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DYNAMICS FOR VORTICES OF AN EVOLUTIONARY GINZBURG-LANDAU EQUATIONS IN 3 DIMENSIONS**

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Abstract

This paper studies the asymptotic behavior of solutions to an evolutionary Ginzburg-Landau equation in 3 dimensions. It is shown that the motion of the Ginzburg-Landau vortex curves is the flow by its curvature. Away from the vortices, the author uses some measure theoretic arguments used by F. H. Lin in [16] to show the strong convergence of solutions.

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§1. Introduction

We consider the following problem

$$\frac{\partial u_\varepsilon}{\partial t} = \Delta u_\varepsilon + \frac{1}{\varepsilon^2} u_\varepsilon (\beta^2(x) - |u_\varepsilon|^2) \text{ in } Q \times R_+, \tag{1.1}$$

$$u_\varepsilon(x,0) = \beta \cdot u_\varepsilon^0(x), \quad x \in Q, \tag{1.2}$$

$$u_\varepsilon(x,t) = \beta \cdot g(x), \quad x \in \Sigma, \quad t \geq 0, \tag{1.3}$$

$$\frac{\partial u_\varepsilon}{\partial z} = 0 \text{ for } z = 0, l, \tag{1.4}$$

where $Q = \Omega \times [0,l]$, $\Omega \subset R^2$ is a bounded smooth domain, $g : \Sigma = \partial\Omega \times [0,l] \to S^1$ is a $C^{1,\alpha}$-map such that $\deg(g, \partial\Omega_z) = d > 0$ for all $0 \leq z \leq l$. Here $\Omega_z = \Omega \times \{z\}$. $\beta : Q \to R$ is a smooth function (say $C^3(\overline{Q})$) with positive lower bound. $u_\varepsilon : Q \times R_+ \to R^2$.

The aim of this article is to understand the dynamics of vortices, or zeros, of solutions u of (1.1)–(1.4). Its importance to the theory of superconductivity and applications are addressed in many earlier works[9,13,20,21,24]. The following is our main theorem.

Let Γ_0 be a collection of d embedded C^2-curves in Q with $\partial\Gamma_0 \subset \Omega \times \{0,l\}$. Moreover, we assume Γ_0 intersects $\Omega \times \{0,l\}$ orthogonally along $\partial\Gamma_0$. Note that the last assumption is compatiable with the assumption $\frac{\partial u_\varepsilon^0}{\partial z} = 0$ for $z = 0, l$ (that is the natural compatibilty condition for problem (1.1)–(1.4). Similarly, we also assume that $u_\varepsilon^0 = g$ on Σ).

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For the initial data u_ε^0, we make the following assumptions:

(H1) $\int_Q \rho^2(x)\Big[|\nabla u_\varepsilon^0|^2 + \frac{\beta^2}{2\varepsilon^2}(|u_\varepsilon^0|^2-1)\Big]dx \le K$ for all $0<\varepsilon\le 1$. Here $\rho(x)=\mathrm{dist}(x,\Gamma_0)$;

(H2) u_ε^0 converges as $\varepsilon\to 0^+$ in the C^0-norm away from Γ_0 to a map with its image in S^1;

(H3) Let Γ_0^i, $i=1,\cdots,k$, be connected components of Γ_0, and let $\delta>0$ be chosen so that the sets $\Gamma_0^i(\delta)$, $i=1,\cdots,k$, are pairwise disjoint. Here $\Gamma_0^i(\delta)=\{x\in Q:\mathrm{dist}(x,\Gamma_0^i)\le\delta\}$.

Let $T>0$, and $\{\Gamma_t\}$, $0\le t\le T$, be a family of embedded C^2-curves inside Q with boundaries $\{\partial\Gamma_t\}$ contained in $\Omega\times\{0,l\}$, Γ_t intersect $\Omega\times\{0,l\}$ orthogonally along $\partial\Gamma_t$, which are obtained from Γ_0 by the following equations in R^3:

$$\begin{cases} \frac{dx(p,t)}{dt} = \vec{H}(x(p,t),t) - \pi\frac{\nabla\beta^2(x(p,t))}{\beta^2(x(p,t))}, \\ x(p,0)=p\in\Gamma_0, \end{cases} \tag{1.5}$$

where π is the projection onto the normal space of Γ_t, the mean curvature $\vec{H}$ of Γ_t is characterized by the property

$$\int_{\Gamma_t}\mathrm{div}^{\Gamma_t}\phi\, d\mathcal{H}^1 = -\int_{\Gamma_t}\vec{H}\cdot\phi\, d\mathcal{H}^1, \quad \forall\phi\in(\phi_1,\phi_2,\phi_3)\in C^1(R^3,R^3).$$

Here $\mathrm{div}^{\Gamma_t}\phi = d_i^{\Gamma_t}\phi_i$ is the tangencial divergence of ϕ. In case $\beta=1$, the equation (1.5) denotes the flow by mean curvature with codimension 2 in R^3.

Theorem 1.1. *Assume that $\beta\in C^3(\overline{Q})$ and $\beta_0=\min_{\overline{Q}}\beta>0$. Under the assumptions* (H1)–(H3) *and that for each t, $0\le t\le T$, one has (by taking subsequences if necessary) that $u_\varepsilon(x,t)\to u_*(x,t)$ strongly in $H^1_{\mathrm{loc}}(\overline{Q}\backslash\Gamma_t)$. Here $u_*(x,t)$ satisfies*

$$\partial_t u_* - \Delta u_* = \beta^{-2}\Big(|\nabla u_*|^2 - \Delta\Big(\frac{1}{2}\beta^2\Big)\Big)u_* \text{ in } Q\backslash\Gamma_t. \tag{1.6}$$

The system (1.1)–(1.4) can be viewed as a simplified evolutionary Ginzburg-Landau equation in the theory of superconductivity of inhomonogence[24].

Now we briefly describe some mathematical advances concerning this problem. In two space dimensions, $\beta=1$, the dynamical law for vortices was formally derived in [9,19]. The first rigorous mathematical proof of this dynamical law, which is of form $\frac{d}{dt}a(t)=-\nabla w(a(t))$, was given by F. H. Lin in [13,14] (see also [15, Lecture 3]). In [13,14], one allows vortices of degree ±1 and assumes that they have the same sign. For vortices of degree ±1 (possibly of different signs), the same type dynamical law was shown lately[11]. We refer to [16] for vortex dynamics under the Neumann boundary conditions for pinning conditions. In three space dimensions, $\beta=1$, a similar dynamical law was also established in [16] for nearly parallel filaments. The short-time dynamical law for codimension 2 interfaces in higher dimensions was shown in [16]. In two space dimensions, $\beta\not\equiv 1$, the dynamical law was established in [12,18] under the first boundary condition and the Neumann boundary condition respectively.

The rest of the paper is organized as follows. In Section 2, we collect some basic facts on the curve-shortening flow. In Section 3, we prove the weak convergence. In Section 4, we study the strong convergence.

§2. Mean Curvature Flow with Codimension 2

Given a set $E\subset R^3$, we set

$$\eta_E(x) = \frac{1}{2}(\mathrm{dist}(x,E))^2.$$

The following results on the square distance function have been proved in [4]. Let γ be a smooth embeded curve in R^3; then η_γ is smooth in a suitable tubular neighbourhood Ω of γ. $-\Delta\nabla\eta_\gamma$ coincides, on γ, with the curvature vector $\overrightarrow{H}$ of γ.

