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BOUNDARY REGULARITY FOR WEAK HEAT FLOWS**

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Abstract

The partial regularity of the weak heat flow of harmonic maps from a Riemannian manifold M with boundary into general compact Riemannian manifold N without boundary is considered. It is shown that the singular set Sing(u) of the weak heat flow satisfies $H^n_\rho(\mathrm{Sing}(u)) = 0$, with n = dimension$M$. Here H^n_ρ is Hausdorff measure with respect to parabolic metric $\rho((x,t),(y,s)) = \max\{|x-y|, \sqrt{|t-s|}\}$.

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§1. Introduction

Let (M, g) be a compact smooth Riemannian manifold of dimension n with C^2 boundary ∂M, and (N, h) be a smooth compact Riemannian manifolds of dimension k. Assume that (N, h) without boundary is isometrically embedded into the Euclidean space $(R^m, \langle .,. \rangle)$.

We assume that Sobolev space

$$H^1(M,N) = \{u \in H^1(M;R^m) | u(x) \in N \text{ for a.e.} x \in M\}$$

and for every $u \in H^1(M;N)$, define the energy of u,

$$E(u) = \int_M |\nabla u|^2 dv, \tag{1.1}$$

where in local coordinate $|\nabla u|^2 = g^{\alpha\beta} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^i}{\partial x^\beta}$, $dv = \sqrt{\det(g_{\alpha\beta})} dx^1 \cdots dx^n$ and $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$, and $(g_{\alpha\beta})$ is the metric of M. Here and in the following, repeated indices mean to sum.

A map $u \in H^1(M,N)$ is called a weakly harmonic map if it satisfies Euler-Lagrange equations for the energy $E(u)$

$$-\Delta_M u = A(u)(\nabla u, \nabla u) \tag{1.2}$$

in weak sense, where $A(u)(\nabla u, \nabla u)$ denotes the second fundamental form of N in R^m at the point u.

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We will consider in this note the corresponding evolution problem with boundary condition for this equation:

$$\partial_t u - \Delta u = A(u)(\nabla u, \nabla u) \quad \text{in} \quad M \times (0, \infty), \tag{1.3}$$

$$u|_{t=0} = u_0(x) \quad \text{in} \quad M, \tag{1.4}$$

$$u = f, \quad \partial_\nu u = 0 \quad \text{on } \partial M, \quad t > 0, \tag{1.5}$$

where $f \in C^2(\partial M; N)$, $u_0 \in H^1(M; N)$, and $u_0|_{\partial M} = f$.

Define the space

$$W^{1,2}_{\text{loc}}(M \times R_+; N) = \{u \in H^1_{\text{loc}}(M \times R_+; N) \mid u_t \in L^2(M \times R_+; R^m), \\ \nabla u \in L^2_{\text{loc}}(R_+; L^2(M; P^{n\times m}))\},$$

where $P^{n\times m}$ is the space of $n \times m$ matrices.

The global existence of weak solutions to the heat flow of harmonic maps between compact Riemannian manifold was first shown by Chen-Struwe[7]. Chen-Lin[5] then generalized this to the case with Dirichlet boundary condition, where M has a C^2 boundary, i.e. the problem (1.3)-(1.5) has a global weak solution.

In our previous article[20] we have considered interior regularity in the case that the target N is a general compact manifold. In this note we will consider the partial regularity at boundary. The main difficulties are to estimate energy decay. And for the reason of boundary, we must modify the energy at boundary, this turns out more complicated to estimate. This energy modifying was used in [21] (see also [1]). In the argument of blow up, it is essential that C^2 norm of boundary function is bounded.

Our main result is the following theorem.

Theorem 1.1. *Let (M^n, g) and (N^k, h) be two smooth compact Riemannian manifolds, M with C^2 boundary, while N without boundary. Assume that $u \in W^{1,2}_{\text{loc}}(M \times R_+; N)$ is a global weak solution of (3)-(5) with $u_0 \in H^1(M; N)$ which satisfies the monotonicity inequality (10) and the energy inequality (22). Then there exists a closed subset $Sing(u) \subset \overline{M} \times (0, \infty)$ such that $H^n_\rho(Sing(u)) = 0$ and $u \in C^\infty(\overline{M} \times (0, \infty) \backslash Sing(u); N)$, if $f \in C^\infty(\partial M; N)$. Here H^n_ρ is n-dimensional Hausdorff measure with respect to the parabolic metric* $\rho((x, t), (y, s)) = \max\{|x - y|, \sqrt{|t - s|}\}$.

This paper is organized as follows: In Section 2, the monotonicity inequality and the energy inequality at boundary is given. In Section 3 we prove energy decay lemma, thus we can complete the proof of Theorem 1.1.

For the sake of simplicity, we will assume that M is flat, g is the standard metric of R^n, $\overline{M} = R^n_+ = \{x = (x_1, \cdots, x_n) \in R^n | x_n \geq 0\}$. So the problem (1.3)-(1.5) is of the form:

$$\partial_t u - \Delta u = A(u)(\nabla u, \nabla u) \quad \text{in } R^n_+ \times (0, \infty), \tag{1.6}$$

$$u|_{t=0} = u_0(x) \quad \text{on } R^n_+, \tag{1.7}$$

$$u|_{x_n=0} = f(x', 0), \quad t > 0, x' = (x_1, \cdots, x_{n-1}). \tag{1.8}$$

§2. Monotonicity inequality and Energy inequality

For the regularity of weak heat flow of harmonic maps, the monotonicity inequality is very important. Struwe[26,27] found it out firstly in the studying of heat flow.

In the following we adopt the same notations as those in [27] and [6]. Let $z = (x, t) \in R^n \times R$. For $z_0 = (x_0, t_0)$ and $r > 0$, write

$$B_r(x_0) = \{x \in R^n || x_0 - x| < r\},$$

$$P_r(z_0) = \{z = (x, t) \in R^n \times R || x - x_0| < r, |t - t_0| < r^2\},$$

$$T_r(z_0) = \{z = (x, t) \in R^n \times R | t_0 - 4r^2 < t < t_0 - r^2\}.$$

The fundamental solution of the backward heat equation

$$G_{z_0}=\frac{1}{[4\pi(t_0-t)]^{n/2}}\exp\Big(-\frac{|x-x_0|^2}{4(t_0-t)}\Big),\quad t<t_0.$$

Let $G(z)=G_{(0,0)}(z)$, $T_r=T_r(0,0)$, $P_r=P_r(0,0)$, $B_r=B_r(0)$, and write $B_r^+(x_0)=B_r(x_0)\cap R_+^n$, $P_r^+(z_0)=P_r(z_0)\cap R_+^n$, $T_r^+(z_0)=T_r(z_0)\cap R_+^n$.

For $z_0\in\partial R_+^n\times R_+$, we define

$$\Psi_\beta^+(R,u,z)=\frac12\int_{T^+_{\beta R}(z)}|\nabla u|^2G_z\varphi_\beta^2, \tag{2.1}$$

where $\varphi_\beta(x)=\varphi(\frac{x-x_0}{\varphi})$, and $\varphi\in C_0^\infty(B_{1/2}(0))$ is a cut-off function such that $0\le\varphi\le1$ and $\varphi=0$ on $B_{1/4}$, and $\beta>0$ is any fixed constant.

We have

Lemma 2.1 (Monotonicity inequality). *There exists a $c(n,\|f\|_{C^2})$ such that for any $z_0=(x_0,t_0)\in\partial R_+^n\times R_+$ and any $0<R_1<R_2\le\min\Big(\frac{\sqrt{t_0}}{2\beta},\frac14\Big)$*

$$\Psi_\beta^+(R_1,u,z_0)\le\exp(c(R_2-R_1))\Psi_\beta^+(R_2,u,z_0)+c(R_2-R_1)\Big(\beta^2+\beta^{-n}\int_{P^+_{\frac{\beta}{2}(z_0)}}|\nabla u|^2\Big). \tag{2.2}$$

Remark. Our monotonicity inequality at boundary differs from that in [3].

