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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO THE COMPRESSIBLE ADIABATIC FLOW THROUGH POROUS MEDIA WITH BOUNDARY EFFECTS***

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Abstract

The initial boundary value problems (IBVP) for the system of compressible adiabatic flow through porous media and the IBVP for its corresponding reduced system through Darcy' laws on $[0,1] \times [0,+\infty)$ are considered respectively. The global existence of smooth solutions to the IBVP problems for two systems are proved, and their large-time behavior is analyzed. The time-asymptotic equivalence of these two systems are investigated, the decay rate of the difference of solutions between these two systems are shown to be determined explicitly by the initial perturbations and boundary effects. It is found that the oscillation of the specific volume can be cancelled by that of entropy, i.e., the oscillation of the specific volume and entropy is not required to be small.

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§1. Introduction

The system of compressible adiabatic gas flow through porous media can be modeled by the following system

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v,s)_x = -\alpha u, \qquad\qquad \alpha > 0, \\ \left[e(v,s) + \frac{1}{2}u^2\right]_t + (pu)_x = -\alpha u^2. \end{cases} \tag{1.0}$$

Here, v denotes the specific volume, u is velocity, s stands for entropy, p denotes pressure with $p_v(v,s) < 0$ for $v > 0$, and e is the specific internal energy for which $e_s \neq 0$ and $e_v + p = 0$ holds due to the second law of thermodynamics. For smooth solutions, the system (1.0) is equivalent to the following system

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v,s)_x = -\alpha u, \quad \alpha > 0, \\ s_t = 0, \end{cases} \tag{1.1}$$

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which is strictly hyperbolic with eigenvalues $-\lambda_1 = \lambda_3 = \sqrt{-p_v}$, and $\lambda_2 = 0$.

By approximating $(1.1)_2$ with Darcy's law, we obtain the following system

$$\begin{cases} \widetilde{v}_t = -\frac{1}{\alpha} p(\widetilde{v}, s)_{xx}, \\ \widetilde{u} = -\frac{1}{\alpha} p(\widetilde{v}, s)_x, \\ s_t = 0. \end{cases} \tag{1.2}$$

For the isentropic flow, namely s = const., (1.1) and (1.2) take the following forms respectively

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = -\alpha u, \quad \alpha > 0, \end{cases} \tag{1.1$'$}$$

which can be viewed as the Euler equations with friction term added to the momentum equation in Lagrangian coordinates, and

$$\begin{cases} \widetilde{v}_t = -\frac{1}{\alpha} p(\widetilde{v})_{xx}, \\ \widetilde{u} = -\frac{1}{\alpha} p(\widetilde{v})_x. \end{cases} \tag{1.2$'$}$$

It was first shown in [3] that the smooth solutions of the Cauchy problem for $(1.1)'$ with initial data

$$(v, u)(x, 0) = (v_0, u_0)(x), \quad v_0(\pm\infty) = v_\pm,$$

tend time-asymptotically to the solutions of $(1.2)'$ with initial data

$$\widetilde{v}(x, 0) = v^*(x + b),$$

where v^* is the similarity solution to $(1.2)'_1$ with $v^*(\pm\infty) = v_\pm$, and b a constant. Namely, the nonlinear diffusive phenomena of smooth solutions to $(1.1)'$ occurs due to the damping mechanism. Based on the resolution of perturbated Riemann problem for $(1.1)'$ in [10] and [11], the nonlinear diffusive phenomena of entropy weak solutions to (1.1) was shown in [6]. There are other related results to [3], such as [2, 4, 5, 8, 9, 18, 21] for smooth solutions, and [1, 14, 16, 20] for weak solutions.

We are interested in the question concerning the global smooth solutions to the hyperbolic system (1.1) and the reduced decoupled system (1.2) with boundary effects, and in comparing their large-time behaviors. In the present paper, we investigate the initial boundary value problems (IBVP) for (1.1) on $\Omega \equiv [0, 1] \times [0, +\infty)$ with initial data

$$(v, u)(x, 0) = (v_0, u_0)(x), \quad x \in [0, 1], \tag{1.3}$$

and boundary values given by one of the followings

$$u(0, t) = u_1(t), \quad u(1, t) = u_2(t), \quad t \geq 0, \tag{1.4$_1$}$$

$$u(0, t) = u_1(t), \quad p(1, t) = p_2(t), \quad t \geq 0, \tag{1.4$_2$}$$

$$p(0, t) = p_1(t), \quad p(1, t) = p_2(t), \quad t \geq 0, \tag{1.4$_3$}$$

where $u_i(t) \to 0$ $(i = 1, 2)$ as $t \to +\infty$, and $p = p(v, s)(x, t)$. Also, we consider the IBVP for (1.2) with initial data

$$\widetilde{v}(x, 0) = \widetilde{v}_0(x), \quad x \in [0, 1], \tag{1.5}$$

and the corresponding boundary values given by

$$\widetilde{p}_x(0, t) = f_1(t), \quad \widetilde{p}_x(1, t) = f_2(t), \quad t \geq 0 \quad (\text{for } i = 1), \tag{1.6$_1$}$$

$$\widetilde{p}_x(0, t) = f_1(t), \quad \widetilde{p}(1, t) = p_2(t), \quad t \geq 0 \quad (\text{for } i = 2), \tag{1.6$_2$}$$

$$\widetilde{p}(0, t) = p_1(t), \quad \widetilde{p}(1, t) = p_2(t), \quad t \geq 0 \quad (\text{for } i = 3), \tag{1.6$_3$}$$

respectively, where $\widetilde{p} = p(\widetilde{v}, s)$ and $f_i = -(\alpha u_i + u_i')(t)$ $(i = 1, 2)$.