Lemma 2.1[4, Lemma 3.7] *Let $(\Gamma_t)_{t\in[0,T]}$ be a smooth flow. Then, there exists $\sigma > 0$ such that the function*

$$\eta(x,t) := \frac{1}{2}\mathrm{dist}^2(x,\Gamma_t)$$

is smooth in $\{(x,t) \in R^3 \times [0,T] : \eta \le \sigma\}$. Moreover, the displacement of the flow is given by

$$\frac{dx(p,t)}{dt} = -\nabla\eta_t(x(p,t),t), \quad \forall t \in [0,T], \quad p \in \Gamma_0.$$

In particular, $(\Gamma_t)_{t\in[0,T]}$ is a smooth curvature flow defined by (1.5) *if and only if*

$$\nabla\eta_t = \Delta\nabla\eta - \nabla^2\eta\frac{\nabla\beta^2}{\beta^2}, \quad \text{on } \Gamma_t.$$

Short time existence for curvature flow of smooth initial space curves is a consequence of a general theorem proved in [1,10,26].

Lemma 2.2. *Assume that γ_0 is an embedded C^2-curve in Q with $\partial\gamma_0 \in \Omega\times\{0,l\}$. And, we assume Γ_0 intersects $\Omega\times\{0,l\}$ orthogonally along $\partial\gamma_0$. Then there exist a positive number $t_0 > 0$ and a family of embedded C^2-curves inside Q with $\partial\Gamma_t \subset \Omega\times\{0,l\}$ such that the following system of equalities holds on γ_t :*

$$\frac{\partial\nabla\eta_\gamma}{\partial t}(t,p) - \Delta\nabla\eta_\gamma(t,p) + \nabla^2\eta\frac{\nabla\beta^2}{\beta^2}(p) = 0, \quad t\in[0,t_0], \quad p \in \gamma_t,$$

and γ_t intersects $\Omega\times\{0,l\}$ orthogonally along $\partial\gamma_t$.

§3. Some Estimates and Weak Convergence

Let $v_\varepsilon = \frac{1}{\beta}u_\varepsilon$. Then v_ε satisfies

$$v_{\varepsilon t} = \Delta v_\varepsilon + \frac{\nabla\beta^2}{\beta^2}\cdot\nabla v_\varepsilon + \frac{\Delta\beta}{\beta}v_\varepsilon + \frac{1}{2\varepsilon^2}\beta^2(1-|v_\varepsilon|^2)v_\varepsilon \text{ in } Q\times R_+, \tag{3.1}$$

$$v_\varepsilon(x,0) = u_\varepsilon^0(x), \quad x\in\Omega, \tag{3.2}$$

$$v_\varepsilon(x,0) = g(x), \quad \in\Sigma, \quad t\ge 0, \tag{3.3}$$

$$\beta\frac{\partial v_\varepsilon}{\partial z} + \frac{\partial\beta}{\partial z}v_\varepsilon = 0, \quad \text{for } z = 0,l. \tag{3.4}$$

Lemma 3.1. *Assume* (H1)–(H2). *We have*

$$|u_\varepsilon| \le M_1 \text{ in } \overline{Q}\times[0,T], \tag{3.5}$$

where $M = \max_{\overline{Q}}|\beta(x)|$, M_1 depends only on M, u_ε^0.

Proof. Let $\overline{u} = \frac{u_\varepsilon}{M}$, $\omega = |\overline{u}|^2 - M_1^2$, $M_1 = M(1+K)$, $|u_\varepsilon^0| \le K$. Then

$$\partial_t\omega - \Delta\omega + 2M^2\varepsilon^{-2}|\overline{u}|^2\omega \le 0 \quad \text{in } Q\times(0,T),$$

$$\frac{\partial\omega}{\partial z} = 0 \quad \text{for } z = 0,l,$$

$$\omega \le 0, \quad x\in\Sigma, \quad t\ge 0,$$

$$\omega \le 0, \quad x\in Q, \quad t = 0.$$

Thus, by maximum principle, we have $\omega \le 0$ in $\overline{Q}\times[0,T]$. Hence $|u_\varepsilon(x,t)| \le M_1$ in $\overline{Q}\times[0,T]$.

Lemma 3.2 (Energy Inequality).

$$\sup_{t\ge 0}\Big[\int_0^t\int_Q |v_t|^2 + E(v(\cdot,t))\Big] \le CE(u_0^\varepsilon) + C, \tag{3.6}$$

where C is independent of ε, and

$$E(v(\cdot,t)) = \frac{1}{2}\int_Q \Big[|\nabla v|^2(x,t) + \frac{1}{2\varepsilon^2}\beta^2(1-|v|^2(x,t))^2\Big]dx.$$

Proof. $|v_t|^2 = \frac{1}{\beta^2}\mathrm{div}(\beta^2\nabla v)\cdot v_t + \frac{\Delta\beta}{\beta}vv_t + \frac{\beta^2}{2\varepsilon^2}(1-|v|^2)v\cdot v_t$,

$$\int_Q |v_t|^2 = \int_Q \frac{1}{\beta^2}\mathrm{div}(\beta^2\nabla v)v_t + \frac{\Delta\beta}{2\beta}\frac{\partial}{\partial t}(|v|^2) - \frac{\beta^2}{4\varepsilon^2}\frac{\partial}{\partial t}(1-|v|^2)^2,$$

$$\begin{aligned}
&\int_0^t\int_Q |v_t|^2\\
&= \int_0^t\int_Q \mathrm{div}(\beta^2\nabla v)\Big(\frac{1}{\beta^2}v_t\Big) + \frac{1}{2}\int_0^t\int_Q \frac{\partial}{\partial t}\Big[\frac{\Delta\beta}{\beta}|v|^2 - \frac{\beta^2}{2\varepsilon^2}(1-|v|^2)^2\Big]\\
&= -\int_0^t\int_Q \frac{\partial}{\partial t}|\nabla v|^2 + \beta^2 v_t\cdot\nabla v\cdot\nabla\Big(\frac{1}{\beta^2}\Big) + \int_0^t\Big[\int_{\Omega_0} v_t\Big(\frac{1}{\beta}\frac{\partial\beta}{\partial z}\Big)v + \int_{\Omega_l} v_t\Big(-\frac{1}{\beta}\frac{\partial\beta}{\partial z}\Big)v\Big]\\
&\quad + \frac{1}{2}\int_0^t\int_Q \frac{\partial}{\partial t}\Big[\frac{\Delta\beta}{\beta}|v|^2\Big] - \frac{\partial}{\partial t}[(1-|v|^2)^2],\\
&\int_0^t\int_Q |v_t|^2 + \int_{Q(t)}\Big[|\nabla v|^2 + \frac{\beta^2}{2\varepsilon^2}(1-|v|^2)^2\Big]\\
&\le \int_{Q(0)}\Big[|\nabla u_\varepsilon^0|^2 + \frac{\beta^2}{2\varepsilon^2}(1-|u_\varepsilon^0|^2)^2\Big] + 2\int_{Q(t)} v\nabla v\cdot\frac{\nabla\beta}{\beta}\\
&\quad + 2\int_{Q(0)} v\nabla v\cdot\frac{\nabla\beta}{\beta} + \int_0^t\int_Q \frac{\partial}{\partial t}\Big(|v|^2\cdot\frac{\Delta\beta}{\beta}\Big)\\
&\quad + \frac{1}{2}\int_{\Omega_0}\frac{1}{\beta}\frac{\partial\beta}{\partial z}|v|^2\,\Big|_{t=0}^{t=t} - \frac{1}{2}\int_{\Omega_l}\frac{1}{\beta}\frac{\partial\beta}{\partial z}|v|^2\,\Big|_{t=0}^{t=t},
\end{aligned}$$

where $Q(t) = Q\times\{t\}$.