Proof of Lemma 2.1. We may assume $z_0=(0,0)$. Defining $u_R(x,t)=u(Rx,R^2t)$, we have $\Psi_\beta^+(R,u,0)=\Psi_\beta^+(1,u_R,0)$. As in [3] we have

$$\begin{aligned}\frac{d\Psi_\beta^+(R,u,0)}{dR}&=\int_{T_\beta^+}\nabla u_R\nabla\Big(\frac{du_R}{dR}-x'\nabla_{x'}f(Rx',0)\Big)G\varphi_\beta^2\\&\quad+\int_{T_1^+}\nabla u_k\nabla(x'\nabla_{x'}f(Rx',0))G\varphi_\beta^2+\int_{T_\beta^+}|\nabla u_k|^2G\varphi_\beta(Rx)\nabla\varphi\Big(\frac{Rx}{\beta}\Big)\frac{x}{\beta}\\&=I+II+III.\end{aligned}\tag{2.3}$$

We have that

$$\begin{aligned}I&=-\int_{T_\beta^+}\Big(\Delta u_R+\frac{x\nabla u_R}{2t}\Big)\Big(\frac{du_R}{dR}-x'\nabla_{x'}f(Rx',0)\Big)G\varphi_\beta^2\\&=\int_{T_\beta^+}\frac{-R}{2t}\Big|\frac{du_R}{dR}\Big|^2G\varphi_\beta^2+\int_{T_\beta^+}\frac{R}{2t}\frac{du_R}{dR}(x'\nabla_{x'}f(Rx',0))G\varphi_\beta^2\\&\quad-\int_{T_\beta^+}A(u_R)(\nabla u_R,\nabla u_R)(x'\nabla_{x'}f(Rx',0))G\varphi_\beta^2\\&\quad-2\int_{T_\beta^+}\nabla u_R\frac{du_R}{dR}G\varphi_\beta(Rx)\nabla\varphi\Big(\frac{Rx}{\beta}\Big)\frac{R}{\beta}\\&\quad+2\int_{T_\beta^+}\nabla u_RG\varphi_\beta(Rx)\nabla\varphi\Big(\frac{Rx}{\beta}\Big)\frac{2R}{\beta}(x'\nabla_{x'}f(Rx',0))\\&=I_1+I_2+I_3+I_4+I_5,\end{aligned}\tag{2.4}$$

$$\begin{aligned}I_2&\le\frac{I_1}{5}+c\int_{T_\beta^+}\frac{R}{2|t|}|x'\nabla_{x'}f(Rx',0)|G\varphi_\beta^2\\&\le\frac{I_1}{5}+c\int_{-4\beta^2}^{-\beta^2}\int_{|x|\le\frac{\beta}{2R}}|x'|^2|\nabla f(Rx',0)|^2G\frac{R^2}{|t|}.\end{aligned}\tag{2.5}$$

Since

$$\int_{-4\beta^2}^{-\beta^2}\int_{|x|\le\frac{\beta}{2R}}|x'|^2|\nabla f(Rx',0)|^2G\frac{R^2}{|t|}$$
$$\le c(n,\|f\|_{C^2})\int_{-4\beta^2}^{-\beta^2}|t|^{-\frac{n+2}{2}}\int_{|x|\le\frac{\beta}{2R}}|x'|^2\exp\Big(-\frac{|x|^2}{4|t|}\Big)$$
$$\le c\int_{-4\beta^2}^{-\beta^2}\int_0^{\frac{\beta}{4R\sqrt{|t|}}}\exp(-r^2)r^{n+1}\le c\beta^2, \tag{2.6}$$

we have

$$I_2\le\frac{I_1}{5}+c(n,\|f\|_{C^2})\beta^2, \tag{2.7}$$
$$I_3\le c\int_{T_\beta^+}|\nabla u_R|^2G|x||\nabla f|R\varphi_\beta^2\le\int_{-4\beta^2}^{-\beta^2}\int_{|x|\le\frac{\beta}{2R}}|\nabla u_R|^2G|x||\nabla f|R$$
$$\le c(n,N,\|f\|_{C^2})\beta\Psi_\beta^+. \tag{2.8}$$

Similarly we may obtain

$$I\ge -c\Psi_\beta^+-c\beta^2-c\beta^{-n}\int_{P^+_{\beta/2}}|\nabla u|^2, \tag{2.9}$$
$$II\le\Psi_\beta^++c\beta^2, \tag{2.10}$$
$$III\le\Psi_\beta^++c\beta^{-n}\int_{P^+_{\beta/2}}|\nabla u|^2, \tag{2.11}$$

so that

$$\frac{d\Psi_\beta^+}{dR}\ge -c\Psi_\beta^+-c\beta^2-c\beta^{-n}\int_{P^+_{\beta/2}}|\nabla u|^2, \tag{2.12}$$

where we use the fact $I_1\ge 0$. This easily finishes the proof of the lemma.

Set $E_0^+(r,u,z)=\frac{1}{r^n}\int_{P_r^+(z)}|\nabla u|^2$.

Lemma 2.2. *There exists a constant $K>0$, depending only on n, such that*

$$E_0^+(r,u,z)\le KE_0^+(r_1,u,z_1). \tag{2.13}$$

For $z\in P^+_{ar_1}(z_1)$ and $0<r<br_1$, where a and b are positive constants satisfying $a+2b<1/2$.

Proof. When we have the monotonicity inequality (2.2), the proof of (2.13) is easy (see Lemma 2.2 in [6]).

The following lemma is an energy inequality at boundary.

Lemma 2.3 (Energy inequality). For any $z_0\in\partial R^n_+\times R_+$ and $\phi\in C_0^\infty(R^n)$ and $0\le t_1\le t_2<\infty$, it is true that

$$\int_{R^n_+\times[t_1,t_2]}|u_t|^2\phi^2+\Big(\int_{R^n_+}\phi^2|\nabla u|^2\Big)(t_2)\le\Big(\int_{R^n_+}\phi^2|\nabla u|^2\Big)(t_1)+4\int_{R^n_+\times[t_1,t_2]}|Du|^2|\nabla\phi|^2. \tag{2.14}$$

Proof. Since $u_t\in T_uN$ and $A(u)(\nabla u,\nabla u)\perp T_uN$, we have

$$\int_{t_1}^{t_2}\int_{R^n_+}u_t\bullet u_t\phi^2-\Delta u\bullet u_t\phi^2=0,$$

but

$$-\int_{t_1}^{t_2}\int_{R_+^n}\Delta u\bullet u_t\phi^2=\int_{t_1}^{t_2}\int_{R_+^n}\frac{1}{2}\partial_t|\nabla u|^2\phi^2+2\nabla u\bullet\nabla\phi u_t\phi-\int_{t_1}^{t_2}\int_{\partial R_+^n}\partial_\nu u u_t\phi^2$$
$$\le\frac{1}{2}\Big(\int_{R_+^n}|\nabla u|^2\Big)(t_2)-\frac{1}{2}\Big(\int_{R_+^n}|\nabla u|^2\Big)(t_1)$$
$$+2\Big(\int_{R_+^n\times[t_1,t_2]}|u_t|^2\phi^2\Big)^{\frac{1}{2}}\Big(\int_{R_+^n\times[t_1,t_2]}|\nabla u|^2|\nabla\phi|^2\Big)^{\frac{1}{2}}. \tag{2.15}$$

From the energy inequality above we can get

Lemma 2.4. *There exists a constant* $c(n)$ *such that*

$$r^2\int_{P^+_{\frac{r}{2}}(z)}|\partial_t u|^2\le c(n)\int_{P_r^+(z)}|\nabla u|^2, \tag{2.16}$$

for $z=(x,t)\in\partial R_+^n\times R_+$, $0<r\le\sqrt{t}$.

Proof. Via Fubini's theorem we have

$$\int_{P_r^+(z)}|\nabla u|^2=\int_{B_r^+(x)}\int_{t-r^2}^{t+r^2}|\nabla u|^2\ge\int_{B_r^+(x)}\int_{t-(\frac{1}{2}r)^2}^{t-(\frac{3}{4}r)^2}|\nabla u|^2$$
$$\ge\Big(\int_{B_r^+(x)}|\nabla u|^2\Big)(t-\alpha^2r^2)r^2\frac{5}{16},$$

where $\alpha\in\left(\frac{1}{2},\frac{3}{4}\right)$. Choose a smooth cut-off function $\phi\in C_0^\infty(R^n)$ such that $\phi=1$ in $B_{\alpha r}(x)$, and $\phi=0$ in outside $B_r(x)$, and $0\le\phi\le1$, and $|\nabla\phi|\le\frac{c}{r}$. From energy inequality we obtain

$$\int_{P^+_{\frac{r}{2}}(z)}|\partial_t u|^2\le\int_{P^+_{\alpha r}(z)}|\partial_t u|^2\phi^2\le\int_{t-(\alpha r)^2}^{t+(\alpha r)^2}\int_{R_+^n}|u_t|^2\phi^2$$
$$\le\Big(\int_{R_+^n}|\nabla u|^2\Big)(t-(\alpha r)^2)+4\int_{t-(\alpha r)^2}^{t+(\alpha r)^2}\int_{R_+^n}|u_t|^2|\nabla\phi|^2\le\frac{c}{r^2}\int_{P_r^+(z)}|\nabla u|^2.$$

The lemma is proved.

§3. Energy Decay and Compactness Lemma

As the usual blow-up argument, the key part for the proof of Theorem 1.1 is the following small energy decay lemma. Set $E^+(r,u,z)=\frac{1}{r^n}\int_{P_r^+(z)}|\nabla u|^2+r$. We have

Lemma 3.1. *There exist constants* $0<\epsilon_0$, $\tau<1$ *such that if* $E^+(r,u,z)\le\epsilon_0^2$, *then*

$$E^+(r,u,z)\le\frac{1}{2}E^+(r,u,z), \tag{3.1}$$

for any $z\in R_+^n\times R_+$ *and* $0<r<\sqrt{t}$.