First, we consider the IBVP (1.1), (1.3) and $(1.4)_1$ and the IBVP (1.2), (1.5) and $(1.6)_1$ respectively. For simplicity, we consider a typical case $p(v, s) = a_0 v^{-\gamma} e^{s(x)}$ with $a_0 > 0$ and

$\gamma \geq 1$, and assume that $\alpha = 1$. In addition, we assume that the compatibility conditions of the initial and boundary values at $(0,0)$ and $(1,0)$ hold.

Integrating $(1.1)_1$ over $[0,1]\times[0,t]$, we obtain

$$\int_0^1 v(x,t)dx = \int_0^1 v_0(x)dx + \int_0^t (u_2 - u_1)(\tau)d\tau \triangleq Q(t). \tag{1.7}$$

In order to avoid the appearance of $v = 0$ or $v = +\infty$, we assume that there are positive constants Q_1 and Q_2 such that

$$0 < Q_1 \leq Q(t) \leq Q_2 < +\infty, \quad t \geq 0. \tag{1.8}$$

Integrating $(1.2)_1$ over $[0,1]\times[0,t]$, we obtain

$$\begin{aligned}\int_0^1 \widetilde{v}(x,t)dx = \int_0^1 \widetilde{v}_0(x)dx + \int_0^t (u_2 - u_1)(\tau)d\tau \\ + (u_2 - u_1)(t) - (u_2 - u_1)(0) \triangleq \widetilde{Q}(t).\end{aligned} \tag{1.9}$$

Similarly, we assume that for positive constants $\widetilde{Q}_1$ and $\widetilde{Q}_2$ it holds that

$$0 < \widetilde{Q}_1 \leq \widetilde{Q}(t) \leq \widetilde{Q}_2 < +\infty, \quad t \geq 0. \tag{1.10}$$

Let $v_c = Q(+\infty)$ and $\widetilde{v}_c = \widetilde{Q}(+\infty)$, then it holds that $\widetilde{v}_c = v_c$, provided that

$$\int_0^1 \widetilde{v}_0(x)dx - (u_2 - u_1)(0) = \int_0^1 v_0(x)dx. \tag{1.11}$$

The main result in this case shows that the global smooth solutions to these two IBVP exist and are equivalent time-asymptotically, in the sense that they have the same asymptotic state and decay rate, in H^2 norm for any $\gamma \geq 1$, provided that $\|(v_0 - \widehat{v}, \widetilde{v}_0 - \widehat{v})(\cdot)\| + \|(u_{0x}, (s - \gamma \ln v_0)_x)\|_1 + \|(f_1, f_2)\|_2 + \|(f_1, f_2)\|_{L^1}$ is small enough and (1.11) holds. Here

$$\widehat{v} = v_c e^{\frac{1}{\gamma}s(x)} \Big(\int_0^1 e^{\frac{1}{\gamma}s(x)}dx\Big)^{-1}. \tag{1.12}$$

The other IBVP for (1.1) and (1.2) (namely, the IBVP (1.1), (1.3), $(1.4)_i$ and the IBVP (1.2), (1.5), $(1.6)_i$ $(i = 2,3)$) can be dealt with also. If we set

$$p_i(t) \to \overline{p} > 0, \quad \text{as} \quad t \to +\infty, \quad i = 1,2, \tag{1.13}$$

then the main results for those cases will show that the smooth solutions to the IBVP for (1.1) and (1.2) are still equivalent time-asymptotically in H^2 norm, also in the sense that they have the same asymptotic state and decay rate, provided that $\|(v_0 - \widehat{v}_1, \widetilde{v}_0 - \widehat{v}_1)(\cdot)\|_2 + \|(u_{0x}, (s - \gamma \ln v_0)_x)\|_1 + \|f_1\|_2 + \|(p_1 - \overline{p}, p_2 - \overline{p})\|_3$ is small enough, where

$$\widehat{v}_1(x) = \Big(a_0 \overline{p}^{-1} e^{s(x)}\Big)^{\frac{1}{\gamma}}. \tag{1.14}$$

Recently, the IBVP for (1.1) was discussed in [15] and [19] with different kinds of boundary effects on a quarter plane, and in [7] on a strip Ω with fixed endpoints, i.e., $u(0,t) = u(1,t) = 0, t \geq 0$. The large-time behavior of solutions and the relation to the corresponding nonlinear diffusive waves are analysed therein. In [7], the characteristic method was used to control the uniform upper and lower bounds of $v(x,t)$, which are determined by the initial bounds of $v_0(x)$ and are crucial in proving the existence of global smooth solutions to (1.1). In addition, the oscillations of $v_0(x)$ and $s(x)$ are required to be small, and only the case $1 < \gamma < 3$ was discussed. However, we should point out that the method used in [7] does not work in the present paper for general $\gamma \geq 1$ and general boundary conditions because the bounds of $v(x,t)$ may not be controlled by the initial data $v_0(x)$ due to the influence of boundary effects and entropy $s(x)$. Instead, the energy method is used in the present paper to overcome difficulties. As making the energy estimates, a new weight function