$$\begin{aligned}
&\int_0^t\int_Q |v_t|^2 + \int_{Q(t)}\Big[|\nabla v|^2 + \frac{1}{2\varepsilon^2}\beta^2(1-|v|^2)^2\Big]\\
&\le \int_{Q(0)}\Big[|\nabla v|^2 + \frac{1}{2\varepsilon^2}\beta^2(1-|v|^2)^2\Big] + 2\int_{Q(t)} v\nabla v\cdot\frac{\nabla\beta}{\beta}\\
&\quad + 2\int_{Q(0)} v\nabla v\cdot\frac{\nabla\beta}{\beta} + \int_{Q(t)}|v|^2\frac{\Delta\beta}{\beta} - \int_{Q(0)}|v|^2\frac{\Delta\beta}{\beta} + C,
\end{aligned}$$

where we have used the fact that $|v|\le C$ in $\overline{Q}\times[0,T]$. Hence

$$\begin{aligned}
\int_0^t\int_Q |v_t|^2 + E(v(\cdot,t)) &\le CE(v(\cdot,0)) + \int_{Q(t)}|v|^2 + C\\
&\le C(E(v(\cdot,0)) + 1).
\end{aligned}$$

Define

$$e_\varepsilon(v)=\frac{1}{2}\Big[|\nabla v|^2+\frac{1}{2\varepsilon^2}\beta^2(|1-|v|^2)^2\Big],$$

$$G_{z_0}(x,t)=[4\pi(t_0-t)]^{-3/2}\exp\Big[-\frac{|x-x_0|^2}{4(t_0-t)}\Big],$$

where $t<t_0$, $z_0=(x_0,t_0)$,

$$G(x,t)=G_0(x,t),$$
$$S_R(z_0)=\{z=(x,t):t=t_0-R^2\},$$
$$P_R(z_0)=\{z=(x,t):|x-x_0|<R,|t-t_0|<R^2\},$$
$$T_R(z_0)=\{z=(x,t):x\in \mathrm{R}^3,t_0-4R^2\le t\le t_0-R^2\},$$
$$T_1=T_1(0),$$
$$\Psi(R)=\int_{T_R(z_0)}e_\varepsilon(v)G_{z_0}\phi^2dxdt,$$
$$\Phi(R)=R^2\int_{S_R(z_0)}e_\varepsilon(v)G_{z_0}\phi^2dxdt,$$

where $\phi\in C_0^\infty(B_{\rho_0}(x_0))$, $0\le\phi\le 1$, $\phi\equiv 1$ for $|x-x_0|\le\frac{\rho_0}{2}$ and $|\nabla\phi|\le\frac{C_0}{\rho_0}$, $x_0\in Q$, $0<\rho_0<\mathrm{dist}(x_0,\partial Q)$.

Lemma 3.3 (Monotonicity Formular).

$$\Phi(R)\le\exp(C(R_0-R))\Phi(R_0)+C(E_0+1)(R_0-R),\tag{3.7}$$
$$\Psi(R)\le\exp(C(R_0-R))\Psi(R_0)+C(E_0+1)(R_0-R),\tag{3.8}$$

where $E_0=E(v(\cdot,0))$.

Proof. See [5, 18].

Lemma 3.4 (Small Energy Regularity Theorem). *There exists a constant* $\theta_0\in(0,1/2)$ *depending only on* β *such that if for some* $0<R<\sqrt[2]{t_0}/2$, $z_0=(x_0,t_0)$, $v=v_\varepsilon$ *satisfies*

$$\Psi(R)=\frac{1}{2}\int_{T_R(z_0)}\Big[|\nabla v|^2+\frac{1}{2\varepsilon^2}\beta^2(1-|v|^2)^2\Big]G_{z_0}dxdt<\theta_0,\tag{3.9}$$

then

$$\sup_{P_{\delta R}(z_0)}\Big\{|\nabla v|^2+\frac{1}{2\varepsilon^2}\beta^2(1-|v|^2)^2\Big\}\le C(\delta R)^{-2}\tag{3.10}$$

with a constant $\delta\in(0,1/2)$ *depending only on* E_0, $\inf\{R,1\}$ *and an absolute constant* C.

Proof. The proof of Lemma 3.4 is identical to [5,Theorem 2.1] (or, see [18]).

Lemma 3.5 (Uniformly Estimate).

$$\int_{Q\setminus\Gamma_t(\delta)}[|v_{\varepsilon t}|^2+e_\varepsilon(v_\varepsilon)]dxdt\le C(\delta,T,\sigma,K),$$

where $\sigma>0$ *is such that the sets* $\Gamma_t^i(4\sigma)$, $i=1,\cdots,k$ *are pairwise disjoint for all* $0\le t\le T$, $0<\delta\le\sigma$.

Proof. Let $\phi_\sigma:R_+\to R_+$ be a smooth monotone function such that

$$\phi_\sigma(r)=\begin{cases}r^2 & \text{if } r\le\sigma\\ 4\sigma^2 & \text{if } r\ge 2\sigma\end{cases}\qquad(\sigma>0).$$

Define $\rho(x,t)=\mathrm{dist}(x,\Gamma_t)$. Assume that

$$\min\{|x-y|:x\in\Gamma_t,y\in\Sigma,0\le t\le T\}\ge 4\sigma.$$

$$\begin{aligned}&\frac{d}{dt}\int_Q\frac{1}{2}\phi_\sigma(\rho(x,t))\beta^2\Big[|\nabla v|^2+\frac{1}{2\varepsilon^2}\beta^2(1-|v|^2)^2\Big]\\ =&\int_Q\frac{1}{2}\phi_t\beta^2\Big[|\nabla v|^2+\frac{1}{2\varepsilon^2}\beta^2(1-|v|^2)^2\Big]\\ &+\int_Q\phi\beta^2\Big[\nabla v\cdot\nabla v_t+\frac{1}{2\varepsilon^2}\beta^2(1-|v|^2)\cdot(-2vv_t)\Big]\\ =:&\ I+II.\end{aligned}$$

$$\begin{aligned}II=&\int_Q\phi\beta^2\Big[-\Delta v-\frac{1}{\varepsilon^2}\beta^2(1-|v|^2)^2v\Big]v_t\\ &-\int_Q\nabla(\phi\beta^2)\nabla v\cdot v_t+\int_{\Omega_l}\phi\beta^2\frac{\partial v}{\partial z}v_t-\int_{\Omega_0}\phi\beta^2\frac{\partial v}{\partial z}v_t\\ =&\int_Q-\phi\beta^2|v_t|^2-\int_Q\beta^2\nabla\phi\nabla v\cdot v_t-\int_Q\phi(\nabla\beta^2\cdot\nabla v)v_t\\ &+\frac{1}{2}\frac{\partial}{\partial t}\Big[\int_{\Omega_l}\phi\beta\frac{\partial\beta}{\partial z}|v|^2-\int_{\Omega_0}\phi\beta\frac{\partial\beta}{\partial z}|v|^2\Big].\end{aligned}$$

Now we calculate the expression $\beta^2\nabla\phi_\sigma\nabla vv_t$. We shall use the summation convection, and simplify notation. We shall also set $\phi_\sigma=\phi, v_\varepsilon=v$.