Proof. We argue by contradiction. Fixed $\tau\in(0,1)$, were (3.1) false, there would exist $\{z_k\}\subset\partial R_+^n\times R_+$ and $0<r_k<\sqrt{t_k}$ such that

$$E^+(r_k,u,z_k)=\lambda_k^2\to0, \tag{3.2}$$

whereas

$$E^+(\tau r_k,u,z_k)>\frac{1}{2}\lambda_k^2. \tag{3.3}$$

We rescale our variables to the unit parabolic half ball $P_1^+ \subset R_+^n \times R$ as follows. If $z=(x,t)\in P_1^+$, write

$$v_k(x,t) \equiv \frac{u_k(x,t)-f_k}{\lambda_k}, \tag{3.4}$$

where $u_k(x,t) = u(y,s) = u(x_k + r_k x, t_k + r_k^2 t)$, $f_k = f(y',0) = f(x_k' + r_k x', 0)$, and $y = x_k + r_k x$, $s = t_k + r_k^2 t$.

Since

$$\nabla v_k = \frac{1}{\lambda_k}(\nabla u r_k - \nabla_y f(y',0) r_k), \partial_t v_k = \frac{1}{\lambda_k}\partial_s u r_k,$$

we see that

$$\begin{aligned}\int_{P_1^+} |\nabla v_k|^2 &\le \frac{2}{\lambda_k^2}\Big[\frac{1}{r_k^n}\int_{P_{r_k}^+(z_k)} |\nabla u| + \int_{P_{r_k}^+(z_k)} |\nabla f|^2\Big] \\ &\le \frac{2}{\lambda_k^2}(\lambda_k^2 + c(\|f\|_{C^2}) r_k^2) \le c(\|f\|_{C^2}),\end{aligned} \tag{3.5}$$

where we use the fact that $\frac{r_k}{\lambda_k^2} \le 1$, $r_k \le 1$. And since $u_k(0,t) - f_k(0,0) = 0$, we have

$$|u_k(x,t) - f_k(x',0)| \le r_k |\nabla u| + r_k^2 |\partial_s u| + r_k |\nabla f|,$$

so that

$$\int_{P_{1/2}^+} |\nabla v_k|^2 \le \frac{2}{\lambda_k^2}\frac{1}{r_k^n}\int_{P_{\frac12 r_k}^+(z_k)} |\nabla u|^2 + |\nabla f|^2 + r_k^2 |\partial_s u|^2 \le c(\|f\|_{C^2}), \tag{3.6}$$

where we use Lemma 2.4. Similarly we obtain

$$\int_{P_{1/2}^+} |\partial_t v_k|^2 \le c(n). \tag{3.7}$$

But

$$\begin{aligned}\frac{1}{\tau^n}\int_{P_\tau^+} |\nabla v_k|^2 &\ge \frac{1}{\lambda_k^2}\frac{1}{(\tau r_k)^n}\int_{P_{\tau r_k}^+(z_k)} \frac{1}{2}|\nabla u|^2 - |\nabla f|^2 \\ &> \frac{1}{4} - \frac{\tau r_k}{2\lambda_k^2} - \frac{\|f\|_{C^2}^2 (\tau r_k)^2}{\lambda_k^2} \ge \frac14 - \frac{\tau}{2} - \tau^2 \|f\|_{C^2}^2 \ge \frac{1}{12},\end{aligned} \tag{3.8}$$

where we assume $\tau \le \alpha_0 \le \frac14$ for some α_0 such that $\frac{\alpha_0}{2} + \alpha_0^2 \|f\|_{C^2}^2 \le \frac{1}{16}$. Hence there exists a subsequence such that

$$v_k \to v \quad \text{strongly in } L^2(P_{1/2}^+; R^m), \tag{3.9}$$

$$\nabla v_k \rightharpoonup \nabla v \quad \text{weakly in } L^2(P_{1/2}^+; P^{n\times m}), \tag{3.10}$$

$$\partial_t v_k \rightharpoonup \partial_t v \quad \text{weakly in } L^2(P_{1/2}^+; R^m). \tag{3.11}$$

Since v_k is a weak solution of the equation on $P_{1/2}^+$,

$$\begin{aligned}\partial_t v_k - \Delta v_k &= \lambda_k A(u_k)(\nabla v_k, \nabla v_k) + 2 r_k A(u_k)(\nabla v_k, \nabla f) \\ &\quad + \frac{r_k^2}{\lambda_k} A(u_k)(\nabla f, \nabla f) + \frac{r_k^2}{\lambda_k}\Delta f, \\ v_k|_{P_{1/2}^+ \cap \partial R_+^n} &= 0.\end{aligned}$$

Combining (3.4)-(3.10) and letting $\frac{r_k}{\lambda_k^2} \le 1$, $r_k \to 0$ one can get that v is a weak and smooth solution of the following equation

$$\partial_t v - \Delta v = 0 \qquad \text{in } P_{1/2}^+, \quad v|_{P_{1/2}^+ \cap \partial R_+^n} = 0. \tag{3.12}$$

Moreover

$$\|\nabla v\|_{L^\infty(P^+_{1/4})} \le c\int_{P^+_{1/2}} |\nabla v|^2 \le c. \tag{3.13}$$

Thus we have that for τ sufficient small

$$\frac{1}{\tau^n}\int_{P^+_\tau} |\nabla v|^2 \le c\tau^2 < \frac{1}{12}. \tag{3.14}$$

If we can prove

Lemma 3.2 (Compactness).

$$\nabla v_k \to \nabla v \quad in\ L^2(P^+_{\frac{1}{8}}), \tag{3.15}$$

then from (3.15) and (3.3) we get a contradiction with (3.14).

In a way similar to [20] we can prove the compactness lemma. For the reader's convenience we give a complete proof in Appendix B.

Finally using Lemma 3.1 and interior partial regularity (see [20]), we can finish the proof of Theorem 1.1 by the standard method as in [14] or [6].

Appendix A

In this appendix, we introduce the parabolic Hardy space.

Definition A.1. *Let Ω be an open set of R^{n+1} and $f \in L^1_{\rm loc}(\Omega; R^m)$. We call that f belongs to the local Hardy space $\mathcal{H}^1_{\rm loc}(\Omega; R^m)$ if for any compact subset $K \subset\subset \Omega$ there exists an $\epsilon > 0$ such that $\int_K \sup\limits_{0<r<\epsilon}\sup\limits_{\phi\in\Lambda} |\phi_r * f| < \infty$, where*

$$\Lambda = \Big\{\phi \in C_0^\infty(R^{n+1}) : supp(\phi) \subset P_1, \int_{R^{n+1}} \phi = 1\Big\},$$

and for $\phi \in \Lambda$, $\phi_r(z) \equiv r^{-(n+2)}\phi\Big(\frac{x-y}{r}, \frac{t}{r^2}\Big)$ for $r > 0$, $z = (x,t) \in R^n \times R$.

Because $\phi_r * f$ is well defined, and $\mathrm{supp}(\phi * f) \subset \Omega$ for all $r < \rho(z, \partial\Omega)$, we associate for each $K \subset\subset \Omega$

$$\|f\|_{\mathcal{H}^1(K)} = \int_K \sup_{0<r<\epsilon<\rho(z,\partial\Omega)} \sup_{\phi\in\Lambda} |\phi_r * f|. \tag{A.1}$$

When $\Omega = R^{n+1}$, ϵ can be taken to be infinity, then the parabolic Hardy space

$$\mathcal{H}^1(R^{n+1}; R^m) = \Big\{f \in L^1_{\rm loc}(R^{n+1}) : \|f^*\|_{L^1} < \infty\Big\},$$

where $f^*(z) = \sup\limits_{0<r<\infty}\sup\limits_{\phi\in\Lambda} |\phi_r * f|(z)$.

Definition A.2. *For $f \in L_{\rm loc}(R^{n+1}; R^m)$, set*

$$f_*(z) = \sup_{r>0} \frac{1}{|P_r(z)|}\int_{P_r(z)} |f - (f)_{z,r}|.$$

We call that f belongs to $BMO_\rho(R^{n+1}; R^m)$ if $f_*(z) \in L^\infty(R^{n+1})$ and define the BMO norm of f, $\|f\|_{\rm BMO} \equiv \|f_*\|_{L^\infty(R^{n+1})}$, where

$$(f)_{z,r} = \frac{1}{|P_r(z)|}\int_{P_r(z)} |f - (f)_{z,r}|,$$

$|P_r(z)|$ is $(n+1)$-dimensional Lebesgue measure of $P_r(z)$.

Fefferman's duality theorem claims:

$$\mathcal{H}^1_\rho(R^{n+1})^* = \mathrm{BMO}_\rho(R^{n+1}). \tag{A.2}$$

Now let $\Omega \subset R^n$ and $0 < T < \infty$, and write $\Omega_T = \Omega \times [0,T]$. Let $f, g \in L^2(R^1; H^1(R^n))$, and write

$$\begin{aligned}\{f,g\}_{ij} &\equiv (f(x,t)g(x,t)_{x^i})_{x^j} - (f(x,t)g(x,t)_{x^j})_{x^i} \\ &= f(x,t)_{x^j} g(x,t)_{x^i} - f(x,t)_{x^i} g(x,t)_{x^j}. \end{aligned} \tag{A.3}$$

Then $\{f,g\}_{ij} \in L^1(R; L^1(R^n))$, and thanks to its divergence structure, we have the following lemma.