p is selected to overcome the difficulties, and the IBVP problems for (1.1) and (1.2) are reformulated differently from those in [7]. Though the value of $v(x,t)$ may not be controlled by the bounds of $v_0(x)$ due to the boundary effects and entropy, the new variable p can express the common influence of boundary effects, initial perturbations, and entropy. Based on this observation, we take the expected time-asymptotic state $\widehat{p}$ into consideration with $\widehat{p}=\overline{p}$ or $\widehat{p}=p(\widehat{v},s)$, as making energy estimates, and assume that $0<\overline{p}_-<\widehat{p}<\overline{p}_+$, where $\overline{p}_-$ and $\overline{p}_+$ are the a priori bounds of $p(x,t)$, to which the a priori bounds of p_x (or $(s-\gamma\ln v)_x$) and p_t (or u_x) are involved. Thus, the oscillations of $v_0(x)$ and $s(x)$ are not required to be small separately. The analysis also works for the IBVP (1.2), (1.5) and (1.6). Moreover, we find that the boundary effects also influence the time-convergence rate between the solutions to IBVP (1.1) and (1.2) (see Theorem 4.1). Our results contain that obtained in [7] (see Remark 2.4). The idea to introduce a new variable and the above analysis can also be used to solve the Cauchy problem for (1.1) with $s(+\infty)\neq s(-\infty)$ which is unsolved in [5] (see Theorem5.1).

This paper is arranged as follows. The results for IBVP (1.1), (1.3) and (1.4) will be stated and proved in Section 2. The corresponding results for IBVP (1.2), (1.5) and (1.6) and the relation between solutions to the IBVP for (1.1) and (1.2) are given in Section 3. Another suitable weight variable is also introduced in Section 4 to solve Cauchy problem for (1.1) with $s(+\infty)\neq s(-\infty)$.

Notation. From now on, L^2 will denote the usual space of square integrable functions with norm $\|\cdot\|$ on $I=[0,1]$,$R_+=(0,+\infty)$, or $R=(-\infty,+\infty)$, and $H^l(l\geq 0)$ the usual Sobolev space with norm $\|\cdot\|_l$ on $I=[0,1]$, R_+, or R. We assume that for any of the above norms written as $\|\cdot\|_n$, the norm of a vector valued function (g_1,g_2,g_3) is given by

$$\|(g_1,g_2,g_3)\|_n=\sum_{i=1}^{3}\|g_i\|_n.$$

§2. IBVP for the Original System

We will prove that the IBVP (1.1), (1.3) and (1.4) has global smooth solutions and analyze their large-time behavior.

Assume $(v_0,u_0)\in(H^2)^2$ and $v_0(x)>0$. Let, for positive constant $\overline{p}$,

$$N_1=\|(u_{0x},(s-\gamma\ln v_0)_x)\|_1,$$

$$\mu_0=\|(f_1,f_2)\|_2,\quad \mu_1=\|f_1\|_2+\|p_2-\overline{p}\|_3,\quad \mu_2=\|(p_1-\overline{p},p_2-\overline{p})\|_3,$$

and denote

$$A_1(t)=\sqrt{\int_t^{t+1}\sum_{i=0}^{3}\Big(\Big|\frac{d^iu_1(t)}{dt^i}\Big|^2+\Big|\frac{d^iu_2(t)}{dt^i}\Big|^2\Big)d\tau},$$

$$A_2(t)=\sqrt{\int_t^{t+1}\Big(\sum_{i=0}^{3}\Big|\frac{d^iu_1(t)}{dt^i}\Big|^2+(p_2-\overline{p})^2+p_{2t}^2+p_{2tt}^2+p_{2ttt}^2\Big)d\tau},$$

$$A_3(t)=\sqrt{\sum_{i=1}^{2}\int_t^{t+1}((p_i-\overline{p})^2+p_{it}^2+p_{itt}^2+p_{ittt}^2)d\tau}.$$

Let C be a generic positive constant. Then, we have

Theorem 2.1. *Assume that $u_i\in H^3$ and $f_i\in L^1$ $(i=1,2)$. Then, there is a $\delta_0>0$ such that if $\|(v_0-\widehat{v})(\cdot)\|+N_1+\|(f_1,f_2)\|_{L^1}+\mu_0\leq\delta_0$, a global solution (v,u) to (1.1), (1.3) and $(1.4)_1$ exists and satisfies, for some $c_0>0$, that*

$$\|(v-\widehat{v},u)(\cdot,t)\|_2^2\sim C\{e^{-c_0t}+A_1(t)\}\to 0,\quad as\quad t\to+\infty,$$

where $\widehat{v}$ *is given by* (1.12).

Theorem 2.2. *Assume that* $u_1 \in H^3$, $f_1 \in L^1$, *and* $(p_2 - \overline{p}) \in H^3$. *Then, there is a* $\delta_1 > 0$ *such that if* $\|(v_0 - \widehat{v}_1)(\cdot)\| + N_1 + \mu_1 \leq \delta_1$, *a global solution* (v, u) *to* (1.1), (1.3) *and* $(1.4)_2$ *exists and satisfies, for some* $c_0 > 0$, *that*

$$\|(v - \widehat{v}_1, u)(\cdot, t)\|_2^2 \sim C\{e^{-c_0 t} + A_2(t)\} \to 0, \quad as \quad t \to +\infty,$$

where $\widehat{v}_1(x)$ *is given by* (1.14).