$$\begin{aligned}\beta^2\nabla\phi\nabla v\cdot v_t=&\ \nabla\phi\nabla v\Big[\mathrm{div}(\beta^2\nabla v)+\beta\Delta\beta v+\frac{1}{2\varepsilon^2}\beta^4(1-|v|^2)v\Big]\\ =&\Big[(\beta^2v_j)_jv_i+\beta^4\frac{1}{\varepsilon^2}v(1-|v|^2)v_i\Big]\phi_i+(\beta\Delta\beta)v(v_i\phi_i)\\ =&\ \phi_i(\beta^2v_iv_j)_j-\phi_i\beta^2v_jv_{ij}+(\beta\Delta\beta)\phi_i\Big(\frac{|v|^2}{2}\Big)_i\\ &-\Big[\beta^4\cdot\frac{1}{4\varepsilon^2}(1-|v|^2)^2\Big]_i\phi_i+(\beta^4)_i\frac{1}{4\varepsilon^2}(1-|v|^2)^2\phi_i\\ =&\ \phi_i(\beta^2v_iv_j)_j-\Big(\beta^2\cdot\frac{|v_j|^2}{2}\Big)_i\phi_i+(\beta^2)_i\phi_i\Big(\frac{|v_j|^2}{2}\Big)\\ &+(\beta\Delta\beta)\phi\Big(\frac{|v|^2}{2}\Big)_i-\Big[\beta^4\cdot\frac{1}{4\varepsilon^2}(1-|v|^2)^2\Big]_i\phi_i+(\beta^4)_i\frac{1}{4\varepsilon^2}(1-|v|^2)^2\phi_i,\end{aligned}$$

where $v_i=\frac{\partial v}{\partial x_i}$, $(\beta)^2_i=\frac{\partial\beta^2}{\partial x_i}$. Hence

$$\begin{aligned}&\int_Q\beta^2\nabla\phi\nabla v\cdot v_t\\ =&\int_Q-\beta^2\phi_{ij}v_iv_j+\int_{\partial Q}\beta^2\phi_iv_iv_j\nu_j+\int_Q\frac{1}{2}\Big[(\Delta\phi)\beta^2|\nabla v|^2+\Delta\phi\cdot\frac{\beta^4}{2\varepsilon^2}\cdot(1-|v|^2)^2\Big]\\ &-\int_{\partial Q}\beta^2\phi_i\nu_i\frac{1}{2}|\nabla v|^2+\int_{\partial Q}\frac{1}{4\varepsilon^2}\beta^4(1-|v|^2)^2\phi_i\nu_i\\ &+\int_Q(\nabla\beta^4\cdot\nabla\phi)\frac{1}{4\varepsilon^2}(1-|v|^2)^2+\nabla\beta^2\cdot\nabla\phi\cdot\frac{1}{2}|\nabla v|^2+\int_Q(\beta\Delta\beta)\nabla\phi\cdot\nabla\Big(\frac{|v|^2}{2}\Big)\end{aligned}$$

$$= \int_Q -\beta^2\phi_{ij}v_iv_j + \int_Q \beta^2\Delta\phi[|\nabla v|^2 + \frac{1}{2\varepsilon^2}\beta^2(1-|v|^2)^2]$$
$$+ \int_Q \frac{\nabla\beta^2}{\beta^2}\nabla\phi\Big[\frac{1}{2}\beta^2|\nabla v|^2 + \frac{1}{4\varepsilon^2}\beta^4(1-|v|^2)^2\Big] + \int_Q (\beta\Delta\beta)\nabla\phi\cdot\nabla\Big(\frac{|v|^2}{2}\Big).$$

Let $e_\varepsilon(v) = \frac{1}{2}\beta^2[|\nabla v|^2 + \frac{1}{2\varepsilon^2}\beta^2(1-|v|^2)^2]$. Then

$$\frac{d}{dt}\int_Q \phi e_\varepsilon(v)$$
$$= \int_Q \phi_t e_\varepsilon(v) - \int_Q \phi\beta^2|v_t|^2 - \int_Q (\Delta\phi)e_\varepsilon(v) - \int_Q \Big(\frac{\nabla\beta^2}{\beta^2}\cdot\nabla\phi\Big)e_\varepsilon(v)$$
$$+ \int_Q \beta^2\phi_{ij}v_iv_j - \int_Q (\beta\Delta\beta)\nabla\phi\cdot\nabla\Big(\frac{|v|^2}{2}\Big) - \int_Q \phi(\nabla\beta^2\cdot\nabla v)v_t$$
$$= \int_Q \Big(\phi_t - \Delta\phi - \frac{\nabla\beta^2}{\beta^2}\cdot\nabla\phi\Big)e_\varepsilon(v) + \int_Q \beta^2\phi_{ij}v_iv_j - \int_Q \phi\beta^2|v_t|^2$$
$$- \int_Q 2\phi\beta\nabla\beta\nabla vv_t - \int_Q (\beta\Delta\beta)v\nabla v\cdot\nabla\phi + \frac{1}{2}\frac{\partial}{\partial t}\Big[\int_{\Omega_l} \phi\beta\frac{\partial\beta}{\partial z}|v|^2 - \int_{\Omega_0} \phi\beta\frac{\partial\beta}{\partial z}|v|^2\Big]. \tag{3.11}$$

Next we observe that on the set $\{x \in Q : \rho(x,t) < \sigma\}$, $(\phi_{ij}) \le I$ in the sence that $\phi_{ij}\xi_i\xi_j \le |\xi|^2$ for all $\xi \in R^3$. Also on Γ_t, we have $\phi_t = 0$, $\Delta\phi = 2$. Since Γ_t is obtained from Γ_0 by curvature flow (1.5), by Lemma 2.1, we have $\nabla\Big(\phi_t - \Delta\phi - \frac{\nabla\beta^2}{\beta^2}\cdot\nabla\phi\Big) = 0$ on Γ_t. Thus

$$\phi_t - \Delta\phi - \frac{\nabla\beta^2}{\beta^2}\cdot\nabla\phi \le -2 + C_0\rho^2(x,t) = -2 + C_1\phi. \tag{3.12}$$

Combining (3.19) and (3.20) with the fact that $(\phi_{ij}) \le I$, we have

$$\frac{d}{dt}\int_Q \frac{1}{2}\phi e(v) \le C(\sigma)\int_Q \frac{1}{2}\phi e(v) + C(\sigma) + \frac{1}{2}\frac{\partial}{\partial t}\Big[\int_{\Omega_l} \phi\beta\frac{\partial\beta}{\partial z}|v|^2 - \int_{\Omega_0} \phi\beta\frac{\partial\beta}{\partial z}|v|^2\Big]. \tag{3.13}$$

So

$$\int_Q \frac{1}{2}\phi e(v)\Big|_{t=t} \le C(\sigma)\int_0^t\int_Q \frac{1}{2}\phi e(v) + \int_Q \frac{1}{2}\phi e(v)\Big|_{t=0} + C(\sigma)t + C.$$

Now we use the assmption (H1) to obtain

$$\int_Q \phi_\sigma(\rho(x,t))e_\varepsilon(v_\varepsilon)dx$$
$$\le e^{C(\sigma)t}\int_Q \phi_\sigma(\rho(x,0))e_\varepsilon(u_\varepsilon^0) + C(\sigma)te^{C(\sigma)t}$$
$$\le e^{C(\sigma)T}K + Te^{C(\sigma)T}\cdot C(\sigma), \quad 0 \le t \le T. \tag{3.14}$$

The last inequality implies that

$$\int_{Q\setminus\Gamma_t(\delta)} e_\varepsilon(v_\varepsilon)(x,t)dx \le C(\delta,\sigma,T,K) \tag{3.15}$$

for all $0 \le t \le T$ and $0 < \varepsilon \ll 1$.