Lemma A.1. *Let $g \in L^2(0,T; H^1(\Omega, R^m))$, $f \in H^1(\Omega_T; R^m)$. Then*

$$\{f,g\}_{ij} \in \mathcal{H}^1_{\rm loc}(\Omega_T, R^m)$$

and for every $K \subset\subset \Omega_T$, it is true that

$$\|\{f,g\}_{ij}\|_{\mathcal{H}^1(K)} \le c(K,n)\Big(\int_{\Omega_T} |\nabla g|^2\Big)^{1/2}\Big(\int_{\Omega_T} |\nabla f|^2 + |\partial_t f|^2\Big)^{1/2}. \tag{A.4}$$

Proof. For any $\phi \in \Lambda$, $z = (x,t) \in K$ and $r > 0$, we have

$$\begin{aligned}
&\phi_r * \{f,g\}_{ij}(z) \\
&= r^{-(n+2)} \int_{R^{n+1}} \phi\Big(\frac{x-y}{r}, \frac{t-\tau}{r^2}\Big)((f(y,\tau)g(y,\tau)_{y^i})_{y^j} - (f(y,\tau)g(y,\tau)_{y^j})_{y^i})dyd\tau \\
&= r^{-(n+1)} \int_{R^{n+1}} (f - (f)_{z,r})(g_{y^j}\phi_{y^i} - g_{y^i}\phi_{y^j})(y,\tau)dyd\tau \\
&\le cr^{-(n+3)}\Big(\int_{P_r(z)} |f-(f)_{z,r}|^p\Big)^{\frac1p}\Big(\int_{P_r(z)} |\nabla g|^{p'}\Big)^{\frac1{p'}} \\
&\le cr^{-(n+3)}\Big(\int_{P_r(z)} |\nabla g|^{p'}\Big)^{\frac1{p'}}\Big[\Big(\int_{P_r(z)} |\nabla f|^{p^*}\Big)^{\frac1{p^*}} + \Big(\int_{P_r(z)} r^{p^*}|\partial_t f|^{p^*}\Big)^{\frac1{p^*}}\Big] \\
&\le c\Big(\int_{P_r(z)} |\nabla g|^{p'}\Big)^{\frac1{p'}}\Big[\Big(\int_{P_r(z)} |\nabla f|^{p^*}\Big)^{\frac1{p^*}} + \Big(\int_{P_r(z)} r^{p^*}|\partial_t f|^{p^*}\Big)^{\frac1{p^*}}\Big],
\end{aligned} \tag{A.5}$$

where we have used Sobolev-Poincaré inequality, and $1 < p < 2$, $\frac1p + \frac1{p'} = 1$, $p^* = \frac{p(n+2)}{(n+2)+p}$. Now we introduce Hardy-Littlewood and generalized Hardy-Littlewood maximal functions

$$M(f)(z) = \sup_{P_r(z)\subset\Omega_T} \Big\{\frac{1}{|P_r(z)|}\int_{P_r(z)} |f|\Big\}, \quad M_\delta(f) = \sup_{P_r(z)\subset\Omega_T}\Big\{\frac{r^\delta}{|P_r(z)|}\int_{P_r(z)} |f|\Big\}.$$

From Theorem 3 in [22] or [23] we have that

$$\|M_\delta(f)\|_{L^k} \le c\|f\|_{L^l}, \quad \text{if } 1 < l \le k, \frac1k = \frac1l - \frac{\delta}{n+2}. \tag{A.6}$$

Hence using (3.15) we obtain from (2.14)

$$\begin{aligned}
&\int_K \sup_{0<r<\epsilon<\rho(z,\partial\Omega_T)} \sup_{\phi\in\Lambda} |\phi_r * f|(z) \\
&\le c(n)\int_K M(|\nabla g|^{p'})^{\frac1{p'}}[M(|\nabla f|^{p^*})^{\frac1{p^*}} + M_{p^*}(|\partial_t f|^{p^*})^{\frac1{p^*}}] \\
&\le c(n)\Big[\Big(\int_K M(|\nabla g|^{p'})^{\frac2{p'}}\Big)^{1/2}\Big(\int_K M(|\nabla f|^{p^*})^{\frac2{p^*}}\Big)^{1/2} \\
&\quad + \Big(\int_K M(|\nabla g|^{p'})^{\frac{q'}{p'}}\Big)^{1/q'}\Big(\int_K M_{p^*}(|\partial_t f|^{p^*})^{\frac{q}{p^*}}\Big)^{1/q}\Big].
\end{aligned} \tag{A.7}$$

Now taking $1 < \frac{2(n+2)}{(n+2)+2} < p' < 2$, so $1 < p^* < 2$, we have

$$\|M(|\nabla g|^{p'})\|_{L^{\frac{2}{p'}}}^{\frac{2}{p'}} = \int_K M(|\nabla g|^{p'})^{\frac{2}{p'}} \le c\||\nabla g|^{p'}\|_{L^{\frac{2}{P'}}}^{\frac{2}{p'}} \le \int_{\Omega_T} |\nabla g|^2, \tag{A.8}$$

$$\|M(|\nabla f|^{p^*})\|_{L^{\frac{2}{p^*}}}^{\frac{2}{p^*}} = \int_K M(|\nabla f|^{p^*})^{\frac{2}{p^*}} \le c\||\nabla f|^{p^*}\|_{L^{\frac{2}{P^*}}}^{\frac{2}{p^*}} \le \int_{\Omega_T} |\nabla f|^2. \tag{A.9}$$

Choosing q' such that $\frac{1}{q} + \frac{1}{q'} = 1$ and $q' > p'$, $k = \frac{q}{p^*} > l = \frac{q(n+2+p)}{p(n+2+q)}$, furthermore $p < q < \frac{2(n+2)}{n}$, $q' < 2$, $\frac{q}{p^*} > 1$, $l > 1$, we obtain

$$\|M(|\nabla g|^{p'})\|_{L^{\frac{q'}{p}}}^{\frac{q'}{p'}} \le c\||\nabla g|^{p'}\|_{\frac{q'}{p'}}^{\frac{q'}{p'}} \le c|K|^{1-\frac{q'}{2}}\Big(\int_K |\nabla g|^2\Big)^{\frac{q'}{2}}, \tag{A.10}$$

$$\begin{aligned}\|M_{p^*}(|\partial_t f|^{p^*})\|_{L^{\frac{q}{p^*}}}^{\frac{q}{p^*}} &\le c\Big(\int_K |\partial_t f|^{p^* l}\Big)^{\frac{q}{p^* l}} \le c\Big(\int_K |\partial_t f|^2\Big)^{\frac{q}{p^* l}\frac{p^* l}{2}} |K|^{(1-\frac{p^* l}{2})\frac{q}{lp^*}}\\ &\le c(K)\Big(\int_K |\partial_t f|^2\Big)^{\frac{q}{2}}.\end{aligned} \tag{A.11}$$

Therefore the lemma is proved by virtue of (2.16)-(A.11).

In the end of the section, we introduce the relation of $\mathcal{H}_{\rm loc}$ and $\mathcal{H}$. The following characterization of function $f \in \mathcal{H}^1_{\rm loc}(P)$ is essentially due to Semmes[24].

Lemma A.2. *Let P be an open set in R^{n+1}. Then $f \in \mathcal{H}^1_{\rm loc}(P)$ if and only if for every $\eta \in C_0^\infty(P)$ with $\int \eta \ne 0$, $\eta(f-\nu) \in \mathcal{H}^1(R^{n+1})$ and*

$$\|\eta(f-\nu)\|_{\mathcal{H}^1(R^{n+1})} \le c(K)(1 + \|f\|_{\mathcal{H}^1(K)}),$$

where $\nu = \frac{\int \eta f}{\int \eta}$, and $K = spt(\eta)$.

Appendix B

In this appendix we give the proof of Lemma 3.2.

Since $v_k, v = 0$ on the boundary $P_{1/2} \cap \partial R^{n+1}$, we may extend the v_k and v in $P_{1/2}$ by zero; the functions extended are denoted yet by v_k and v.

let ζ be a smooth function from R^{n+1} to R_+ such that $\zeta = 1$ on $P_{1/4}$, $0 \le \zeta \le 1$, $\zeta \in C_0^\infty(P_{1/2})$. For every l, let $w_l(u_k)$ be the 1-form defined on $P_{1/2}$ by

$$w_l(u_k) = e_l \wedge d(v_k - v),$$

and $\widetilde{w_l}(u_k)$ be the 1-form on R^{n+1} defined by $\widetilde{w}_l(u_k) = e_l \wedge d(\zeta(v_k - v))$. On $P_{1/4}$, $\widetilde{w}_l = w_l$.