Theorem 2.3. *Assume that* $(p_i - \overline{p}) \in H^3$ $(i = 1, 2)$. *Then, there is a* $\delta_2 > 0$ *such that if* $\|(v_0 - \widehat{v}_1)(\cdot)\| + N_1 + \mu_2 \leq \delta_2$, *a global solution* (v, u) *to* (1.1), (1.3) *and* $(1.4)_3$ *exists and satisfies, for some* $c_0 > 0$, *that*

$$\|(v - \widehat{v}_1, u)(\cdot, t)\|_2^2 \sim C\{e^{-c_0 t} + A_3(t)\} \to 0, \quad as \quad t \to +\infty, \tag{2.1}$$

where $\widehat{v}_1(x)$ *is given by* (1.14).

Remark 2.1. (1) It is easy to verify that for the IBVP (1.1), (1.3) and (1.4) it holds that for any $x \in [0, 1]$, $\int_0^x (v(y, t) - v_d(y))dy \to 0$, as $t \to +\infty$, where $v_d(x) = \widehat{v}(x)$ or $\widehat{v}_1(x)$.

(2) If for $i = 1, 2$ and $j = 0, 1, 2, 3$ it holds that

$$\frac{d^j u_i(t)}{dt^j} \sim Ce^{-\eta_1 t}, \quad \frac{d^j (p_i(t) - \overline{p})}{dt^j} \sim Ce^{-\eta_1 t}, \quad t \geq 0,$$

with a positive constant η_1, then there is a constant $\eta_2 > 0$ such that $A_i(t) \sim Ce^{-\eta_2 t}(t \geq 0, i = 1, 2, 3)$, which implies that for IBVP (1.1), (1.3) and (1.4)

$$\|(v - v_d)(\cdot, t)\|_{H^2}^2 + \|u(\cdot, t)\|_{H^2}^2 \leq Ce^{-\eta_3 t}, \quad t \geq 0,$$

with a positive constant η_3 and $v_d(x)$ given above.

(3) The condition that $\|(s - \gamma \ln v_0)_x\|_1 \ll 1$ implies that there is cancellation between the oscillations of entropy and the initial specific volume.

We only prove Theorem 2.3. Theorems 2.1–2.2 can be proved by the same approach except the different resolution on boundary terms.

Representing v by p and s, i.e., $v = (a_0 e^{s(x)} p^{-1})^{1/\gamma} = a(x) p^{-\frac{1}{\gamma}}$, the IBVP (1.1), (1.3) and $(1.4)_3$ is reformulated into

$$\begin{cases} a(x)(p^{-\frac{1}{\gamma}})_t - u_x = 0, \\ u_t + p_x = -u, \\ p(0, t) = p_1(t), \quad p(1, t) = p_2(t), \\ (p, u)(x, 0) = (p_0, u_0)(x), \end{cases} \tag{2.2}$$

where $a(x) = (a_0 e^{s(x)})^{1/\gamma}$ and $p_0(x) = p(v_0(x), s(x)) > 0$.

Let us define the work space for (2.2) (equivalently (1.1), (1.3) and $(1.4)_3$) as

$$X(0, T) = \{(p, u); (p - \overline{p}, u) \in C^0(0, T; (H^2)^2), p_- \leq p \leq p_+, x \in [0, 1]\},$$

where $\overline{p}$ is determined by (1.13), p_- and p_+ satisfy

$$p_- < \min_{x \in [0,1]} \{\overline{p}, p_0(x)\}, \quad p_+ > \max_{x \in [0,1]} \{\overline{p}, p_0(x)\},$$

and assume apriorily, for $(p, u) \in X(0, T)$, that

$$\eta \equiv \Big(\frac{1}{\gamma} + 1\Big) p^{-1}(|p_t| + |p_x| + |p_+ - p_-|) \ll 1.$$

Since the local existence of smooth solutions to (2.2) (or (1.1), (1.3) and $(1.4)_3$) can be proved by the standard method in [12] and [13], what should be done is to obtain the expected a priori estimates, i.e., Lemmas 2.1–2.3. To prove them, we first assume that (p, u) (or (v, u)) has enough regularity, i.e., $(p, u) \in C^0(0, T; (H^3)^2)$, then with the help of the Friedrich's mollifier, we can verify that they are true for $(p, u) \in X(0, T)$.