Next, for $0 \le t_1 \le t \le t_2 \le T$, let $\eta(x)$ be a smooth cutoff function supported in

$Q\setminus \bigcup_{t_1\le t\le t_2} \Gamma_t$. Then

$$\begin{aligned}
&\frac{d}{dt}\int_Q \eta^2(x)e_\varepsilon(v_\varepsilon)(x,t)dx\\
&=\int_Q \eta^2(x)\Big[\nabla v\cdot\nabla v_t+\frac{1}{2\varepsilon^2}\beta^2(1-|v|^2)(-2v\cdot v_t)\Big]\\
&=-\int_Q \eta^2(x)\Big[\Delta v\cdot v_t+\frac{1}{\varepsilon^2}\beta^2(1-|v|^2)v\cdot v_t\Big]\\
&\quad -2\int_Q \eta\nabla\eta\nabla v\cdot v_t+\int_{\Omega_0}\eta^2\frac{\partial v}{\partial z}\cdot v_t-\int_{\Omega_l}\eta^2\frac{\partial v}{\partial z}v_t\\
&=-\int_Q \eta^2(x)|v_t|^2-2\int_Q \eta\nabla\eta v\cdot v_t+\int_{\Omega_0}\eta^2\frac{\partial v}{\partial z}v_t-\int_{\Omega_l}\eta^2\frac{\partial v}{\partial z}v_t\\
&\le -\frac{1}{2}\int_Q \eta^2(x)|v_t|^2+4\int_Q |\nabla\eta|^2|\nabla v|^2dx\\
&\quad +\frac{1}{2}\frac{\partial}{\partial t}\Big[\int_{\Omega_0}\eta^2\frac{1}{\beta}\frac{\partial\beta}{\partial z}|v|^2-\int_{\Omega_l}\eta^2\frac{1}{\beta}\frac{\partial\beta}{\partial z}|v|^2\Big].
\end{aligned}\tag{3.16}$$

From Lemma 3.1, (3.23) and (3.24), we obtain that

$$v_\varepsilon\in H^1_{\rm loc}(\overline{Q}\times[0,T]\setminus \bigcup_{0\le t\le T}\Gamma_t).$$

Here, by taking a subsequence if necessary, we obtain that $v_\varepsilon(x,t)\to v_*(x,t)$ weakly in $H^1_{\rm loc}(\overline{Q}\times[0,T]\setminus\bigcup_{0\le t\le T}\Gamma_t)$. It is easy to verify that v_* satisfies

$$\frac{\partial v}{\partial t}=\Delta v+\frac{2\nabla\beta}{\beta}\nabla v+\frac{\Delta\beta}{\beta}v+\frac{1}{\beta^2}(\beta\Delta\beta-\beta^2|\nabla v|^2)v \tag{3.17}$$

and $v_*\in S^1$.

Note that $u_\varepsilon=\beta v_\varepsilon$. We also obtain that $u_\varepsilon\to u_*$ weakly in $H^1_{\rm loc}(\overline{Q}\times[0,T]\setminus\bigcup_{o\le t\le T}\Gamma_t)$. It is easy to verify u_* satisfies (1.6).

§4. Strong Convergence

The aim of this section is to prove strong convergence. The proof of this conclusion is based on the fact that the solutions v_ε to (3.1) satisfy a monotonicity inequality, from which the ε-regularity can be proved. Then, it implies the strong convergence of the sequence of $\{v_\varepsilon\}$.

Theorem 4.1. *Let $v_\varepsilon(x,t)$ be a solution of*

$$\frac{\partial v_\varepsilon}{\partial t}=\Delta v_\varepsilon+\frac{2}{\beta}\nabla\beta\nabla v_\varepsilon+\frac{\Delta\beta}{\beta}v_\varepsilon+\frac{1}{\varepsilon^2}\beta^2(1-|v_\varepsilon|^2)v_\varepsilon \ \textit{in}\ Q_1$$

with

$$\int_0^1\int_{B_1^3}e_\varepsilon(v_\varepsilon)dxdt+\int_0^1\int_{B_1^3}\Big|\frac{\partial v_\varepsilon}{\partial t}\Big|^2dxdt\le M$$

for $0<\varepsilon\ll 1$. Here $v_\varepsilon\in C$, $Q_1=B_1^3\times[0,1]$, $B_1^3=\{x\in R^3:|x|<1\}$.

Suppose also that $\{v_\varepsilon\}$ converges weakly to a map v_* as $\varepsilon\to 0^+$ in $H^1(Q_1)$. Then $\{v_\varepsilon\}$ converges strongly to v_* in $H^1_{\rm loc}(Q_1)$, and v_* satisfies (3.17).

Lemma 4.1. *Suppose v satisfies*

$$\frac{\partial v}{\partial t} = \Delta v + \frac{2}{\beta}\nabla\beta\nabla v + \frac{\Delta\beta}{\beta}v + \frac{1}{\beta^2}(\beta\Delta\beta - \beta^2|\nabla v|^2)v \ in\ Q_1$$

with

$$\int_{Q_1}\Big[|\nabla v|^2 + |\frac{\partial v}{\partial t}|^2\Big]dxdt \le C.$$

Then

$$v \in C_{\rm loc}^{2+\alpha, 1+\frac{\alpha}{2}}(Q_1).$$

Proof. Let $v = e^{i\psi}$. Then $\psi \in H^1(Q_1)$ and

$$\int_{Q_1}|\nabla\psi|^2 dxdt \le C,$$

$$\psi_t = \Delta\psi + \frac{2}{\beta}\nabla\beta\cdot\nabla\psi \text{ in } Q_1.$$

One has

$$\psi \in W_2^{2,1}(Q_1).$$

So, by regularity theory and a bootstrap argument, we have

$$\psi \in C_{\rm loc}^{2+\alpha, 1+\frac{\alpha}{2}}(Q_1).$$

Hence

$$v \in C_{\rm loc}^{2+\alpha, 1+\frac{\alpha}{2}}(Q_1).$$

Proof of Theorem 4.1. The proof is identical to [7, Theorem C]. For the sake of completeness, we sketch it here.

Let P^m denote the m-dimensional Hausdorff measure in R^{n+1} with respect to the parabolic metric $\delta((x,t),(y,s)) = \max\{|x-y|, \sqrt{|t-s|}\}$, and H^{m-2} denote the $m-2$ dimensional Hausdorff measure in R^m with respect to the standard metric.

Now assume $v_\varepsilon \to v_*$ weakly in $H^1(B_1\times(0,1))$. Then $e(v_\varepsilon)dxdt \to \mu = \frac{1}{2}|\nabla v_*|^2 + \nu$ as Radon measure for some Radon measure $\nu \ge 0$. Moreover we define

$$\Sigma = \bigcap_{R>0}\Big\{z \in B_1\times(0,1) : \liminf_{r\to 0}\int_{T_R(z)}\eta^2 e(v_\varepsilon)(x,t)G_z(x,t)dxdt \ge \theta_0\Big\},$$

where θ_0 is as in Lemma 3.4. Then the monotonicity formula Lemma 3.3 implies Σ is closed and $P(\Sigma\cap P_R) < \infty$ for any $R < 1$. Lemma 3.4 implies that $v_\varepsilon \to v_*$ in $C^1(B_1\times(0,1)\backslash\Sigma)\cap H^1(B_1\times(0,1)\backslash\Sigma)$ locally (if needed, passing to subsequences). Note that v_* is smooth, by Lemma 4.1.

Claim 1. ${\rm spt}(\nu) = \Sigma$.

