.12). It follows from (1.9) and (2.7) that $v(0)=\frac{1}{2}$ and $\lim\limits_{h\to\frac{4}{9}}v(h)=+\infty$, which implies the curve $v(h)=\frac{J_0''(h)}{J_0'(h)}$ is the trajectory of (2.12) starting from $(\frac{4}{9},+\infty)$ to the saddle point $(0,\frac{1}{2})$. Since $\dot{v}|_{v=0}=-2<0$, we conclude that the trajectory $v(h)=\frac{J_0''(h)}{J_0'(h)}$ does not intersect h-axis, which yields $J_0''(h)\neq0$ (see Fig.2). Noting $J_0''(0)=\frac{1}{2}J_0'(0)>0$, we have $J_0''(h)>0, h\in(0,\frac{4}{9})$.

Fig.2 The Trajectory $v(h)$ does not Intersect h-Axis

Proposition 2.1. *The ratio* $\omega(h)=\frac{J''_{-1}(h)}{J''_0(h)}$ *satisfies the following Ricatti equation*

$$h(9h-4)\omega' = -h\omega^2 + (-3h+8)\omega - 16. \tag{2.13}$$

Proof. The equation (2.13) follows from (2.5).

§3. Monotonicity of the Ratio $\omega(h)=\frac{J''_{-1}(h)}{J''_0(h)}$

Consider the system

$$\begin{cases} \dot{h} = h(9h-4), \\ \dot{\omega} = -h\omega^2 + (-3h+8)\omega - 16. \end{cases} \tag{3.1}$$

Obviously, it follows from Proposition 2.1 that in the $h\omega$-plane the curve $\omega(h)=\frac{J''_{-1}(h)}{J''_0(h)}$ is a trajectory of the system (3.1), $h\in(0,\frac{4}{9})$. In this section, we will use the system (3.1) to prove the monotonicity of $\omega(h)$.

The zero isoclines $\omega^+(h)$ and $\omega^-(h)$ of (3.1) are determined by the algebraic curve

$$G(h,\omega) \stackrel{\text{def}}{=} -h\omega^2 + (-3h+8)\omega - 16 = 0, \tag{3.2}$$

where

$$\omega^+(h) = \frac{-3h+8+\sqrt{9h^2-112h+64}}{2h}, \tag{3.3}$$

$$\omega^-(h) = \frac{-3h+8-\sqrt{9h^2-112h+64}}{2h}. \tag{3.4}$$

Denote $d(h)=9h^2-112h+64$. Since $d'(h)=18(h-\frac{56}{9})$, we know that $d'(h)<0, h\in(0,\frac{4}{9})$, which implies $9h^2-112h+64>d(\frac{4}{9})>0$, $h\in(0,\frac{4}{9})$.

Lemma 3.1. (i) *Near* $h=0$, *the zero isoclines* $\omega^+(h)$ *and* $\omega^-(h)$ *have the following expansions respectively*:

$$\omega^+(h) = \frac{8}{h} - 5 + o(1), \tag{3.5}$$

$$\omega^-(h) = 2 + \frac{5}{4}h + o(h). \tag{3.6}$$

(ii) $\lim\limits_{h\to 0^+}\omega^+(h)=+\infty$, $\omega^+(\frac{4}{9})=12$, $\omega^-(0)=2, \omega^-(\frac{4}{9})=3$.

Lemma 3.2. $\frac{d\omega^+(h)}{dh}<0$, $\frac{d\omega^-(h)}{dh}>0$.

Proof. Assume $\frac{d\omega^\pm(h)}{dh}=0$ at $h=\widetilde{h}$. Differentiating (3.2) with respect to h, we have

$$-(\omega^\pm(\widetilde{h}))^2 - 3\omega^\pm(\widetilde{h}) = 0,$$

which implies $\omega^{\pm}(\widetilde{h}) = 0$ or $\omega^{\pm}(\widetilde{h}) = -3$. However, $G(\widetilde{h}, 0) = -16 \neq 0$, $G(\widetilde{h}, -3) = -40 \neq 0$. This yields that the points $(\widetilde{h}, 0)$ and $(\widetilde{h}, -3)$ are not on the zero isocline $\omega^{\pm}(h)$. Hence, $\frac{d\omega^{\pm}(h)}{dh} \neq 0$. By Lemma 3.1 (ii), we have $\frac{d\omega^{+}(h)}{dh} < 0$, $\frac{d\omega^{-}(h)}{dh} > 0$.

