Chin. Ann. of Math.
23B:2(2002),149-164.

STABLE AND UNSTABLE IDEAL PLANE FLOWS

C. BARDOS* Y. GUO** W. STRAUSS***

(*Dedicated to the memory of Jacques-Louis Lions*)

Abstract

The authors investigate the stability of a steady ideal plane flow in an arbitrary domain in terms of the L^2 norm of the vorticity. Linear stability implies nonlinear instability provided the growth rate of the linearized system exceeds the Liapunov exponent of the flow. In contrast, a maximizer of the entropy subject to constant energy and mass is stable. This implies the stability of certain solutions of the mean field equation.

Keywords Stable ideal plane flows, Unstable ideal plane flows, Liapunov exponent
2000 MR Subject Classification 35L50, 35B35
Chinese Library Classification O175.27, O175.21 **Document Code** A
Article ID 0252-9599(2002)02-0149-16

§1. Introduction

We consider solutions $u = (u_1(x_1, x_2, t), u_2(x_1, x_2, t))$ of the incompressible two-dimensional Euler equation in a bounded domain $\Omega \subset \mathbb{R}^2$ with a smooth impermeable boundary $\partial\Omega$:

$$\partial_t u + u \cdot \nabla u = -\nabla p \text{ in } \mathbb{R}_t \times \Omega, \quad u \cdot \vec{n} = 0 \text{ on } \mathbb{R}_t \times \partial\Omega\,. \tag{1.1}$$

$\vec{n}$ denotes the outward normal to the boundary. The vorticity

$$\omega = \nabla \wedge u \equiv \partial_{x_1} u_2 - \partial_{x_2} u_1$$

is then transported by the flow according to the equation

$$\partial_t \omega + u \cdot \nabla \omega = 0 \text{ in } \mathbb{R}_t \times \Omega\,. \tag{1.2}$$

We define the operator curl^{-1} by the formula

$$\mathrm{curl}^{-1}\omega = \nabla \wedge \Psi, \text{ with } -\Delta\Psi = \omega \text{ in } \Omega\,,\ \Psi = 0 \text{ on } \partial\Omega. \tag{1.3}$$

Manuscript received October 31, 2001.

*Université Denis Diderot and LAN Université Pierre et Marie Curie, Paris, France.
E-mail: bardos@math.jussieu.fr

**Division of Applied Mathematics, Brown University, Providence, Rhode Island 02912, USA.
E-mail: guoy@cfm.brown.edu

***Department of Mathematics, Brown University, Providence, Rhode Island 02912, USA.
E-mail: wstrauss@math.brown.edu

If the domain Ω is simply connected, then $u = \mathrm{curl}^{-1}\omega$. If it is not simply connected, but $u = \mathrm{curl}^{-1}\omega$ at time $t = 0$, then $u = \mathrm{curl}^{-1}\omega$ at all times. This is the only case that we consider and therefore the Euler equation is equivalent to the equation

$$\partial_t \omega + \mathrm{curl}^{-1}(\omega) \cdot \nabla\omega = 0\,. \tag{1.4}$$

If $u_0(x)$ is a stationary solution of the Euler equation (1.4), then the vorticity $\omega_0(x)$ satisfies the equation

$$0 = u_0 \cdot \nabla\omega_0 = \nabla\Psi_0 \wedge \nabla\omega_0\,, \tag{1.5}$$

which says that the level lines of Ψ_0 and Ω_0 coincide. This condition is satisfied in particular for any solution of the nonlinear elliptic equation

$$-\Delta\Psi = f(\Psi) \text{ in } \Omega, \quad \Psi = 0 \text{ on } \partial\Omega\,. \tag{1.6}$$

The present article is composed of two parts. The first concerns the stability of solutions of the mean field equation which was introduced to the subject by Onsager[15]:

$$-\Delta\Psi = Ce^{-\beta\Psi} \text{ in } \Omega\,, \quad C > 0\,. \tag{1.7}$$

The convexity of the entropy functional

$$S(\omega) = \int_\Omega \omega \log \omega dx \tag{1.8}$$

is used in conjunction with the tools developed in [3] and [4]. In a standard normalization, the stability is proven for $\beta > -8\pi$, extending previous results obtained by Arnold's method[1] for β negative but small in absolute value. The case of $-\beta$ large corresponds to large energy and seems to be the relevant one in the formation of coherent structures.