We use the Hodge decomposition (see [18] or [20])

$$\widetilde{w}_l(u_k) = d\alpha_{lk} + d^*\beta_{lk}, \quad d^*\alpha_{lk} = d\beta_{lk} = 0, \tag{B.1}$$

where the differential forms $\alpha_{lk} \in L^2(R : H^1(R^n; \Lambda^0))$ and $\beta_{lk} \in L^2(R; H^1(R^n; \Lambda^2))$ and

$$\|\alpha_{lk}\|_{L^2(R;H^1(R^n;\Lambda^0))} + \|\beta_{lk}\|_{L^2(R;H^1(R^n;\Lambda^2))} \le c(n)\|\widetilde{w}_l(u_k)\|_{L^2(R;L^2(R^n;\Lambda^l))}. \tag{B.2}$$

Clearly we have

$$|w_l(u_k)| = |\langle \nabla(v_k - v), e_l\rangle| \le c(|\nabla \alpha_{lk}| + |\nabla \beta_{lk}|) \quad \text{in } P_{1/4}. \tag{B.3}$$

Since $d\widetilde{w}_l = dd^*\beta_l$, the coefficients β^{ij}_{lk} of β_{lk} in the standard basis satisfy the equation

$$\Delta \beta^{ij}_{lk} = \{e_l, \zeta(v_k - v)\}_{ij}, \tag{B.4}$$

where we use the notation $\{f, g\}_{ij} \equiv f(x,t)_{x^i} g(x,t)_{x^j} - f(x,t)_{x^j} g(x,t)_{x^i}$. We have the following lemma from [6]

Lemma B.1. *The sequence $\{\zeta v_k\}$ is bounded in $BMO(R^{n+1})$.*

Thus we obtain firstly

Lemma B.2. *There is a constant c, independent of k, such that*

$$\int_{R^{n+1}} |\nabla\beta_{lk}^{ij}|^2 \le c\lambda_k. \tag{B.5}$$

Proof. Multiplying (B.4) by β_{lk}^{ij} and integrating on R^{n+1}, we get

$$\int_{R^{n+1}} |\nabla\beta_{lk}^{ij}| = -\int_{R^{n+1}} \{e_l, \zeta^2(v_k - v)\}_{ij}\beta_{lk}^{ij} = \int_{R^{n+1}} \{e_l, \beta_{lk}^{ij}\}\zeta^2(v_k - v).$$

From Lemma A.1 we know that $\Big\{\lambda_k^{-1}e_l, \beta_{lk}^{ij}\Big\}_{ij} \in \mathcal{H}^1_{\rm loc}(P_{7/16})$ and

$$\begin{aligned}\|\{\lambda_k^{-1}e_l, \beta_{lk}^{ij}\}_{ij}\|_{\mathcal{H}^1_{\rm loc}(P_{7/16})} &\le c\Big(\int_{P_{7/16}} |\nabla\beta_{lk}^{ij}|^2\Big)^{1/2}\Big(\int_{P_{7/16}} |\partial_t v_k|^2 + |\nabla v_k|^2\Big)^{1/2}\\ &\le c\Big(\int_{P_{7/16}} |\nabla\beta_{lk}^{ij}|^2\Big)^{1/2},\end{aligned}$$

where we use that $|\nabla_z e_l| \le c\lambda_k|\nabla_z v_k|$. From Semmes theorem (see Lemma A.2) we have

$$\begin{aligned}&\int_{R^{n+1}} |\nabla\beta_{lk}^{ij}|^2 = \lambda_k \int_{R^{n+1}} \zeta(\Big\{\lambda_k^{-1}e_l, \beta_{lk}^{ij}\Big\}_{ij} - \nu_{lk})\zeta(v_k - v) + \lambda_k \int_{R^{n+1}} \nu_{lk}\zeta^2(v_k - v)\\ &\le \lambda_k\|\zeta(\Big\{\lambda_k^{-1}e_l, \beta_{lk}^{ij}\Big\}_{ij} - \nu_{lk})\|_{\mathcal{H}^1(R^{n+1})}\|\zeta(v_k - v)\|_{\mathrm{BMO}(R^{n+1})} + \lambda_k\int_{R^{n+1}} |\nu_{lk}|\zeta^2|v_k - v|\\ &\le c\lambda_k(1 + \|\Big\{\lambda_k^{-1}e_l, \beta_{lk}^{ij}\Big\}\|_{\mathcal{H}^1(P_{7/16})}) + c\lambda_k\Big(\int_{P_{7/16}} |\nabla\beta_{lk}^{ij}|^2\Big)^{1/2}\int_{R^{n+1}} \zeta^2|v_k - v|\\ &\le c\lambda_k\Big(1 + \Big(\int_{P_{7/16}} |\nabla\beta_{lk}^{ij}|^2\Big)^{1/2}\Big).\end{aligned}$$

Now from Hölder inequality we easily complete the proof.

Secondly we estimate the α_{lk}.

Step 1. For any $z_0 \in P_{7/16}$ and $0 < r < 1/64$ there exists a constant c_1,independent of k, such that

$$\int_{P_r(z_0)} |\nabla\beta_{lk}^{ij}|^2 \le c_1 r^n. \tag{B.6}$$

We have in fact that $\lim\limits_{k\to\infty}\int_{R^{n+1}} |\nabla\beta_{lk}^{ij}|^2 = 0$ implies that $|\nabla\beta_{lk}^{ij}|^2$ has the equicontinuous integral, i.e., for every $r > 0$, there exists $\delta(r) > 0$ such that $\int_E |\nabla\beta_{lk}^{ij}|^2 < r^n$, if the measure of E $|E| < \delta$, and that there exists a subsequence such that $|\nabla\beta_{lk}^{ij}|^2 \to 0$ a.e. Consequently, for this $\delta(r) > 0$, there exists a closed subset $D \subset P_{1/2}$ such that $|P_{1/2}\backslash D| < \delta$ and $|\nabla\beta_{lk}^{ij}|^2 \le 1$ in D, by Yegonoff' theorem. So

$$\int_{P_r(z_0)} |\nabla\beta_{lk}^{ij}|^2 = \Big(\int_{P_r(z_0)\cap D} + \int_{P_r(z_0)\backslash D}\Big)|\nabla\beta_{lk}^{ij}|^2 \le |P_r(z_0)\cap D| + r^n \le cr^{n+2} + r^n \le c_1 r^n.$$

Since $|\nabla\alpha_{lk}| \le c(|\nabla(v_k - v)| + |\nabla\beta_{lk}^{ij}|)$ on $P_{1/4}$, taking $z_1 = (0,0)$, $r_1 = 1$, and $u = u_k$, $a = 7/16$, $b = 1/32$ in Lemma 2.2, we obtain

$$\begin{aligned}&\frac{1}{r^n}\int_{P_r(z_0)} |\nabla\alpha_{lk}|^2 \le \frac{c}{r^n}\int_{P_r(z_0)} |\nabla(v_k - v)|^2 + |\nabla\beta_{lk}^{ij}|^2\\ &\le \frac{c}{r^n}\int_{P_r(z_0)} |\nabla v_k|^2 + |\nabla v|^2 + |\nabla\beta_{lk}^{ij}|^2 \le cK\int_{P_1} |\nabla v_k|^2 + c + c_1 \le c,\end{aligned}$$

where we use (B.6).

Thus

$$\frac{1}{|P_r(z_0)|}\int_{P_r(z_0)}|\alpha_{lk}-(\alpha_{lk})_{P_r(z_0)}|\le\Big(\frac{1}{|P_r(z_0)|}\int_{P_r(z_0)}|\alpha_{lk}-(\alpha_{lk})_{P_r(z_0)}|^2\Big)^{1/2}$$
$$\le\Big(\frac{1}{|P_r(z_0)|}\int_{P_r(z_0)}|\alpha_{lk}-(\alpha_{lk})_{B_r(x_0)}|^2\Big)^{1/2}\le c\Big(\frac{1}{r^n}\int_{P_r(z_0)}|\nabla\alpha_{lk}|^2\Big)^{1/2}\le c,$$

where $(f)_E=\frac{1}{E}\int_E f$.

The John-Nirenberg inequality implies that

$$\{\alpha_{lk}\}\text{ is bounded in }L^p(P_{7/16}),\quad 1\le p<\infty. \tag{B.7}$$

Step 2. Fix $1/8\le\tau<s\le 1/4$. Assume that $\eta\in C_0^\infty(P_s)$, $\eta=1$ in P_τ, $|\nabla_z\eta|\le\frac{c}{s-\tau}$. Then

$$\|\eta\alpha_{lk}\|_{\mathrm{BMO}(R^{n+1})}\le\frac{c}{s-\tau}. \tag{B.8}$$

If $z_o\in P_{s+\epsilon\tau}, r<\epsilon\tau/2, 0<\epsilon<1/9$, then $P_r(z_0)\subset P_{7/16}$. We have

$$\frac{1}{|P_r(z_0)|}\int_{P_r(z_0)}|\eta\alpha_{lk}-(\eta\alpha_{lk})_{P_r(z_0)}|$$
$$\le\frac{1}{|P_r(z_0)|}\int_{P_r(z_0)}\eta|\alpha_{lk}-(\alpha_{lk})_{P_r(z_0)}|+\frac{1}{|P_r(z_0)|}\int_{P_r(z_0)}|(\eta\alpha_{lk})_{P_r(z_0)}-\eta(\alpha_{lk})_{P_r(z_0)}|$$
$$\le\frac{1}{|P_r(z_0)|}\int_{P_r(z_0)}|\alpha_{lk}-(\alpha_{lk})_{P_r(z_0)}|+\int_{P_r(z_0)}\int_{P_r(z_0)}|\eta(z)-\eta(z')||\alpha_{lk}(z)|$$
$$\le c+\frac{cr}{s-\tau}\int_{P_r(z_0)}|\alpha_{lk}|\le c+\frac{c}{r^{n+1}}\Big(\int_{P_r(z_0)}|\alpha_{lk}|^{n+2}\Big)^{1/(n+2)}r^{(n+2)(1-1/(n+2))}$$
$$\le c+\frac{c}{s-\tau}\le\frac{c}{s-\tau},$$

where we use (B.7) with $p=n+2$.