By substituting $(2.2)_1$ into $(2.2)_2$, the IBVP (2.2) can be reformulated into

$$\begin{cases} L_1 \equiv a(x)(p_{tt}+p_t) - \gamma p^{1+\frac{1}{\gamma}} p_{xx} - a(x)\frac{1+\gamma}{\gamma} p^{-1} p_t^2 = 0, & (x,t)\in\Omega, \\ p(0,t)=p_1(t), p(1,t)=p_2(t), & t\ge 0, \\ p(x,0)=p_0(x), p_t(x,0) = -\gamma a_0^{-\frac{1}{\gamma}} p_0^{1+\frac{1}{\gamma}} e^{-\frac{1}{\gamma}s(x)} u_{0x} =: p_3(x), & x\in[0,1]. \end{cases} \tag{2.2$'$}$$

Consider the following equality

$$(p-\overline{p}+2p_t)L_1 = 0. \tag{2.3}$$

Integrating (2.3) over $[0,1]\times[0,t]$, we get, after computation, that

$$\frac{1}{2}E_1(t) + \int_0^t E_2(\tau)d\tau = \frac{1}{2}E_0 + B_1(t), \tag{2.4}$$

where

$$\begin{aligned} E_1(t) &\equiv \int_0^1 \Big\{a(x)[(p-\overline{p})^2 + 2(p-\overline{p})p_t + 2p_t^2] + 2\gamma p^{1+\frac{1}{\gamma}} p_x^2\Big\}dx, \\ E_2(t) &\equiv \int_0^1 \Big\{a(x)\Big[1-\Big(1+\frac{1}{\gamma}\Big)p^{-1}(2p_t+p-\overline{p})\Big]p_t^2 \\ &\quad + \gamma p^{1+\frac{1}{\gamma}}\Big[1+\Big(1+\frac{1}{\gamma}\Big)p^{-1}(p_t+p-\overline{p})\Big]p_x^2\Big\}(x,t)dx, \\ E_0 &\equiv \int_0^1 \Big\{a(x)((p_0-\overline{p})^2 + 2(p_0-\overline{p})p_3 + 2p_3^2) + 2\gamma p_0^{1+\frac{1}{\gamma}} p_{0x}^2\Big\}dx, \\ B_1(t) &\equiv \int_0^t \Big[\gamma(2p_t+p-\overline{p})p^{1+\frac{1}{\gamma}} p_x(x,\tau)\Big]g\Big|_{x=0}^1 d\tau. \end{aligned} \tag{2.5}$$

It is easy to obtain, due to (2.4), (2.5), and $(2.2)'_3$, the following lemma.

Lemma 2.1. *Under the assumptions of Theorem* 2.3, *it holds, for* $(p,u)\in X(0,T)$, *that*

$$\int_0^1 \Big\{a(x)((p-\overline{p})^2+p_t^2) + \gamma p^{1+\frac{1}{\gamma}} p_x^2\Big\}dx + \int_0^t\int_0^1 \Big\{\gamma p^{1+\frac{1}{\gamma}} p_x^2 + a(x)p_t^2\Big\}dxd\tau \le C(N_2 + |B_1(t)|),$$

where $N_2 = N_1^2 + \|v_0 - \widehat{v}_1\|^2$, *provided that* $\eta \ll 1$.

Differentiating $(2.2)'_1$ with t, we get

$$\partial_t L_1 \equiv a(x)p_{ttt} + a(x)p_{tt}\Big(1-\frac{3(\gamma+1)}{\gamma}p^{-1}p_t\Big) - \gamma p^{1+\frac{1}{\gamma}} p_{xxt} + a(x)n_1 p_t = 0, \tag{2.6}$$

where

$$n_1 = n_1(x,t) = -\frac{(1+\gamma)}{\gamma}p^{-1}p_t + \frac{2\gamma+1}{\gamma+1}\Big(\frac{(1+\gamma)}{\gamma}p^{-1}p_t\Big)^2.$$

Consider the following equality

$$(p_t + 2p_{tt})\partial_t L_1 = 0. \tag{2.7}$$

Integrating (2.7) by parts over $[0,1]\times[0,t]$, and using Lemma 2.1, we obtain

Lemma 2.2. *Under the assumptions of Theorem* 2.3, *it holds, for* $(p,u)\in X(0,T)$, *that*

$$\int_0^1 \{a(x)(p_t^2+p_{tt}^2) + \gamma p^{1+\frac{1}{\gamma}} p_{xt}^2\}dx + \int_0^t\int_0^1 \{a(x)p_{tt}^2 + \gamma p^{1+\frac{1}{\gamma}} p_{xt}^2\}dxd\tau \le C(N_2 + |B_1(t)| + |B_2(t)|),$$

where

$$B_2(t) = \int_0^t \Big[\gamma(p_t + 2p_{tt})p^{1+\frac{1}{\gamma}} p_{xt}(x,\tau)\Big] g|_{x=0}^1 d\tau,$$

provided that $\eta \ll 1$.

Then, we have, with the help of Lemmas 2.1–2.2, and $(2.2)_1'$, that

$$\begin{aligned}&\int_0^1 \{a(x)(p_t^2 + p_{tt}^2 + (p-\overline{p})^2) + \gamma p^{1+\frac{1}{\gamma}}(p_x^2 + p_{xx}^2 + p_{xt}^2)\}dx \\ &+ \int_0^t \int_0^1 \{a(x)(p_t^2 + p_{tt}^2) + \gamma p^{1+\frac{1}{\gamma}}(p_x^2 + p_{xt}^2)\}dxd\tau \\ &\le C(N_2 + |B_1(t)| + |B_2(t)|).\end{aligned} \tag{2.10}$$