Our first stability theorem is as follows.

Theorem 1.1. *Consider the variational problem*

$$S(A, E) = \inf_\omega \int_\Omega \omega \log \omega dx \tag{1.9}$$

subject to the mass and energy constraints

$$\omega \geq 0\,, \quad \int_\Omega \omega dx = A\,, \quad \frac{1}{2}\int_\Omega |\mathrm{curl}^{-1}\omega|^2 dx = E. \tag{1.10}$$

Assume that the minimizer μ, which satisfies

$$\int_\Omega \mu \log \mu dx = S(A, E) \tag{1.11}$$

and always exists[3]*, is unique. Consider the family F of initial data defined as the nonnegative functions that belong to a Holder space $C^{0,\alpha}$ for some $\alpha > 0$. Then for all $\epsilon > 0$ there exists $\delta > 0$ such that, for the solutions $\omega(t)$ of the Euler equation* (1.4) *with $\omega(0) \in F$, the implication*

$$\|\omega(0) - \mu\|_{L^2(\Omega)} \leq \delta \quad \Rightarrow \quad \sup_{t\in R} \|\omega(t) - \mu\|_{L^2(\Omega)} \leq \epsilon \tag{1.12}$$

holds.

The proof of this theorem and some variants to handle the case where the minimizer is not unique will be given in Section 2.

In Section 3 it is shown how to deduce nonlinear instability from linearized instability. The method follows closely the program initiated by the second and third authors of this paper (in collaboration with others)[6,7,10], where it was observed that the notion of instability is a robust property. In particular, using a perturbation method, nonlinear instabilities can be deduced from instabilities of the linearized operator.

In the present situation the linearized equation is

$$(\partial_t + u_0 \cdot \nabla)\tilde{\omega} + \tilde{u} \cdot \nabla\omega_0 = 0\,. \tag{1.13}$$

Solutions of (1.13) are described by the group of operators e^{tA_0} with generator

$$A_0\tilde{\omega} = -u_0 \cdot \nabla\tilde{\omega} - \mathrm{curl}^{-1}\omega \cdot \nabla\omega_0\,. \tag{1.14}$$

This generator is the sum of an advection term which generates an isometry group in any $L^p(\Omega)$ $(1 \le p < \infty)$, denoted by $e^{-tu_0\cdot\nabla}$, and a compact perturbation. It follows that the part $\Sigma(A_0)$ of the spectrum of A_0 outside the imaginary axis is purely discrete and that if $\Sigma(A_0) \ne \emptyset$, the type Λ of the semi group e^{-tA_0} in any L^p space is given by

$$\Lambda = \sup_{\lambda\in\Sigma(A_0)} \mathrm{Re}\,\lambda\,. \tag{1.15}$$

Linear instability corresponds to the case when $\Lambda > 0$ (that is, $\Sigma(A_0) \ne \emptyset$). We denote by σ the Liapunov exponent of the autonomous flow associated to u_0. Our instability theorem is as follows.

Theorem 1.2 (From Linear to Nonlinear Instability). *Given a steady flow $u_0 \in C^3(\Omega)$, consider the linearized equation*

$$\partial_t\tilde{\omega} + u_0 \cdot \nabla\tilde{\omega} + \mathrm{curl}^{-1}\tilde{\omega} \cdot \nabla\omega_0 = 0\,. \tag{1.16}$$