Since $\eta=0$ outside of P_s, the same inequality holds for $z_0\in R^{n+2}\backslash P_{s+\epsilon\tau}$ and $0<r<\epsilon\tau/2$. Step 3. We prove that for some $1<q<2$

$$\int_{P_{1/8}}|\nabla\alpha_{lk}|^2\le c\int_{P_{1/4}}|\alpha_{lk}-(\alpha_{lk})_{P_{1/4}}|^2+c\Big(\int_{P_{1/4}}|\partial_t(v_k-v)|^q\Big)^{1/q}+c\lambda_k. \tag{B.9}$$

Similar to the case of interior estimate[20], there are the orthonormal frame $\{e_1,\cdots,e_{\overline{m}}\}$ on TN and 2-form $\omega_{lm}\in L^2((0,\infty);H^1(B_r^+(x);\Lambda^2))$ such that for any $x\in\partial R^n_+, r>0$, on

$B_r^+(x) \times R_+$ it holds that

$$d^*\omega_{lm} = e_l \wedge de_m = e_l \bullet de_m, \tag{B.10}$$

$$\int_{B_r^+} |\nabla e_l|^2 \le \int_{B_r^+} \Big|\frac{\partial u^i}{\partial x_k} \nabla_{e_i} e_l\Big|^2 \le c \int_{B_r^+} |\nabla u|^2, \tag{B.11}$$

$$\int_0^\infty \int_{B_r^+} |\nabla \omega_{lm}|^2 \le c \int_0^\infty \int_{B_r^+} |\nabla u|^2, \tag{B.12}$$

$$\begin{aligned}\langle \nabla u, \nabla e_l\rangle &= \langle \nabla u \bullet e_m, \nabla e_l \bullet e_m\rangle = \langle du \bullet e_m, d^*\omega_{lm}\rangle \\ &= \langle d^*\omega_{lm}, du\rangle \bullet e_m,\end{aligned} \tag{B.13}$$

$$\langle d^*\omega_{lm}, du\rangle = - * \langle d(*\omega_{lm}) \wedge du\rangle = (-1)^{n+1} \sum_{i<j} \Big\{\omega_{lm}^{ij}, u\Big\}_{ij}, \tag{B.14}$$

$$\langle \partial_t u, e_l\rangle - \operatorname{div}\langle \nabla u, e_l\rangle = (-1)^n \sum_m \sum_{i<j} \Big\{\omega_{lm}^{ij}, u\Big\}_{ij} \bullet e_m. \tag{B.15}$$

Thus we can get that on $B_{r_k}^+(x_k) \times R_+$ for every k,

$$\langle \partial_t u, e_l(u)\rangle - \operatorname{div}\langle \nabla u, e_l(u)\rangle = -\langle \nabla u, \nabla e_l(u)\rangle, \tag{B.16}$$

$$\int_{P_{r_k}^+(z_k)} |\nabla e_l(u)|^2 \le c(n) \int_{P_{r_k}^+(z_k)} |\nabla u|^2, \tag{B.17}$$

$$\int_{P_{r_k}^+(x_k,t_k)} |\nabla \omega_{lm}|^2(y,s)dyds \le c(n) \int_{P_{r_k}^+(x_k,t_k)} |\nabla u|^2(y,s)dyds. \tag{B.18}$$

Equivalently on $P_{1/2}^+$

$$\langle \partial_t v_k, e_l(u_k)\rangle - \operatorname{div}\langle \nabla v_k, e_l(u_k)\rangle = -\langle \nabla v_k, \nabla e_l(u_k)\rangle + \frac{r_k^2}{\lambda_k}\langle \Delta f, e_l(u_k)\rangle, \tag{B.19}$$

$$\begin{aligned}\int_{P_1^+} |\nabla e_l(u_k)|^2 &\le c(n) \int_{P_1^+} |\nabla u_k|^2 \\ &\le c(n)\lambda_k^2 \int_{P_1^+} |\nabla v_k|^2 + r_k^2 |\nabla f|^2 \\ &\le c(n, \|f\|_{C^2})\lambda_k^2 \int_{P_1^+} (|\nabla v_k| + 1)^2,\end{aligned} \tag{B.20}$$

where we use that $\frac{r_k}{\lambda_k^2} \le 1$. And

$$\int_{P_1^+} |\nabla \omega_{k,lm}|^2 \le c(n) \int_{P_1^+} |\nabla u_k|^2 \le c(n, \|f\|_{C^2})\lambda_k^2 \int_{P_1^+} (|\nabla v_k| + 1)^2, \tag{B.21}$$

where $\omega_{k,lm}^{ij}(x,t) = \omega_{lm}^{ij}(x_k + r_k x, t_k + r_k^2 t)$. Furthermore, noticing that

$$\nabla u_k = \lambda_k \nabla v_k + \nabla f_k,$$

from (2.5) and (2.6) we have

$$\langle\nabla v_k(x,t),\nabla e_l[u_k(x,t)]\rangle=\frac{r_k^2}{\lambda_k}[\langle\nabla u,\nabla e_l(u)\rangle-\langle\nabla f,\nabla e_l(u)\rangle]$$

$$=\frac{1}{\lambda_k}[(-1)^{n+1}\sum_m\sum_{i<j}\left\{\omega_{k,lm}^{ij},u_k\right\}_{ij}\bullet e_m(x,t)-\frac{r_k^2}{\lambda_k}\langle\nabla f,\nabla e_l(u)\rangle \tag{B.22}$$

$$=(-1)^{n+1}\sum_m\sum_{i<j}\left\{\omega_{k,lm}^{ij},v_k\right\}_{ij}\bullet e_m(x,t)-\frac{r_k^2}{\lambda_k}\langle\nabla f,\nabla e_l(u)\rangle$$
$$+\frac{1}{\lambda_k}(-1)^{n+1}\sum_m\sum_{i<j}\left\{\omega_{k,lm}^{ij},f_k\right\}_{ij}\bullet e_m(x,t). \tag{B.23}$$

Thus on $P_{1/2}^+$ it holds that

$$\begin{aligned}&\langle\partial_t v_k,e_l(u_k)\rangle-\operatorname{div}\langle\nabla v_k,e_l(u_k)\rangle\\&=(-1)^n\sum_m\sum_{i<j}\left\{\omega_{k,lm}^{ij},v_k\right\}_{ij}\bullet e_m(x,t)\\&\quad+\frac{1}{\lambda_k}(-1)^n\sum_m\sum_{i<j}\left\{\omega_{k,lm}^{ij},f_k\right\}_{ij}\bullet e_m(x,t)\\&\quad+\frac{r_k^2}{\lambda_k}\langle\nabla f,\nabla e_l(u)\rangle+\frac{r_k^2}{\lambda_k}\langle\Delta f,e_l(u_k)\rangle.\end{aligned} \tag{B.24}$$

Equivalently from (3.12)

$$\begin{aligned}&\langle\partial_t(v_k-v),e_l(u_k)\rangle-\operatorname{div}\langle\nabla(v_k-v),e_l(u_k)\rangle\\&=(-1)^n\sum_m\sum_{i<j}\left\{\omega_{k,lm}^{ij},v_k\right\}_{ij}\bullet e_m(x,t)\\&\quad+\frac{1}{\lambda_k}(-1)^n\sum_m\sum_{i<j}\left\{\omega_{k,lm}^{ij},f_k\right\}_{ij}\bullet e_m(x,t)+\frac{r_k^2}{\lambda_k}\langle\nabla f,\nabla e_l(u)\rangle\\&\quad+\frac{r_k^2}{\lambda_k}\langle\Delta f,e_l(u_k)\rangle+\langle\nabla v,\nabla e_l(u_k)\rangle.\end{aligned} \tag{B.25}$$

Now

$$d^*\widetilde{w}_l(u_k)=d^*d\alpha_{lk}=\Delta\alpha_{lk}.$$

Since $v_k,v\equiv 0$ in the $P_{1/2}^-$, we may assume the equation above holds in $P_{1/2}$ for the sake of simplicity. In view of the equation above it holds in $P_{1/4}$ that

$$\begin{aligned}-\Delta\alpha_{lk}&=-\operatorname{div}\langle\nabla(v_k-v),e_l\rangle\\&=-\langle\partial_t(v_k-v),e_l(u_k)\rangle+\langle\nabla v,\nabla e_l\rangle\\&\quad+(-1)^{n+1}\sum_m\sum_{i<j}\left\{\omega_{k,lm}^{ij},v_k\right\}_{ij}\bullet e_m\\&\quad+\frac{1}{\lambda_k}(-1)^n\sum_m\sum_{i<j}\left\{\omega_{k,lm}^{ij},f_k\right\}_{ij}\bullet e_m(x,t)+\frac{r_k^2}{\lambda_k}\langle\nabla f,\nabla e_l(u)\rangle\\&\quad+\frac{r_k^2}{\lambda_k}\langle\Delta f,e_l(u_k)\rangle.\end{aligned} \tag{B.26}$$