The terms in the right-hand side of (2.10) can be estimated as follows. We estimate the terms in $B_2(t)$ first. Integrating by part, we have

$$\begin{aligned}B_2(t) =& \Big[\gamma(2p_{tt} + p_t)p^{1+\frac{1}{\gamma}} p_x(x,\tau)\Big] g|_{x=0}^1 g|_{\tau=0}^t \\ &+ \int_0^t \Big[(2p_{ttt} + p_{tt} + (2p_{tt} + p_t)\Big(1 + \frac{1}{\gamma}\Big)p^{-1}p_t)\gamma p^{1+\frac{1}{\gamma}} p_x(x,\tau)\Big]\Big|_{x=0}^1 d\tau.\end{aligned}$$

For any $(x,t) \in [0,1] \times [0,T]$, it holds that

$$\begin{aligned}\gamma p^{1+\frac{1}{\gamma}} p_x^2 &\le \int_0^1 \gamma p^{1+\frac{1}{\gamma}} p_x^2 \Big(1 + \Big(1 + \frac{1}{\gamma}\Big)p^{-1}|p_x|\Big)dx + 2\Big|\int_0^1 \gamma p^{1+\frac{1}{\gamma}} p_{xx}p_x dx\Big| \\ &\le C\int_0^1 \Big(\gamma p^{1+\frac{1}{\gamma}} p_x^2 \Big(1 + \frac{1}{\gamma}a_+ p_-^{-(1+\frac{1}{\gamma})}\Big) + a(x)(p_{tt}^2 + p_t^2)\Big)dx,\end{aligned} \tag{2.11}$$

where we have used the the equality $(2.2)_1'$, $a_+ = \max_{x\in[0,1]} a(x)$, and $\eta \ll 1$. Then, by Cauchy inequality and (2.11), we have

$$\begin{aligned}|B_2(t)| \le& C(\mu_2^2 + N_2) + \frac{1}{4}\int_0^1 \Big(\gamma p^{1+\frac{1}{\gamma}} p_x^2 \Big(1 + \frac{1}{\gamma}a_+ p_-^{-(1+\frac{1}{\gamma})}\Big) + a(x)(p_{tt}^2 + p_t^2)\Big)dx \\ &+ \frac{1}{4}\int_0^t \int_0^1 \Big(\gamma p^{1+\frac{1}{\gamma}} p_x^2 \Big(1 + \frac{1}{\gamma}a_+ p_-^{-(1+\frac{1}{\gamma})}\Big) + a(x)(p_{tt}^2 + p_t^2)\Big)dxd\tau,\end{aligned} \tag{2.12}$$

$$|B_1(t)| \le C\mu_2 + \frac{1}{4}\int_0^1 \{a(x)(p_{tt}^2 + p_t^2) + \gamma p^{1+\frac{1}{\gamma}} p_x^2\}dx. \tag{2.13}$$

Substituting (2.12) and (2.13) into (2.10), we get

Lemma 2.3. *Under the assumptions of Theorem* 2.3, *it holds, for* $(p,u) \in X(0,T)$, *that*

$$\begin{aligned}&\int_0^1 \{a(x)(p_t^2 + p_{tt}^2 + (p-\overline{p})^2) + \gamma p^{1+\frac{1}{\gamma}}(p_x^2 + p_{xx}^2 + p_{xt}^2)\}dx \\ &+ \int_0^t \int_0^1 \{a(x)(p_t^2 + p_{tt}^2) + \gamma p^{1+\frac{1}{\gamma}}(p_x^2 + p_{xt}^2)\}dxd\tau \le C(N_2 + \mu_2^2),\end{aligned}$$

provided that $\eta \ll 1$.

Remark 2.2. For Theorems 2.1–2.2, the boundary terms can be dealt with as follows. For example, we consider Theorem 2.2 and only estimate the boundary terms at $x = 0$, which we denote by

$$B_3(t) = \int_0^t \Big[\gamma p^{1+\frac{1}{\gamma}}((p - \overline{p} + 2p_t)p_x + (p_t + 2p_{tt})p_{xt})\Big](0,\tau)d\tau.$$

Noticing that $p_x(0,t)=f_1(t)$, $p_{xt}(0,t)=f_1'(t)$ and

$$\gamma p^{1+\frac{1}{\gamma}}p_t^2\le\int_0^1\gamma p^{1+\frac{1}{\gamma}}\Big(p_t^2\Big(1+(1+\frac{1}{\gamma}\Big)p^{-1}|p_x|\Big)+|p_{xt}p_t|\Big)dx\le 3\int_0^1\gamma p^{1+\frac{1}{\gamma}}(p_t^2+p_{xt}^2)dx,$$

where we have used that $\eta\ll 1$, we can estimate $B_3(t)$, by Cauchy inquality and Sobolev embedding theorem, as

$$\begin{aligned}|B_3(t)|&\le|2\gamma p^{1+\frac{1}{\gamma}}p_tp_{xt}|_{x=0}^1|+\Big|\int_0^t 2\gamma p^{1+\frac{1}{\gamma}}p_t\Big(p_{xtt}+\Big(1+\frac{1}{\gamma}\Big)p^{-1}p_tp_{xt}\Big)(0,\tau)d\tau\Big|\\&\quad+\Big|\int_0^t\Big[\gamma p^{1+\frac{1}{\gamma}}((p-p_2+p_2-\overline{p}+2p_t)p_x+p_tp_{xt})\Big](0,\tau)d\tau\Big|\\&\le C(\mu_1^2+N_2)+\frac{1}{4}\int_0^1(a(x)p_t^2+\gamma p^{1+\frac{1}{\gamma}}p_{xt}^2)dx\\&\quad+\frac{1}{4}\int_0^t\int_0^1(a(x)p_t^2+\gamma p^{1+\frac{1}{\gamma}}(p_x^2+p_{xt}^2))dxd\tau.\end{aligned}$$

By the local existence theorem and Lemma 2.3, we can prove that the a priori assumption that $\eta\ll 1$ is true if we choose $C(N_2+\mu_2^2)$ small enough.