Assume that the type Λ of the semigroup and the Liapunov exponent σ of the flow generated by the vector field u_0 satisfy the inequality

$$\Lambda > \sigma. \tag{1.17}$$

Then for any $p > 2$ there exist positive constants C, ϵ_0, δ_0 and a family of solutions of the nonlinear Euler equation $\{\omega_\delta\,,\ 0 < \delta \le \delta_0\}$ which satisfy both

$$\|\omega_\delta(0) - \omega_0\|_{W^{1,p}} \le \delta \tag{1.18}$$

and

$$\sup_{0<t\le C|\log\delta|} \|\omega_\delta(t) - \omega_0\|_{L^2} \ge \epsilon_0. \tag{1.19}$$

In two space variables the use of the vorticity equation forces the discreteness of the spectrum of A_0 off the imaginary axis. In contrast, Friedlander and Vishik[5] consider a different linearized operator. They linearize the Euler equation for the velocity to get the operator

$$B_0\tilde{u} = -(u_0 \cdot \nabla)\tilde{u} - (\tilde{u} \cdot \nabla)u_0 - \nabla q$$

on the space $\{\tilde{u} : \tilde{u} \in L^2(\Omega), \nabla \cdot \tilde{u} = 0\}$. They prove[5,18] that the essential spectral radius of e^{tB_0} is equal to the maximal growth rate of the bicharacteristic-amplitude equations

$$\dot{x} = u_0(x),\ \ \dot{\xi} = -(\partial_x u_0)^T\xi,$$
$$\dot{b} = -(\partial_x u_0)b + 2(\xi \cdot \partial_x u_0)b\,\xi/|\xi|^2.$$

Thus the essential spectra of A_0 and B_0 are quite different.

In [6], it is proven that certain flows are nonlinearly unstable in H^s for $s > 2$, by making use of the point spectrum of B_0. In [8], Grenier proves similar results in the space $\{u \in L^2\}$. Here we prove the same kind of theorem for the L^2 norm of the vorticity. This is the very norm for which we obtain stability results. Furthermore, in contrast to [6] and [8], our results are valid (i) with no geometric hypothesis on the shape of the domain Ω and (ii) with less regularity assumed on the steady flow.

Two basic examples are shear flow and simple rotating flow. In both cases, the exponent σ vanishes. Indeed, shear flow is

$$u_0 = \begin{pmatrix} u(x_2) \\ 0 \end{pmatrix} \quad \text{with} \quad X_0(t, x_1, x_2) = \begin{pmatrix} u(x_2)t + x_1 \\ x_2 \end{pmatrix}.$$

Since the flow grows only linearly in time, $\sigma = 0$. Simple rotating flow in a disk is

$$u_0 = \begin{pmatrix} u(r)\sin\theta \\ -u(r)\cos\theta \end{pmatrix} \quad \text{with } X_0(t, x_1, x_2) = \begin{pmatrix} r\cos(\theta - tu(r)/r) \\ r\sin(\theta - tu(r)/r) \end{pmatrix}$$

for which σ is also clearly equal to 0.

There exist classical examples[6] of shear or rotating flows going back to Rayleigh for which the linearized instability is proven. The previous comments imply that for these cases the hypothesis (1.17) is satisfied.

§2. Stability of Solutions of the Mean Field Equation

The solutions of the mean field equation

$$-\Delta\Psi = A\frac{e^{-\beta\Psi}}{\int_\Omega e^{-\beta\Psi}dx} \tag{2.1}$$

are invariant under the Euler flow. They are also closely related to the minimization of the functional

$$S(\omega) = \int_\omega H(\omega)dx \quad \text{ with } H(\omega) = \omega\ln\omega \tag{2.2}$$

subject to the constraints

$$\omega \geq 0,\ A = \int \omega dx,\ E = \frac{1}{2}\int_\Omega |\mathrm{curl}^{-1}\omega|^2 dx. \tag{2.3}$$

The constants A and E are positive while β is the Lagrange multiplier of the energy constraint. More precisely, starting from the convexity of the function $\omega \to H(\omega) \geq e^{-1}$ the following facts are proven in [3, 4, 11] and are restated in the next two theorems.