Multiplying (B.26) by $\eta^2(\alpha_{lk}-(\alpha_{lk})_{P_s})$, we obtain

$$\begin{aligned}\int_{P_s}\eta^2|\nabla\alpha_{lk}|^2 &= -2\int_{P_s\setminus P_\tau}\eta\nabla\eta\nabla\alpha_{lk}(\alpha_{lk}-(\alpha_{lk})_{P_s})\\ &\quad-\int_{P_s}\langle\partial_t(v_k-v),e_l\rangle\eta^2(\alpha_{lk}-(\alpha_{lk})_{P_s})\\ &\quad-\int_{P_s}\langle\nabla v,\nabla e_l\rangle\eta^2(\alpha_{lk}-(\alpha_{lk})_{P_s})\\ &\quad+\int_{P_s}(-1)^{n+1}\sum_m\sum_{i<j}\Big\{\omega^{ij}_{k,lm},v_k\Big\}_{ij}\bullet e_m\eta^2(\alpha_{lk}-(\alpha_{lk})_{P_s})\\ &\quad+\int_{P_s}\frac{1}{\lambda_k}(-1)^n\sum_m\sum_{i<j}\Big\{\omega^{ij}_{k,lm},f_k\Big\}_{ij}\bullet e_m(x,t)\eta^2(\alpha_{lk}-(\alpha_{lk})_{P_s})\\ &\quad+\int_{P_s}+\frac{r_k^2}{\lambda_k}\langle\nabla f,\nabla e_l(u)\rangle\eta^2(\alpha_{lk}-(\alpha_{lk})_{P_s})\\ &\quad+\int_{P_s}\frac{r_k^2}{\lambda_k}\langle\Delta f,e_l(u_k)\rangle\eta^2(\alpha_{lk}-(\alpha_{lk})_{P_s})\\ &= I_1+I_2+I_3+I_4+I_5+I_6+I_7. \end{aligned}\tag{B.27}$$

For I_1, we have from Hölder inequality that

$$I_1\le c\int_{P_s\setminus P_\tau}|\nabla\alpha_{lk}|^2+\frac{c}{(s-\tau)^2}\int_{P_s}|\alpha_{lk}-(\alpha_{lk})_{P_s}|^2.$$

For I_2 and I_3, using Hölder inequality and (B.7) we have

$$\begin{aligned}I_2&\le\Big(\int_{P_s}|\partial_t(v_k-v)|^2\Big)^{1/2}\Big(\int_{P_s}|\alpha_{lk}-(\alpha_{lk})_{P_s}|^2\Big)^{1/2}\\ &\le c\Big[\int_{P_{1/4}}|\partial_tv_k|^2+1\Big]^{1/2}\Big(\int_{P_{1/4}}|\alpha_{lk}-(\alpha_{lk})_{P_{1/4}}|^2\Big)^{1/2}\\ &\le c\Big[\int_{P_{1/2}}|\nabla v_k|^2+1\Big]^{1/2}\Big(\int_{P_{1/4}}|\alpha_{lk}-(\alpha_{lk})_{P_{1/4}}|^2\Big)^{1/2}\\ &\le c\Big(\int_{P_{1/4}}|\alpha_{lk}-(\alpha_{lk})_{P_{1/4}}|^2\Big)^{1/2},\\ I_3&\le c\lambda_k\int_{P_s}|\nabla v_k||\alpha_{lk}-(\alpha_{lk})_{P_s}|\le c\lambda_k.\end{aligned}$$

For I_4,and I_5, in a way similar to the procedure of proving Lemma B.2, by using Lemma A.1 and (B.8) we obtain $I_4+I_5\le c\lambda_k/(s-\tau)$. Here we use that

$$|\nabla f_k|=r_k|\nabla f|\le cr_k\|f\|_{C^2}\le c\lambda_k.$$

For I_6 and I_7, using $r_k \le \lambda_k$ we have $I_6 + I_7 \le c\lambda_k$. Thus we get

$$\begin{aligned}\int_{P_\tau} |\nabla\alpha_{lk}|^2 &\le c\int_{P_s\setminus P_\tau} |\nabla\alpha_{lk}|^2 + \frac{c}{(s-\tau)^2}\int_{P_s} |\alpha_{lk} - (\alpha_{lk})_{P_s}|^2 \\ &\quad + c\Big(\int_{P_{1/4}} |\alpha_{lk} - (\alpha_{lk})_{P_{1/4}}|^2\Big)^{1/2} + c\lambda_k + c\frac{\lambda_k}{s-\tau} \\ &\le c\int_{P_s\setminus P_\tau} |\nabla\alpha_{lk}|^2 + \frac{c}{(s-\tau)^2}\int_{P_{1/4}} |\alpha_{lk} - (\alpha_{lk})_{P_{1/4}}|^2 \\ &\quad + c\Big(\int_{P_{1/4}} |\alpha_{lk} - (\alpha_{lk})_{P_{1/4}}|^2\Big)^{1/2} + \frac{c\lambda_k}{s-\tau}.\end{aligned}$$

Now filling the hole we get

$$\begin{aligned}\int_{P_\tau} |\nabla\alpha_{lk}|^2 &\le \theta\int_{P_s} |\nabla\alpha_{lk}|^2 + \frac{c}{(s-\tau)^2}\int_{P_{1/4}} |\nabla\alpha_{lk} - (\alpha_{lk})_{P_{1/4}}|^2 \\ &\quad + c\Big(\int_{P_{1/4}} |\alpha_{lk} - (\alpha_{lk}|^2\Big)^{1/2} + \frac{c\lambda_k}{s-\tau},\end{aligned}$$

where $\theta = \frac{c}{1+c}$. Using the Lemma 3.1 of Giaquinta[15] and (B.7), we have

$$\begin{aligned}\int_{P_{1/8}} |\nabla\alpha_{lk}|^2 &\le c\int_{P_{1/4}} |\alpha_{lk} - (\alpha_{lk})_{P_{1/4}}|^2 + c\Big(\int_{P_{1/4}} |\alpha_{lk} - (\alpha_{lk})_{P_{1/4}}|^2\Big)^{1/2} + c\lambda_k \\ &\le c\Big(\int_{P_{1/4}} |\alpha_{lk} - (\alpha_{lk})_{P_{1/4}}|^2\Big)^{1/2} + c\lambda_k.\end{aligned} \tag{B.28}$$

Step 4. There exists a constant c, independent of k, such that for some $1 < q < 2$

$$\int_{P_{1/4}} |\alpha_{lk} - (\alpha_{lk})_{P_{1/4}}|^2 \le c\int_{P_{1/4}} |\nabla(v_k - v)|^q)^{1/q} + c\lambda_k. \tag{B.29}$$

Since $\alpha_{lk} \in L^2(R; W^{1,2}(R^n))$, it follows from Sobolev embedding theorem with $q = \frac{2n}{n+2}$ for a.e. t that

$$\begin{aligned}\int_{B_{1/4}} |\alpha_{lk} - (\alpha_{lk})_{B_{1/4}}|^2 &\le c\Big(\int_{B_{1/4}} |\nabla\alpha_{lk}|^q\Big)^{2/q} \\ &\le c\Big(\int_{B_{1/4}} |\nabla(v_k - v)|^q + |\nabla\beta_{lk}^{ij}|^q\Big)^{2/q} \\ &\le c\Big(\int_{B_{1/4}} |\nabla(v_k - v)|^q\Big)^{2/q} + \Big(\int_{B_{1/4}} |\nabla\beta_{lk}^{ij}|^q\Big)^{2/q}.\end{aligned}$$

Write $a=-1/16, b=1/16$. Then using Hölder inequality we get

$$\begin{aligned}&\int_{P_{1/4}}|\alpha_{lk}-(\alpha_{lk})_{P_{1/4}}|^2\le\int_{P_{1/4}}|\alpha_{lk}-(\alpha_{lk})_{B_{1/4}}|^2\le c\Big(\int_{P_{1/4}}|\nabla\alpha_{lk}|^q\Big)^{1/q}\\ &\le c\int_a^b\Big(\int_{B_{1/4}}|\nabla v_k-\nabla v|^q\Big)^{2/q}+\int_a^b\Big(\int_{B_{1/4}}|\nabla\beta_{lk}^{ij}|^q\Big)^{2/q}\\ &\le c\Big(\int_a^b\int_{B_{1/4}}|\nabla v_k-\nabla v|^q\Big)^{1/q}\Big(\int_a^b\Big(\int_{B_{1/4}}|\nabla v_k-\nabla v|^q\Big)^{1/(q-1)}\Big)^{(q-1)/q}\\ &\quad+\int_{P_{1/4}}|\nabla\beta_{lk}^{ij}|^2\\ &=II_1+II_2\end{aligned}$$

Since $II_2\le c\lambda_k$ by (B.5), we only estimate II_1.