Proof. In fact, if $z_0 \in {\rm spt}(\nu)$, then Lemma 3.4 implies that for all $r > 0$,

$$\liminf_{i\to\infty}\rho^{-3}\int_{P_\rho(z_0)} e_{\varepsilon_i}(v_{\varepsilon_i}) \ge \theta_0.$$

Thus $z_0 \in \Sigma$. On the other hand, if $z_0 \notin {\rm spt}(\nu)$, then there is a $\rho > 0$ such that $P_\rho(z_0)\cap{\rm spt}(\nu) = \emptyset$ and $e_\varepsilon(v_\varepsilon)dxdt \to \frac{1}{2}|\nabla v_*|^2 dxdt$ as Radon measure. Because v_* is smooth, we derive for all sufficiently small r, $0 < r < \rho$,

$$\rho^{-3}\int_{P_\rho(z_0)}\frac{1}{2}|\nabla v_*|^2 \le \frac{\theta_0}{2}.$$

Hence

$$\rho^{-3}\int_{P_\rho(z_0)} e_\varepsilon(v_\varepsilon) < \theta_0$$

for sufficiently small ε. Thus $z_0 \notin \Sigma$.

For the measures μ and ν above, we define two density functions

$$\theta^3(\mu, z) = \lim_{R\to 0}\int_{T_R(z)} \eta^2 G_z d\mu,$$

$$\theta^3(\nu, z) = \lim_{R\to 0}\int_{T_R(z)} \eta^2 G_z d\nu$$

for $z \in B_1 \times (0,1)$, if both of the limits exist. Then, from Claim 1, we have

Claim 2. (a) $\theta^3(\mu, z)$ exists for any $z \in \Sigma$ and is upper-semicontinous. (b) $\theta_0 \le \theta^3(\mu, z) \le C(K, r)$ for any $z \in \Sigma \cap P_r$. (c) For P^3 a.e. $z \in \Sigma$, $\theta^3(\nu, z)$ exists and $\theta^3(\nu, z) = \theta^3(\mu, z)$.

Now assume that $e_\varepsilon(v_\varepsilon)dxdt \not\rightharpoonup \frac{1}{2}|\nabla v_*|^2 dxdt$. Then one must have

$$P^3(\Sigma) > 0, \text{ and } \nu(B_1 \times (0,1)) > 0.$$

Moreover, Claim 2 shows that there exists $\widetilde{\Sigma} \subset \Sigma$ with $P^3(\widetilde{\Sigma}) = P^3(\Sigma) > 0$ such that $\theta^3(\mu, z) = \theta^3(\nu, z)$ is approximately continuous for $z \in \widetilde{\Sigma}$. Now, we can choose a $z_0 = (x_0, t_0) \in \widetilde{\Sigma}$ such that (i) $\limsup\limits_{r\to 0} r^{-3}P^3(\Sigma \cap P_r(z_0)) > 0$; (ii) $\theta^3(\mu, z)$ is approximately continuous at z_0; and (iii) $\lim\limits_{r\to 0} r^{-3}\int_{P_r(z_0)} |\nabla v_*|^2 = 0$.

For $r_i \downarrow 0$, define the parabolic dilation D_{r_i} by

$$D_{r_i}(A) = \{(x,t) \in R^4 : (x,t) = (r_i y, r_i^2 s) \text{ for some } (y,s) \in A\},$$

and the rescaling measures $\mu_i(A) = r_i^{-3}\mu(z_0 + D_{r_i}(A))$ for any $A \subset B_1 \times (0,1)$. Then we have $\theta_0 \le \mu_i(B_1 \times (0,1)) \le C(K)$. Hence we can assume that $\mu_i \to \mu_*$ for some Radon measure $\mu_* \ge 0$. By the diagonal process, one can extract subsequence $\varepsilon_i \downarrow 0$,

$$e_{\varepsilon_i}(v_{\varepsilon_i})dxdt \to \mu_*, v_{\varepsilon_i} \to \text{ a constant weakly in } H^1(B_1 \times (0,1)),$$

where $\varepsilon_i = r_i^{-1}\varepsilon$.

Note that Σ_*, the support of μ_*, is given by $\Sigma_* = \bigcup\limits_{t\in(-1,1)} \Sigma_*^t$ and

$$\Sigma_*^t = \bigcap_{R>0}\Big\{x \in B_1 : \liminf_{\varepsilon_i\to 0}\int_{T_R(x,t)} \eta^2 e_{\varepsilon_i}(v_{\varepsilon_i})G_{(x,t)} \ge \theta_0\Big\}.$$

So $(0,0) \in \Sigma_*$, $P^3(\Sigma_*) > 0$, $\mu_*(B_1 \times (-1,1)) \ge \theta_0$.

Claim 3. There exists $t_0 > 0$ such that $\Sigma_*^t \not\equiv \phi$ for any $t \in (-t_0, 0]$.

Proof. Soppose not, for $t_0 > 0$, $\Sigma_*^{t_0} = \phi$. Then for any $x_0 \in B_1$, there exists $r_0 > 0$ such that

$$\liminf_{\varepsilon_i\to 0}\int_{T_{r_0}((x_0,t_0))} \eta^2 e_{\varepsilon_i}(v_{\varepsilon_i})G_{(x_0,t_0)} < \theta_0.$$

So Lemma 3.4 yields

$$\sup_{P_{\delta_{r_0}}(x_0,t_0)} e_{\varepsilon_i}(v_{\varepsilon_i}) \le C(\delta r_0)^{-2}$$

for some $C > 0$ and $\delta > 0$. This implies that, for some $\overline{r} > 0$, $v_{\varepsilon_i} \to$ a constant in $C^2(B_{\frac{1}{2}} \times (t_0 - \overline{r}, t_0 + \overline{r}))$, and $\nu(B_{\frac{1}{2}} \times (t_0 - \overline{r}, t_0 + \overline{r})) = 0$, which implies $(0,0) \notin \Sigma_*$ if we choose t_0 sufficently small. This leads to a contradiction.

From Claim 3, one can see $e_\varepsilon(v_\varepsilon)(x,t)dx \not\to 0$, for $t \in (-t_0, 0)$. On the other hand, there exist nonnegative Radon measures ν_t for $t \in (-t_0, 0)$ such that $e(v_{\varepsilon_i})(x,t) \to \nu_t$. Hence $\nu_t(B_1) > 0$ for $t \in (-t_0, 0)$. It is easy to see that $\mathrm{spt}\nu_t \subset \Sigma_*^t$ for $t \in (-t_0, 0)$. In fact (cf. [17, Claim 6]), one has

Claim 4. If $\nu_0(B_1) > 0$, then $H^1(\mathrm{spt}(\nu_t)) > 0$.

Claim 4 gives $H^1(\Sigma_*^t) > 0$ for any $t \in (-t_0, 0)$. Hence, we can pick another point $(x_1, t_1) \in \Sigma_*^{t_1}$ such that $H^1(\Sigma_*^{t_1}) > 0$ and $\overline{\theta}^1(\Sigma_*^{t_1}, x_1) = \limsup_{r\to 0} r^{-1} H^1(\Sigma_*^{t_1} \cap B_r(x_1)) > 0$. By applying the following Lemma 6 in [16] to $\Sigma_*^{t_1}$ at x_1 we conclude that for $r_j \downarrow 0$ there exist $\{x_1^j\} \subset \Sigma_*^{t_1}$ such that

$$|x_1^j - x_0^j| \geq \delta r_j$$

and

$$\mathrm{dist}(x_1^j - x_j^0, \mathrm{span}\{x_1^j - x_0^j\}) \geq \delta r_j,$$

where $x_0^j = x_1$. Let $\mu_{*,j}(A) = r_j^{-3}\mu_*((x_1, t_1) + D_{r_j}(A))$ for each j and define $v_{\varepsilon_{ij}}(x,t) = v_{\varepsilon_i}((x_1 + r_j x, t_1 + r_j^2 t))$. Then, by the diagonal process again, one can find a subsquence of ε_{ij} (denoted as ε_j) such that, as $\varepsilon_j \downarrow 0$,

$$\mu_{*,j} \to \mu_{**}, \quad e(v_{\varepsilon_j})dxdt \to \mu_{**}.$$

Moreover, if we denote $\Sigma_{**} = \mathrm{spt}\mu_{**}$ and $\Sigma_{**}^t = \Sigma_{**} \cap \{t\}$, then

$$\mathrm{span}\{\zeta_1\} \subset \Sigma_{**}^0,$$

where $\zeta_1 = \lim_{j\to\infty} (x_1^j - x_0^j)/r_j$. Note that $\{\zeta_1\}$ spans a 1-dimensional linear subspace of R^3. One also has $P^3(\Sigma_{**}) > 0, v_{\varepsilon_j} \to$ a constant weakly in H^1 and $\theta^1(\mu_{**}, z)$ is a constant for $z \in \Sigma_{**}$.