Proposition 3.1. *For* $\omega(h) = \frac{J''_{-1}(h)}{J''_0(h)}$, $h \in [0, \frac{4}{9}]$,

(i) $\omega(0) = 2$, $\omega(\frac{4}{9}) = 3$;

(ii) $\omega'(h) > 0$, $2 \leq \omega(h) \leq 3$.

Proof. It follows from (2.7) that

$$\omega(h) = \frac{J''_{-1}(h)}{J''_0(h)} = 2 + \frac{5}{6}h + o(h), \tag{3.7}$$

which implies $\omega(0) = 2, \omega'(0) = \frac{5}{6}$. Similarly, the expansion (1.9) yields $\omega(\frac{4}{9}) = 3$.

Obviously, the system (3.1) has three critical points in the finite plane: an unstable node at $(\frac{4}{9}, 3)$, two saddles at $(0, 2)$ and $(\frac{4}{9}, 12)$ respectively. It follows from $\omega(0) = 2$ and $\omega(\frac{4}{9}) = 3$ that in the $h\omega$-plane the curve $\omega(h) = \frac{J''_{-1}(h)}{J''_0(h)}$ is the trajectory of (3.1) starting from the unstable node $(\frac{4}{9}, 3)$ to the saddle point $(0, 2)$. In the phase plane of (3.1), the region $\{(h, \omega) | 0 \leq h \leq \frac{4}{9}\}$ is divided into three parts by the zero isoclines $\omega^{+}(h)$, $\omega^{-}(h)$ and the invariant lines $h = 0, h = \frac{4}{9}$. Since $\omega^{-}(h)$ is a monotonically increasing function and $\frac{d\omega^{-}(h)}{dh}|_{h=0} = \frac{5}{4} > \frac{5}{6} = \omega'(0)$ (cf. Lemma 3.1 and (3.7)), we conclude that the trajectory $\omega(h) = \frac{J''_{-1}(h)}{J''_0(h)}$ must stay in the region $\{(h, \omega) | 0 < h < \frac{4}{9}, \omega < \omega^{-}(h) < \omega^{+}(h)\}$, which implies

$$\frac{d}{dh}\Big(\frac{J''_{-1}(h)}{J''_0(h)}\Big) = \omega'(h) = \frac{-h(\omega(h) - \omega^{+}(h))(\omega(h) - \omega^{-}(h))}{h(9h - 4)} > 0$$

for $h \in (0, \frac{4}{9})$ (see Fig.3). Hence, $2 \leq \omega(h) \leq 3$, $h \in [0, \frac{4}{9}]$.

Fig.3 The Trajectory $\omega(h)$ Stays in the Region $\{(h, \omega) | 0 < h < \frac{4}{9}, \omega < \omega^{-}(h)\}$

Corollary 3.1. *If* $J(h) \not\equiv 0$, *then either* $J''(h)$ *has at most one zero* (*counted with its multiplicity*) *or* $J(h)$ *has no zero in the interval* $(0, \frac{4}{9})$.

Proof. Assume $\alpha \neq 0$. It follows from Lemma 2.2 that

$$J''(h) = \alpha J''_0(h)\Big\{\frac{3\beta + \gamma}{3\alpha} + \omega(h)\Big\}.$$

By Lemma 2.5, we know that the number of zeros of $J''(h)$ is equal to the number of zeros of the function $q(h) = \frac{3\beta+\gamma}{3\alpha} + \omega(h)$. Since Proposition 3.1 yields $q'(h) > 0$, we obtain the result.

If $\alpha = \beta + \frac{1}{3}\gamma = 0$, then $J''(h) \equiv 0$, which implies $J(h) = ah + b$, where a and b are constants. Since the expansions (2.7) show that $J(0) = 0$, $J'(0) = J'_0(0)(\alpha + \beta + \gamma)$, we obtain $J(h) = -2\beta J_0'(0)h$. Obviously, $J(h)$ has no zero in the interval $(0, \frac{4}{9})$.