In the energy inequality, taking $\phi\in C_0^\infty(B_{r_k}(x_k))$, $0\le\phi\le1$, $\phi=1$ on $B_{r_k/2}(x_k)$, $|\nabla\phi|\le c/r_k$, and letting $|t_i-t_k|\le r_k^2$, $i=1,2$, with $t_1\le t_2$, we get

$$\begin{aligned}\int_{B_{r_k/2}}|\nabla u|^2(t_2)&\le\int_{B_{r_k}}|\nabla u|^2(t_1)+\frac{c}{r_k^2}\int_{t_1}^{t_2}\int_{B_{r_k}(x_k)}|\nabla u|^2\\ &\le\int_{B_{r_k}}|\nabla u|^2(t_1)+\frac{c}{r_k^2}\int_{P_{r_k}(z_k)}|\nabla u|^2.\end{aligned}$$

Set $y=x_k+r_kx,\quad t=t_k+r_k^2\tau$. Then

$$\int_{B_{1/2}}|\nabla v_k|^2(\tau_2)\le\int_{B_1}|\nabla v_k|^2(\tau_1)+c\int_{P_1}|\nabla v_k|^2, \tag{B.30}$$

with $-1\le\tau_1\le\tau_2\le1$.

Since $\int_{P_1}|\nabla v_k|^2=1$, we have the following lemma.

Lemma B.3. *There exist a constant c_2, independent of k, and $\tau_k\in[-1,-1/16]$ such that $\int_{B_1}|\nabla v_k|^2(\tau_k)\le c_2$.*

Proof. Write

$$a_k=\int_{-1}^{-1/16}g_k(\tau),\quad g_k(\tau)=\Big(\int_{B_1}|\nabla v_k|^2\Big)^{1/2}.$$

Then

$$a_k\le c\Big(\int_{P_1}|\nabla v_k|^2\Big)^{1/2}\le c_3,$$

where c_3 is independent of k. So there exists a subsequence such that $a_k\to a$.

If $a=0$, this shows that $g_k\to0$ in L^1. We assume that $g_k\to0$ a.e.. Thus by Yegonoff theorem, for any $\epsilon>0$ there exists a closed subset $D\subset[-1,-1/16]$ such that $|[-1,-1/16]\backslash D|<\epsilon$, and $|g_k(\tau)|\le c$, for some constant c independent of k. Thus we take $t_k\in D$, then $g_k(t_k)\le c$.

If $a>0$, then for any measurable subset $E\subset[-1,-1/16]$, we have

$$\int_E g_k\le\Big(\int_E g_k^2\Big)^{1/2}|E|^{1/2}\le c|E|^{1/2},$$

so that for any k, $\int_E g_k\le a/3,\quad$ if $|E|\le\frac{a^2}{9c^2}\equiv\delta_0$. For any N, write

$$E_{Nk}\equiv\big\{\tau;g_k(\tau)\ge N\big\}\cap[-1,-1/16],\quad E_{Nk}^c=[-1,-1/16]\backslash E_{Nk}.$$

Since

$$N|E_{Nk}| \le \int_{E_{Nk}} g_k \le a_k \le c_3,$$

i.e., $|E_{Nk}| \le c_3/N$, we can take N_0 large enough, so that $|E_{N_0k}| \le \delta_0$.

On the other hand, since $a_k \to a$, there exists a k_0 such that $a_k > 2a/3$ for $k \ge k_0$. So we have that

$$\int_{E^c_{N_0k}} g_k = a_k - \int_{E_{N_0k}} g_k \ge 2a/3 - a/3 = a/3.$$

This implies that $|E^c_{N_0k}| > 0$ for $k \ge k_0$. Then we can take $t_k \in E^c_{N_0k} \subset [-1, -1/16]$ such that $g_k(t_k) \le N_0$. Thus we complete the proof.

Combining (B.30) with Lemma 3.5 we get

$$\int_{B_{1/2}} |\nabla v_k|^2(\tau) \le \int_{B_1} |\nabla v_k|^2(\tau_k) + c\int_{P_1} |\nabla v_k|^2 \le c_2 + c = c_4, \tag{B.31}$$

where $1 \ge \tau \ge -1/16 \ge \tau_k$ and c_4 is independent of k..

We can estimate II_1 as follows:

$$\begin{aligned}
II_1 &\le c\Big(\int_a^b \int_{B_{1/4}} |\nabla v_k - \nabla v|^q\Big)^{1/q} \Big(\int_a^b \Big(\int_{B_{1/4}} |\nabla v_k - \nabla v|^q\Big)^{1/(q-1)}\Big)^{(q-1)/q} \\
&\le c\Big(\int_{P_{1/4}} |\nabla v_k - \nabla v|^q\Big)^{1/q} \Big(\int_a^b \Big(\int_{B_{1/4}} |\nabla v_k - \nabla v|^2\Big)^{q/(2(q-1))}\Big)^{(q-1)/q} \\
&\le c(c_2^{(2-q)/(2q)} + 1)\Big(\int_{P_{1/4}} |\nabla v_k - \nabla v|^q\Big)^{1/q} \Big(\int_{P_{1/4}} |\nabla v_k - \nabla v|^2\Big)^{(q-1)/q} \\
&\le c\Big(\int_{P_{1/4}} |\nabla v_k - \nabla v|^q\Big)^{1/q}.
\end{aligned}$$

Thus we prove (B.29).

Now we have from (B.3), (B.5), (B.28) and (B.29) that for some $1 < q < 2$

$$\begin{aligned}
\int_{P_{1/8}} \langle \nabla(v_k - v), e_l\rangle^2 &\le c\int_{P_{1/8}} |\nabla \alpha_{lk}|^2 + |\nabla \beta^{ij}_{lk}|^2 \\
&\le c\Big(\int_{P_{1/4}} |\alpha_{lk} - (\alpha_{lk})_{P_{1/4}}|^2\Big)^{1/2} + c\lambda_k \\
&\le c\Big(\Big(\int_{P_{1/4}} |\nabla(v_k - v)|^q\Big)^{1/q} + \lambda_k\Big)^{1/2} + c\lambda_k.
\end{aligned} \tag{B.32}$$

Assume that $\{e_\alpha(u_k)\}$ is the normal frame of N at $u_k(x,t)$. Since

$$\langle \partial_t v, e_\alpha(u_k)\rangle - \operatorname{div}\langle \nabla v, e_\alpha(u_k)\rangle = -\langle \nabla v, \nabla e_\alpha\rangle \quad \text{on } P_{1/2},$$

multiplying it by $\langle v - v_k, e_\alpha\rangle\zeta^2$, where $\zeta \in C^\infty_0(P_{1/2}), \zeta = 1$ on $P_{1/4}$, we have

$$\begin{aligned}
\int_{P_{1/2}} \langle \nabla v, e_\alpha(u_k)\rangle^2\zeta^2 = &\int_{P_{1/2}} \langle \partial_t v, e_\alpha\rangle\langle v_k - v, e_\alpha\rangle\zeta^2 + \int_{P_{1/2}} \langle \nabla v, e_\alpha\rangle\langle v_k - v, \nabla(e_\alpha\zeta^2)\rangle \\
&+ \int_{P_{1/2}} \langle \nabla v, \nabla e_\alpha\rangle\langle v_k - v, e_\alpha\rangle\zeta^2.
\end{aligned}$$

Here we use that $\nabla v_k \perp e_\alpha(u_k)$. Using (3.6)-(3.8) and (3.10) we get

$$\int_{P_{1/4}} \langle \nabla v, e_\alpha\rangle^2 = o(1) \quad \text{as } k \to \infty. \tag{B.33}$$

Finally We have the following compact lemma.

Lemma B.4. *Suppose that* $\{v_k\}_{k=1}^\infty$ *are bounded in* $L^\infty((0,T);W^{1,p}(M;R^m))$, $\{\partial_t v_k\}_{k=1}^\infty$ *are bounded in* $L^2((0,T);L^2(M;R^{n+1}))$, *and* $\{g_k\}_{k=1}^\infty$ *are bounded in* $L^1((0,T);L^1(M;R^m))$, *and suppose that* $\{v_k\}$ *satisfy the following equations in the sense of distribution*

$$\partial_t v_k - \frac{1}{\sqrt{det(g)}}\frac{\partial}{\partial x_\alpha}(|\nabla v_k|^{p-2}g^{\alpha\beta}\sqrt{det(g)}\frac{\partial v_k}{\partial x_\beta}) = g_k, \quad (t,x)\in(0,T)\times M. \tag{B.34}$$

Here (M,g) *is a compact Riemannian manifold.*

Then $\{v_k\}_{k=1}^\infty$ *are precompact in* $L^q((0,T);W^{1,q}(M;R^{n+1}))$ *for every* $1\le q<p$.

Its proof can be found in [4] for $p\ge 2$ and in [19] for $1<p\le 2$.

In view of (B.31)-(B.33) and Lemma B.4 with $M=P_{1/4}$ and $p=2$ and

$$g_k = \lambda_k A(u_k)(\nabla v_k,\nabla v_k) + 2r_k A(u_k)(\nabla v_k,\nabla f) + \frac{r_k^2}{\lambda_k}A(u_k)(\nabla f,\nabla f) + \frac{r_k^2}{\lambda_k}\Delta f,$$

there exists a subsequence such that

$$\lim_{k\to\infty}\int_{P_{1/8}}|\nabla(v_k-v)|^2 = \lim_{k\to\infty}\int_{P_{1/8}}\sum_l\langle\nabla(v_k-v),e_l\rangle^2 + \sum_\alpha\langle\nabla v,e_\alpha\rangle^2 = 0. \tag{B.35}$$

This completes the proof of Lemma 3.2.

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