Theorem 2.1. (i) *Denote*

$$P(E, A) = \Big\{\omega \geq 0 \text{ a.e. in } \Omega,\ \int_\Omega \omega(x)dx = 1,\ \frac{1}{2}\int_\Omega |\mathrm{curl}^{-1}\omega|^2 dx = E\Big\}. \tag{2.4}$$

Then for any pair (E, A) of positive constants there is at least one $\omega_{\min}$ which solves the Microcanonical Variation Principle

$$S(E, A) \equiv \inf_{\omega \in P(E,A)} S(\omega) = S(\omega_{\min}). \tag{2.5}$$

Furthermore, any nonnegative solution ω of (2.5) *is strictly positive and there exists at least one β such that $\omega = -\Delta\Psi$ with Ψ solving the corresponding β-mean field equation* (2.1).

(ii) *Conversely, any solution of the β-mean field equation* (2.1) *is a solution of* (2.5) *with a given mass and energy $E(\beta)$.*

With appropriate scalings, the following renormalizations are assumed for the domain Ω and for the solution of the mean field equation:

$$|\Omega| = \text{meas } \Omega = 1 \text{ and } A = \int_\Omega \omega(x)dx = 1. \tag{2.6}$$

The set of minimizers in (2.5) is denoted by $\mathcal{M}(E, A)$. In particular, $\mathcal{M}(E, 1)$, $P(E, 1)$ and $S(E, 1)$ are denoted by $\mathcal{M}(E), P(E)$ and $S(E)$.

Theorem 2.2. (i) *For any $\beta > -8\pi$ there is at least one solution of* (2.1); *this solution is always unique for $\beta > 0$.*

(ii) *This solution is also unique for $\beta > -8\pi$ if the domain Ω is simply connected. In this setting the mapping $\beta \mapsto E(\beta)$ is well defined. It is also strictly decreasing and is onto from the interval $(-8\pi, \infty)$ to the interval $(0, E_c)$, where*

$$E_c = \lim_{\beta > -8\pi, \beta \to -8\pi} E(\beta).$$

(iii) *For starshaped domains there exists a number $\beta_c \le -8\pi$ such that the mean field equation has no solution for $\beta < \beta_c$. In particular if Ω is a disk, $E_c = \infty$ and $\beta_c = -8\pi$.*

In fact, the situation of the disk corresponds to the ideal case where for any energy $E \in (0, \infty)$ any minimizer corresponds to a unique temperature $\beta \in (-8\pi, \infty)$. As a consequence, the minimizer is uniquely defined. In general, the situation turns out to be more complicated, either when the domain is not simply connected (no uniqueness of the solution of the mean field equation for given β), or when $E_c < \infty$ and $\beta_c < -8\pi$ in which case several values of β may correspond to the same energy.

For our purposes observe that for a simply connected domain a solution of the mean field equation

$$-\Delta \Psi_\beta = \frac{e^{-\beta \Psi}}{\int_\Omega e^{-\beta \Psi_\beta} dx}\,, \quad \beta - 8\pi \tag{2.7}$$

provides a minimizer of $S(\omega_\beta)$ with the constraint $E(\omega) = E(\beta) \in (0, E_c)$. Furthermore if $\omega_{\beta'}$ is another minimizer with the same energy,

$$\text{either } \beta' = \beta \text{ or } \beta' < -8\pi\,. \tag{2.8}$$

Now we prove the stability theorem stated in the introduction.

Proof of Theorem 1.1. The hypothesis $\omega(0) \in F$ implies (cf. [20, 21]) that the corresponding solution is well defined and smooth. In particular one has for $\omega(t)$ and $u(t) = \mathrm{curl}^{-1}\omega(t)$ the invariance

$$E(t) = \frac{1}{2}\int_\Omega |u(x,t)|^2 dx = \frac{1}{2}\int_\Omega |u(x,0)|^2 dx = E(0), \tag{2.9}$$

and for any continuous function f

$$\int_\Omega f(\omega(x,t))dx = \int_\Omega f(\omega(x,0))dx\,. \tag{2.10}$$

By contradiction, it is easy to see that the statement of the theorem is equivalent to the following one. For any sequence of initial data $\omega_n(0) \in F$ and for any sequence of times t_n, the limit

$$\lim_{n\to\infty} \|\omega_n(0) - \mu\|_{L^2(\Omega)} = 0 \tag{2.11}$$

implies the existence of a subsequence n_j such that

$$\lim_{n\to\infty} \|\omega_{n_j}(t_{n_j}) - \mu\|_{L^2(\Omega)} = 0\,. \tag{2.12}$$