If $\alpha = 0, \beta + \frac{1}{3}\gamma \neq 0$, then $J''(h) = (\beta + \frac{1}{3}\gamma)J_0''(h) \neq 0$. The result follows.

Proposition 3.2. *If $J(h) \not\equiv 0$, then $J(h)$ has at most two zeros in the interval $(0, \frac{4}{9})$ (counted with their multiplicities).*

Proof. To get the result, we are goning to prove the following three assertions hold:

Assertion 1. *$J(h)$ has at most two zeros in the interval $(0, \frac{4}{9})$.*

It follows from Corollary 3.1 that $J(h)$ has at most one inflection point (counted with its multiplicity), which yields this assertion.

Assertion 2. *If $h = h_0$ is the zero of $J(h)$ with multiplicity at least two, then $h = h_0$ is the unique zero of $J(h)$.*

It follows from Lemma 2.4 that $J(0) = 0$. Since $J(0) = J(h_0) = 0$, we conclude that there exists $h^* \in (0, h_0)$ such that $J'(h^*) = J'(h_0) = 0$, which implies $J''(h)$ has at least one zero in the open interval (h^*, h_0) by using mean-value theorem. Assume $J(h)$ has another zero $h = h_1$. Without loss of generality, suppose $h_1 > h_0$. Using the same arguments as above, we conclude that $J''(h)$ has at least one zero in the interval (h_0, h_1). Therefore, $J''(h)$ has at least two zeros in $(0, \frac{4}{9})$, which contradicts Corollary 3.1.

Assertion 3. *If $h = h_0$ is the unique zero of $J(h)$ in $(0, \frac{4}{9})$, then its multiplicity is at most two.*

Indeed, if the multiplicity of $h = h_0$ is great than two, then Corollary 3.1 yields $J(h_0) = J'(h_0) = J''(h_0) = 0$, $J'''(h_0) \neq 0$. Noting $J(0) = J(h_0) = 0$, using the same arguments as in the proof of Assertion 2, we conclude that $J''(h)$ has at least one zero $h = h^* \in (0, h_0)$. Therefore, $J''(h)$ has at least two zeros $h = h_0$ and $h = h^*$, which contradicts Corollary 3.1.

These three assertions yield the result.

By Lemma 1.1, Proposition 3.2 shows that the system $(1.5)_\epsilon$ has at most two limit cycles of type (1).

§4. Proof of the Main Result

In this section, we will prove our main result of this paper by using Lemmas 1.1–1.3. By Lemma 1.3 and Lemma 2.4, we have

Lemma 4.1. *The Abelian integral $J(h)$ has the following expansion near $h = 0$:*

$$J(h) = J'(0)\Big\{(\alpha + \beta + \gamma)h + \frac{1}{12}(6\alpha + 3\beta + \gamma)h^2 + \frac{5}{72}(7\alpha + 3\beta + \gamma)h^3 + \cdots\Big\}. \tag{4.1}$$

(i) *If $\alpha + \beta + \gamma = 0$, $6\alpha + 3\beta + \gamma \neq 0$, then the system $(1.5)_\epsilon$ has at most one limit cycle which tends to the center $(1, 0)$ of $(1.5)_0$.*

(ii) *If $\alpha + \beta + \gamma = 6\alpha + 3\beta + \gamma = 0$, then the system $(1.5)_\epsilon$ has at most two limit cycles which tend to $(1, 0)$.*

(iii) *The condition $\alpha + \beta + \gamma = 6\alpha + 3\beta + \gamma = 7\alpha + 3\beta + \gamma = 0$ holds if and only if $\alpha = \beta = \gamma = 0$.*

Lemma 4.2. *If $J(h) \not\equiv 0$, $\alpha J_{-1}(\frac{4}{9}) + \beta J_0(\frac{4}{9}) + \gamma J_1(\frac{4}{9}) = 3\alpha + \beta + \frac{1}{3}\gamma = 0$, then the system $(1.5)_\epsilon$ has at most two limit cycles in the finite plane.*

Proof. By Lemma 1.2, we know that the system $(1.5)_\epsilon$ has at most two limit cycles which tend to $\Gamma_{4/9}$. In what follows we are going to prove that the system $(1.5)_\epsilon$ has no limit cycle which emerges from Γ_h or tends to $(1, 0)$. We split the proof into three cases.