Now, we turn to the proof of the existence of global smooth solution to IBVP (2.2). Integrating $(2.2)_2$ over $[0,1]\times[0,t]$, we have, after a computation, that

$$|u(x,t)|\le e^{-t}\Big|\int_0^1 u_0dx\Big|+\Big|\int_0^t(p_2-p_1)(\tau)e^{-(t-\tau)}d\tau\Big|+C\sqrt{\int_0^1 u_x^2dx}.$$

which yields

$$\int_0^1 u^2(x,t)dx\le C\Big(e^{-t}+\|u_x\|^2+\Big(\int_0^t(|p_2-\overline{p}|+|p_1-\overline{p}|)(\tau)e^{-(t-\tau)}d\tau\Big)^2\Big).\tag{2.14}$$

Therefore, with the help of Lemma 2.3, (2.14) and (2.2), we can prove, by the standard continuity argument, that the global smooth solutions to IBVP (2.2) exist, provided that $\delta_2\ll 1$.

Moreover, the proof of (2.1) then follows from the Sobelov imbedding theorem, Grownwall's lemma and a complicated analysis on $B_i(t)$ and $B_2(t)$. Thus, the proof of Theorem 2.3 is completed.

§3. The IBVP for (1.2) and Comparision

We will prove that the IBVP (1.2), (1.5) and (1.6) has global smooth solutions and analyze their large-time behavior in this section.

Assume $\widetilde{v}_0\in H^3$ and $\widetilde{v}_0(x)>0$. Let $\widehat{v}_c(x)=\widetilde{v}_ce^{\frac{1}{\gamma}s(x)}\Big(\int_0^1 e^{\frac{1}{\gamma}s(x)}dx\Big)^{-1}$.

We have, corresponding to Theorems 2.1–2.3, the following theorems.

Theorem 3.1. *Assume that $u_i\in H^3$ and $f_i\in L^1$ $(i=1,2)$. Then, there is a $\delta_3>0$ such that if $\|\widetilde{v}_0-\widetilde{v}_c\|+\|(s-\gamma\ln\widetilde{v}_0)_x\|_2+|(f_1,f_2)|_{L^1}+\mu_0\le\delta_3$, a global solution $(\widetilde{v},\widetilde{u})$ to (1.2), (1.5) and $(1.6)_1$ exists and satisfies, for some $c_1>0$, that*

$$(\|(\widetilde{v}-\widetilde{v}_c)(\cdot,t)\|_3^2+\|\widetilde{u}(\cdot,t)\|_2^2)\sim C\{e^{-c_1t}+A_1(t)\}\to 0,\quad as\quad t\to+\infty.$$

Theorem 3.2. *Assume that $u_1\in H^3$, $f_1\in L^1$, and $(p_2-\overline{p})\in H^3$. Then, there is a $\delta_4>0$ such that if $\|\widetilde{v}_0-\widetilde{v}_1\|+\|(s-\gamma\ln\widetilde{v}_0)_x\|_2+\mu_1\le\delta_4$, a global solution $(\widetilde{v},\widetilde{u})$ to (1.2), (1.5) and $(1.6)_2$ exists and satisfies, for some $c_1>0$, that*

$$(\|(\widetilde{v}-\widehat{v}_1)(\cdot,t)\|_3^2+\|\widetilde{u}(\cdot,t)\|_2^2)\sim C\{e^{-c_1t}+A_2(t)\}\to 0,\quad as\quad t\to+\infty,$$

where $\widehat{v}_1(x)$ is given by (1.14).

Theorem 3.3. *Assume that* $(p_i - \overline{p}) \in H^3 (i = 1, 2)$. *Then, there is a* $\delta_5 > 0$ *such that if* $\|\widetilde{v}_0 - \widetilde{v}_1\| + \|(s - \gamma \ln \widetilde{v}_0)_x\|_2 + \mu_2 \leq \delta_5$, *a global solution* $(\widetilde{v}, \widetilde{u})$ *to* (1.2), (1.5) *and* $(1.6)_3$ *exists and satisfies, for some* $c_1 > 0$, *that*

$$(\|(\widetilde{v} - \widehat{v}_1)(\cdot, t)\|_3^2 + \|\widetilde{u}(\cdot, t)\|_2^2) \sim C\{e^{-c_1 t} + A_3(t)\} \to 0, \quad as \quad t \to +\infty, \tag{3.1}$$

where $\widehat{v}_1(x)$ *is given by* (1.14).

Remark 3.1. It is easy to verify that for the IBVP (1.2), (1.5) and (1.6) it holds that as $t \to +\infty$, $\int_0^x (\widetilde{v}(y, t) - v_d(y))dy \to 0$ for any $x \in [0, 1]$, where $v_d(x) = \widetilde{v}_c(x)$ or $\widetilde{v}_1(x)$.