Applying the monotonicity formula Lemma 3.3 at centers $(0,0), (\zeta_1, 0)$ and using the fact that $\theta^1(\mu_{**}, z)$ is constant for $z \in \Sigma_{**}$, we have for any $r > 0$,

$$\int_0^1 RdR \int_{T_1} |t|^{-1}\eta^2 |v_{j,R}^k|^2 G_{(\zeta_k,0)} + \eta^2 \frac{(\beta_R^k)^2}{\varepsilon_j^2}(1 - |v_j^k|^2)^2 G_{(\zeta_k,0)} dxdt \to 0, \quad \text{as } j \to \infty, \tag{4.1}$$

for $0 \leq k \leq 1$. Here $\zeta_0 = (0,0)$ and $v_{j,R}^k = \frac{d}{dR} v_{\varepsilon_j}((\zeta_k, 0) + (Rx, R^2 t)), \beta_R^k(x,t) = \beta((\zeta_k, 0) + (Rx, R^2 t))$. Hence, by Fatou's Lemma, one has, for $0 \leq k \leq 1$,

$$\lim_{\varepsilon_j \to 0} \int_{T_1} \eta^2 |v_{j,R}^k|^2 G_{(\zeta_k,0)} dxdt + \eta^2 \frac{(\beta_R^k)^2}{2\varepsilon_j^2}(1 - |v_{j,R}^k|^2)^2 G_{(\zeta_k,0)} dxdt = 0, \quad \forall R \in (0,1). \tag{4.2}$$

Let $\{0\} \times R^1$ be the $\mathrm{span}\{\zeta_1\} = \{0, 0, y_3) \in R^3\}$. Then (4.2) implies

$$\lim_{\varepsilon_j \downarrow 0} \int_{-t_1^2}^{-t_0^2} \int_{R^3} \eta^2 \Big[|\nabla_T v_{\varepsilon_j}|^2 + \frac{1}{\varepsilon_j^2} (\beta_R^k)^2 (1 - |v_{j,R}^k|^2)^2 \Big] dxdt = 0, \tag{4.3}$$

for any $0 < t_0 < t_1 < \infty$. Here $T \in \{0\} \times R^1$, the $\mathrm{span}\{\zeta_1\} = \mathrm{span}\left\{\frac{\partial}{\partial y_3}\right\}$.

Claim 5.

$$\mu_{**}(x,y,t) = \theta^3(\mu_{**}, (x,y,t))(H^1 L\{0\} \times R^1) \times P^2 LS).$$

Here $S = \bigcup_{j=1}^{l} \{(x,t) \in R^2 \times R_- : x = c_j\sqrt{-t}\}$ for some $1 \leq l < \infty$ and $c_j \in R^2 \times \{0\}$. Moreover, if $(x,y,t) \in (\{0\} \times R^1) \times S$, then $\theta^3(\mu_{**}(x,y,t)) = \theta^3(\mu, (x_0, t_0))$.

Proof. We note that $P^3(\Sigma_{**}\cap P_R)<\infty$ for any $R>0$. Note also that

$$\int_{P_R(z)}|\partial_t v_\varepsilon|^2dxdt\le CR^{-2}\int_{P_2R(z)}e(v_\varepsilon)dxdt \tag{4.4}$$

for any $P_R(z)\subset B_1\times(0,1)$. Passing (4.4) to the limit, we see that

$$(D_r)_\#(\mu_{**})=\mu_{**},\quad \forall r>0;$$

therefore $\Sigma_{**}=D_r(\Sigma_{**})$ and we can write $\Sigma_{**}=\{(c\sqrt{-t},t):c\in\Sigma_{**},t\in R_-\}$. Now we need to show that $\Sigma_{**}^{-1}=\{0\}\times R^1\times S$ with S as in the claim. Let $\phi\in C_0^\infty(R^2)$ and for $k=3$, $0<t_0<t_1<\infty$, we compute

$$\begin{aligned}
&\frac{\partial}{\partial y_3}\int_{-t_1^2}^{-t_0^2}\int_{R^2}\phi^2(x)e(v_{\varepsilon_j})(x,y,t)dxdt\\
&=\int_{-t_1^2}^{-t_0^2}\int_{R^2}\phi^2\Big[\frac{\partial v_{\varepsilon_j}}{\partial x}\frac{\partial}{\partial x}\Big(\frac{\partial v_{\varepsilon_j}}{\partial y_3}\Big)+\frac{\partial v_{\varepsilon_j}}{\partial y_3}\cdot\frac{\partial}{\partial y_3}\Big(\frac{\partial v_{\varepsilon_j}}{\partial y_3}\Big)\\
&\quad-\frac{1}{\varepsilon_j^2}(\beta_R^k)^2(x,y_3)(1-|v_{\varepsilon_j}|^2)v_{\varepsilon_j}\cdot\frac{\partial v_{\varepsilon_j}}{\partial\partial y_3}+\frac{1}{\varepsilon_j^2}(1-|v_{\varepsilon_j}|^2)^2\cdot2(\beta_R^k)\cdot\frac{2(\beta_R^k)}{\partial y_3}\Big]dxdt\\
&=-\int_{-t_1^2}^{-t_0^2}\int_{R^2}\frac{\partial\phi^2}{\partial x}\frac{\partial v_{\varepsilon_j}}{\partial x}\cdot\frac{\partial v_{\varepsilon_j}}{\partial y_3}+\frac{\partial}{\partial\partial y_3}\int_{-t_1^2}^{-t_0^2}\int_{R^2}\phi^2\Big|\frac{\partial v_{\varepsilon_j}}{\partial y_3}\Big|^2\\
&\quad+\int_{-t_1^2}^{-t_0^2}\int_{R^2}\phi^2(-\Delta v_{\varepsilon_j}-\frac{1}{\varepsilon_j^2}(\beta_R^k)^2\cdot(1-|v_{\varepsilon_j}|^2)v_{\varepsilon_j})\frac{\partial v_{\varepsilon_j}}{\partial y_3}+\frac{2}{\varepsilon_j^2}(1-|v_{\varepsilon_j}|^2)^2\beta_R^k\cdot\frac{\partial\beta_R^k}{\partial y_3}\\
&=-\int_{-t_1^2}^{-t_0^2}\int_{R^2}\Big(\frac{\partial\phi^2}{\partial x}\frac{\partial v_{\varepsilon_j}}{\partial\partial x}+\phi^2\Big(\frac{\partial v_{\varepsilon_j}}{\partial t}-2\frac{\nabla\beta_R^k}{\beta_R^2}\nabla v_{\varepsilon_j}-\frac{\Delta\beta_R^k}{\beta_R^k}v_{\varepsilon_j}\Big)\Big)\frac{\partial v_{\varepsilon_j}}{\partial y_3}\\
&\quad+\frac{\partial}{\partial y_3}\int_{-t_1^2}^{-t_0^2}\int_{R^2}\phi^2|\frac{\partial v_{\varepsilon_j}}{\partial y_3}|^2+\int_{-t_1^2}^{-t_0^2}\int_{R^2}\frac{2}{\varepsilon_j^2}\beta_R^k\frac{\partial\beta_R^k}{\partial y_3}(1-|v_{\varepsilon_j}|^2)^2.
\end{aligned}\tag{4.5}$$