Case 1. $\alpha \neq 0$, $\alpha+\beta+\gamma \neq 0$.

Without loss of generality, suppose $\alpha = 1$. Since $\beta+\frac{1}{3}\gamma = -3$, it follows from Lemma 2.2, Lemma 2.5 and Proposition 3.1 that $J''(h) = J''_{-1}(h)+(\beta+\frac{1}{3}\gamma)J''_0(h) = J''_0(h)(\omega(h)-3) < 0$, i.e., the curve $J(h)$ has no inflection point in $(0, \frac{4}{9})$ in the hJ plane, which implies that $J'(h)$ has at most one zero in $(0, \frac{4}{9})$. Since $J(0) = J(\frac{4}{9}) = 0$, we conclude that $J(h)$ has no zero in the open interval $(0, \frac{4}{9})$ by using the same arguments as in the proof of Proposition 3.2, which shows that the system $(1.5)_\epsilon$ has no limit cycle which emerges from $\Gamma_h, h \in (0, \frac{4}{9})$.

Since $\alpha+\beta+\gamma \neq 0$, the sytem $(1.5)_\epsilon$ has no limit cycle which tends to $(1, 0)$.

Case 2. $\alpha \neq 0$, $\alpha+\beta+\gamma = 0$.

Without loss of generality, suppose $\alpha = 1$. Since $J''(h) = J''_0(h)(\omega(h) - 3) < 0$, we have $J'(h) < J'(0) = 0$. This yields $J(h) > J(\frac{4}{9}) = 0$, $h \in [0, \frac{4}{9})$, which contradicts the equality $J(0) = 0$. Hence, if $\alpha \neq 0, \alpha J_{-1}(\frac{4}{9}) + \beta J_0(\frac{4}{9}) + \gamma J_1(\frac{4}{9}) = 3\alpha + \beta + \frac{1}{3}\gamma = 0$, then we have $\alpha + \beta + \gamma \neq 0$.

Case 3. $\alpha = 0$.

In this case, it follows froms Lemma 2.2 that $J''(h) \equiv 0$, which yields $J(h) = ah + b$. Since $J(0) = J(\frac{4}{9}) = 0$, we get $a = b = 0$, i.e., $J(h) \equiv 0$. This contradicts the assumption.

The lemma is proved.

Lemma 4.3. *If* $\alpha J_{-1}(\frac{4}{9})+\beta J_0(\frac{4}{9})+\gamma J_1(\frac{4}{9}) = 0$, $3\alpha+\beta+\frac{1}{3}\gamma \neq 0$, *then the system* $(1.5)_\epsilon$ *has at most two limit cycles in the finite plane.*

Proof. By Lemma 1.2, the system $(1.5)_\epsilon$ has at most one limit cycle which tends to $\Gamma_{4/9}$. In what follows we are going to prove that the system $(1.5)_\epsilon$ has at most one limit cycle which tends to the center $(1, 0)$ or the period annulus Γ_h of system $(1.5)_0$, $h \in (0, \frac{4}{9})$. We split the proof into three parts.

Casa 1. $\alpha + \beta + \gamma \neq 0$.

In this case, the system $(1.5)_\epsilon$ has no limit cycle that tends to the center $(1, 0)$ of $(1.5)_0$. It follows from Corollary 3.1 that the curve $J(h)$ in hJ plane has at most one inflection point, which shows $J'(h)$ has at most two zeros, i.e., $J(h)$ has at most one maximal point and one minimal point in $(0, \frac{4}{9})$. Since $J(0) = J(\frac{4}{9}) = 0$, we conclude that $J(h)$ has at most one zero in the open interval $(0, \frac{4}{9})$. Therefore, the system $(1.5)_\epsilon$ has at most one limit cycle which emerges from $\Gamma_h, h \in (0, \frac{4}{9})$.

Case 2. $\alpha + \beta + \gamma = 0$, $6\alpha + 3\beta + \gamma \neq 0$.