Moreover, for IBVP (1.2), (1.5) and (1.6), it holds that $A_i(t) \sim e^{-\eta_4 t}$ $(i = 1, 2, 3)$ under the same assumptions in Remark 2.1.

The proof of Theorems 3.1–3.3 is similar to that of Theorems 2.1–2.3, we omit the details here.

With the help of Theorems 2.1–2.3 and Theorems 3.1–3.3, we can compare the asymptotic behavior of the IBVP problems for (1.1) and (1.2).

Theorem 3.4. *Let* (v, u) *and* $(\widetilde{v}, \widetilde{u})$ *be the smooth solutions to IBVP* (1.1), (1.3) *and* (1.4) *and IBVP* (1.2), (1.5) *and* (1.6) *respectively. Assume that* (1.11) *holds. Then, there is a* $\varepsilon > 0$ *such that if* $\sum\limits_{i=0}^{5} \delta_i \leq \varepsilon$, *it holds that*

$$(\|(v - \widetilde{v})(\cdot, t)\|_2^2 + \|(u - \widetilde{u})(\cdot, t)\|_2^2) \sim C\{e^{-c_3 t} + R(t)\} \to 0, \quad as \quad t \to +\infty,$$

where c_3 *is a given positive constant, and* $R(t) = A_1(t)$, $A_2(t)$, *or* $A_3(t)$, *corresponding to different kinds of boundary conditions.*

Remark 3.2. Theorem 3.4 shows that the IBVP problem (1.2), (1.3)–(1.4) can be well approximated by the IBVP (1.2), (1.5) and (1.6). The convergence rate between their smooth solutions is affected by the boundary effects, i.e., $R(t)$.

§4. Remarks to Cauchy Problem

Consider the Cauchy problems for (1.1) and (1.2) with initial data given by

$$(v, u)(x, 0) = (v_3, u_3)(x), \quad u_3(\pm\infty) = u_\pm, \quad v_3(\pm\infty) = v_\pm, \quad x \in R, \tag{5.1}$$

$$\widetilde{v}(x, 0) = \widetilde{v}_3(x), \quad \widetilde{v}_3(\pm\infty) = v_\pm, \quad x \in R, \tag{5.2}$$

respectively.

The Cauchy problems for (1.1) and (1.2) with initial data given by (5.1) and (5.2) respectively were considered in [5], where the large-time behavior of global smooth solutions and their relation are analysed under the assumption that $(s(x) - \overline{s})$ has a compact support. Now, Based on the idea to introduce the new variable p, we can investigate the case of $s(-\infty) \neq s(+\infty)$.

Assume $s(x) \in C^3(R)$ with $s(\pm\infty) = s_\pm$ and $s_+ \neq s_-$. Set $p_\pm = p(v_\pm, s_\pm)$ and

$$\widehat{v}_3(x) = v_- e^{\frac{1}{\gamma}(s(x) - s_-)}, \quad x \in R,$$

$$z_0(x) = \int_{-\infty}^{x} \Big(p_3^{\frac{1}{\gamma}}(y) - p_-^{\frac{1}{\gamma}}\Big)dy, \quad \widetilde{z}_0(x) = \int_{-\infty}^{x} \Big(\widetilde{p}_3^{\frac{1}{\gamma}}(y) - p_-^{\frac{1}{\gamma}}\Big)dy, \tag{5.3}$$

where $p_3(x) = p(v_3, s)(x) > 0$ and $\widetilde{p}_3(x) = p(\widetilde{v}_3, s)(x) > 0$.

We have the following theorem.

Theorem 4.1. *Assume that* $p_- = p_+$, $(z_0, \widetilde{z}_0, u_{3x}) \in H^1$, $(v_3 - \widehat{v}_3) \in H^2$, *and* $(\widetilde{v}_3 - \widehat{v}_3) \in H^3$. *Then, there is a* $\delta_6 > 0$ *such that if*

$$\|(z_0, \widetilde{z}_0)\|_1 + \|(u_{3x}, (s - \gamma \ln v_3)_x)\|_1 + \|(s - \gamma \ln \widetilde{v}_3)_x\|_2 \leq \delta_6,$$

the global smooth solution (v, u) *of* (1.1) *and* (5.1), *and the global smooth solution* $(\widetilde{v}, \widetilde{u})$ *of* (1.2) *and* (5.2) *exist respectively and satisfy*

$$(\|(v - \widetilde{v})(\cdot, t)\|_2^2 + \|(u - \widetilde{u})(\cdot, t)\|_2^2) \to 0, \quad as \quad t \to +\infty. \tag{5.4}$$

Theorem 4.1 can be proved by a method similar to that used in Theorem 3.4. However, it can also be proved based on the orginal idea in [3] to obtain a "wave" equation for new variable p. We omit the details.

Remark 4.1. (1) The variable w is only useful in dealing with the IBVP (1.1), (1.3) and $(1.4)_i$ and the IBVP (1.2), (1.5) and $(1.6)_i$ for $i = 1, 2$. It is not used for the IBVP (1.1), (1.3) and $(1.4)_3$ and the IBVP (1.2), (1.5) and $(1.6)_3$ since the term $(u(0,t) + \partial_t u(0,t))$, $t \geq 0$, in reformulating the IBVP problems for (1.1) and (1.2), is not known.

(2) It is interesting to investigate the time-decay rate in (5.4) by the methods in [2, 3, 18].

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