Combining (4.5) with (4.3), we have

$$\frac{\partial}{\partial y_3}\int_{-t_1^2}^{-t_0^2}\int_{R^2}\phi^2(x)d\mu_{**}(x,y_3,t)=0, \tag{4.6}$$

in the sense of distribution for all $y_3\in\{0\}\times R^1$. Thus $\mu_{**}(x,y_3,t)=\nu_{**}(x,t)dy$. Hence if we denote $\overline{\Sigma}_{**}\subset R^2\times\{0\}\times R_-$ as sptν_{**}, then $\Sigma_{**}=\overline{\Sigma}_{**}\times(\{0\}\times R^1)$, $\overline{\Sigma}_{**}=\bigcup_{i=j}^{l}\{(c_j\sqrt{-t},t):t\in R_-\}$ and $c_i\in R^2\times\{0\}$ for some $1\le l\le\infty$. The proof of the claim is completed.

From Claim 5, we may assume that v_{ε_i} converges strongly to a constant in $H^1(R^3\times R_-\backslash(\{0\}\times R^1)\times S)$ locally.

Without loss of generality, we will assume $l=1$ and denote $c_1=c$. From (4.5), by applying the weak L^1-estimates of the local Hardy-Littlewood maximal function with respect to the parabolic distance in R^4 (cf. [25]) we conclude that there exists $A_j\subset(\{0\}\times R^1)\times S$ with $P^3(A_j)>0$ such that for any $(c\sqrt{-t_j},y_j,t_j)\in A_j$,

$$\sup_{r\in(0,\frac14)}r^{-3}\int_{P_r^1(y_j,t_j)}f_j\to0,\ \text{as } j\to\infty, \tag{4.7}$$

where $f_j=\int_{B_1^2(c\sqrt{-t_j})}\Big|\frac{\partial v_\varepsilon}{\partial y_3}\Big|^2dx$. Now, pick up $(y_j,t_j)\in A_j\cap\{0\}\times R^1\times S$ such that $|y_j|\le\frac12$ and $-\frac{t_1^2}{2}\le t_j\le-\frac{t_0^2}{2}$ for some $0<t_0<t_1$.

Let $\delta_j \downarrow 0$ and $x_j \in B^2_{\frac{1}{4}}(c\sqrt{-t_j})$ such that

$$\delta_j^{-2}\int_{B^2_{\delta_j}(x_j)\times(t_j-\delta_j^2,t_j)} e(v_{\varepsilon_j})(x,y_j,t)dxdt = \frac{\theta_0}{C(m)}$$
$$= \max\Big\{\delta_j^{-2}\int_{B^2_{\delta_j}(z)\times(t_j-\delta_j^2,t_j)} e(v_{\varepsilon_j})(\cdot,y_j,\cdot) : z \in B^2_{\frac{1}{2}}(c\sqrt{-t_j})\Big\}. \tag{4.8}$$

Define $v_j(x,y,t) = v_{\varepsilon_j}((x_j,y_j,t_j) + (\delta_j(x,y),\delta_j^2 t))$, on $\Omega_j = \delta_j^{-1}((B^2_{\frac{1}{2}}(c\sqrt{-t_j})) \times B^1_2) \times (-\delta_j^{-2}(-2t_1^2+t_0^2),0)$. Then v_j satisfies

$$\partial_t v_j - \Delta v_j - \frac{\nabla\beta_j^2}{\beta_j^2}\nabla v_j - \frac{\Delta\beta_j}{\beta_j}v_j + \frac{1}{\varepsilon_j^2}\beta_j^2(1-|v_j|^2)v_j = 0 \text{ in } \Omega_j, \tag{4.9}$$

where $\beta_j(x,y) = \beta((x_j,y_j)+\delta_j(x,y))$.

$$\int_{B^2_1\times(-1,0)} e(v_j)(x,0,t)dxdt = \frac{\theta_0}{C(m)}$$
$$= \max\Big\{\int_{B^2_1(z)\times(-1,0)} e(v_j)(x,0,t)dxdt : z \in \delta_j^{-1}(B^2_{\frac{1}{2}}(c\sqrt{-t_j}))\Big\}. \tag{4.10}$$

$$\sup_{r\in(0,(4\delta_j)^{-1})} r^{-3}\int_{P_r(0)}\int_{B^2_{\frac{1}{2\delta_j}}(0)} \Big|\frac{\partial v_j}{\partial y_3}\Big|^2 \to 0. \tag{4.11}$$

Claim 6. For any $z \in \delta_j^{-1}\Big(B^2_{\frac{1}{2}}(c\sqrt{-t_j})\Big)$ and $t \in (-\infty,0]$, we have

$$\int_{(B^2_2(z)\times B^1_2(0))\times(t-1,t)} e(v_j)dxdydt \le \frac{4\theta_0}{C(m)}. \tag{4.12}$$

Proof. For the proof of Claim 6, see [16].

Therefore, by choosing sufficiently large $C(m)$, one has

$$2^{-3}\int_{B^2_2(z)\times B^1_2(0)\times(t-2,t))} e(v_j)dxdyds \le \theta_0, \tag{4.13}$$

for $(z,t) \in \delta_j^{-2}(B^2_{\frac{1}{2}}(c\sqrt{-t_j})) \times R_-$. From the local H^1 boundedness of v_j in $R^3 \times R_-$, we may assume that $v_j \to v_\infty$ weakly in $H^1_{\rm loc}(R^3\times R_-,R^2)$. Hence (4.5) implies

$$\int_{R^3\times R_-}\Big|\frac{\partial v_\infty}{\partial y}\Big|^2 = 0,$$

which yields $v_\infty(x,y,t) = v_\infty(x,t)$ for $(x,y,t) \in R^3\times R_-$. On the other hand, from Claim 6, we can apply small energy regularity theorem to get

$$v_j \to v_\infty \quad \text{in } C^1_{\rm loc}(R^2\times(B^1_2)\times R_-,R^2).$$

Combining this with (4.10), we have

$$\int_{B^2_1\times(-1,0)} e(v_\infty)dxdt = \frac{\theta_0}{C(m)}.$$

Hence $e(v_\infty)$ is either $\frac{1}{2}|\nabla v_\infty|^2 + \frac{1}{c^2}\beta^2(x_0,y_0)(1-|v_\infty|^2)^2$ or $\frac{1}{2}|\nabla v_\infty|^2$, where $(x_0,y_0,t_0) = \lim_{j\to\infty}(x_j,y_j,t_j)$. Hence v_∞ is not a constant. Moreover, v_∞ satisfies either

$$\varepsilon_j \downarrow c > 0, \partial_t v_\infty - \Delta v_\infty + \frac{1}{c^2}\beta^2(x_0,y_0)(1-|v_\infty|^2)v_\infty = 0 \text{ in } R^2\times R_-,$$

or $\varepsilon \downarrow 0, \quad |v_\infty| = 1$, and

$$\partial_t v_\infty - \Delta v_\infty - |\nabla v_\infty|^2 v_\infty = 0.$$

By Theorem 5.2 in [17], one has $v_\infty \equiv$ constant. Therefore one yields a contradiction and the proof of Theorem 4.1 is complete.

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