In this case, $J(\frac{4}{9}) = J(0) = J'(0) = 0$, the system $(1.5)_\epsilon$ has at most one limit cycle that tends to $(1, 0)$. Since the curve $J(h)$ contacts h-axis at $(0, 0)$ and Corollary 3.1 shows $J(h)$ has at most one inflection point, we conclude that $J(h)$ has no zero in the interval $(0, \frac{4}{9})$, i.e., the system $(1.5)_\epsilon$ has no limit cycle which emerges from $\Gamma_h, h \in (0, \frac{4}{9})$. Therefore, The system $(1.5)_\epsilon$ has at most two limit cycles that tend to $\Gamma_{4/9}$ or $(1, 0)$.

Case 3. $\alpha + \beta + \gamma = 6\alpha + 3\beta + \gamma = 0$, $7\alpha + 3\beta + \gamma \neq 0$.

In this case, if $\alpha = 0$, then $\alpha = \beta = \gamma = 0$, which contradicts the assumption. Asumme $\alpha \neq 0$. Without loss of generality, suppose $\alpha = 1$. It follows from Lemma 2.2, Lemma 2.5 and Proposition 3.1 that $J''(h) = J''_0(h)(\omega(h) - 2) > 0$, which implies $J'(h) > J'(0) = 0$, $h \in (0, \frac{4}{9})$. Hence, $J(h) > J(0) = 0$, $h \in (0, \frac{4}{9})$. This contradicts the equality $J(\frac{4}{9}) = 0$. Therefore, if $\alpha+\beta+\gamma = \alpha J_{-1}(\frac{4}{9})+\beta J_0(\frac{4}{9})+\gamma J_1(\frac{4}{9}) = 0$, then we must have $6\alpha+3\beta+\gamma \neq 0$.

Lemma 4.4. *Assume* $\alpha J_{-1}(\frac{4}{9})+\beta J_0(\frac{4}{9})+\gamma J_1(\frac{4}{9}) \neq 0$. *If one of the following conditions holds, then the system* $(1.5)_\epsilon$ *has at most two limit cycles in the finite plane*:

(i) $\alpha+\beta+\gamma=0$, $6\alpha+3\beta+\gamma\neq 0$.

(ii) $\alpha+\beta+\gamma=6\alpha+3\beta+\gamma=0$, $7\alpha+3\beta+\gamma\neq 0$.

Proof. Since $\alpha J_{-1}(\frac{4}{9})+\beta J_0(\frac{4}{9})+\gamma J_1(\frac{4}{9})\neq 0$, the system $(1.5)_\epsilon$ has no limit cycle which tends to $\Gamma_{4/9}$.

(i) In this case, the curve $J(h)$ contacts h-axis at $h=0$ in the hJ plane. Since the curve $J(h)$ has at most one inflection point (cf. Corollary 3.1), we know that $J(h)$ has at most one zero in $(0,\frac{4}{9})$. Hence, the system $(1.5)_\epsilon$ has at most two limit cycles: one of them emerges from Γ_h and another tends to $(1,0)$.

(ii) In this case, if $\alpha=0$, then $\beta=\gamma=0$, which contradicts the assumption. Without loss of generality, suppose $\alpha=1$. Using the same arguments as in the proof of Lemma 4.3, we have $J''(h)=J_0''(h)(\omega(h)-2)>0$, which yields $J'(h)>J'(0)=0$. Therefore, $J(h)>J(0)=0$, i.e., $J(h)$ has no zeros in $(0,\frac{4}{9})$, which implies the system $(1.5)_\epsilon$ has no limit cycle which tends to $\Gamma_h, h\in(0,\frac{4}{9})$. By Lemma 1.3, the system $(1.5)_\epsilon$ has at most two limit cycles which tend to the center $(1,0)$ of $(1.5)_0$.

Proof of Therrem 1.1. If $\alpha J_{-1}(\frac{4}{9})+\beta J_0(\frac{4}{9})+\gamma J_1(\frac{4}{9})\neq 0$, $\alpha+\beta+\gamma\neq 0$, then the result of this theorem follows from Proposition 3.2 and Lemma 1.1. For other cases, we have proved the result in Lemmas 4.2–4.4.

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