So we assume (2.11).

We claim that

$$\lim_{n\to\infty} \int_\Omega H(\omega_n(x,0))dx = \int_\Omega H(\mu)dx \equiv S(\mu)\,. \tag{2.13}$$

To prove (2.13), the strict positivity of μ as stated in Theorem 2.1 is used and a positive constant η_0 such that $\mu(x) \geq 2\eta_0$ in Ω is introduced. Let $0 \leq \eta \leq \min(\eta_0, e^{-1})$ and observe the inclusion $\{x \in \Omega \backslash |\omega_n(x,0)| < \eta\} \subset \{x \in \Omega \backslash |\omega_n(x,0) - \mu(x)| > \eta\}$. Then we have

$$\begin{aligned}
&\Big|\int_\Omega H(\omega(x,0)dx - S(\mu))\Big| \\
&\leq \int_{\Omega \cap \{\omega_n(x,0) \geq \eta\}} |H(\omega_n(x,0) - H(\mu(x))|dx + \int_{\Omega \cap \{\omega_n(x,0) < \eta\}} (|H(\omega_n(x,0)| + |H(\mu(x))|)dx \\
&\leq \frac{2}{\eta} \int |\omega_n(x,0)dx - \mu(x))|dx + \eta|\log\eta| + \|H(\mu)\|_{L^\infty(\Omega)} \int_{|\omega_n(x,0)-\mu(x)| \geq \eta} dx\,. \qquad (2.14)
\end{aligned}$$

To complete the proof of the claim, choose η to make $\eta|\log\eta|$ less than $\epsilon/2$ and then choose n large enough to ensure, with the strong L^2 convergence of $\omega_n(0)$ to μ, that the sum of the two other terms is less than $\epsilon/2$.

Then there exists a subsequence n_j denoted below by n such that $\omega_n(t_n)$ converges weakly in $L^2(\Omega)$ to a function $\nu(x)$. With the notation $u_n(t_n) = \mathrm{curl}^{-1}\omega_n(t_n)$, $u_\nu = \mathrm{curl}^{-1}\nu$, the relations (2.9), (2.10) and the "entropic convergence lemma" (Proposition 3.1 of [2]) one has, for $\nu \in L^2(\Omega), \nu \geq 0$, the following properties:

$$\int_\Omega \nu(x)dx = \lim \int_\Omega \omega_n(x,t_n)dx = \lim \int_\Omega \omega_n(x,0)dx = 1\,, \qquad (2.15)$$

$$\frac{1}{2}\int_\Omega |u_\nu(x)|^2 dx = \lim \int_\Omega |u_n(x,t_n)|^2 dx = E\,, \qquad (2.16)$$

$$\int_\Omega H(\nu(x))dx \leq \lim \int_\Omega H(\omega_n(x,t_n)dx = H(\mu)\,. \qquad (2.17)$$

Thus ν is a minimizer. The hypothesis concerning the uniqueness of the minimizer implies the relation $\nu = \mu$ and the strong $L^2(\Omega)$ convergence because

$$\begin{aligned}
\int_\Omega |\nu(x)|^2 dx &\leq \lim \int_\Omega |\omega_n(x,t_n)|^2 dx = \lim \int_\Omega |\omega_n(x,0)|^2 dx \\
&= \int_\Omega |\mu(x)|^2 dx = \int_\Omega |\nu(x)|^2 dx\,. \qquad (2.18)
\end{aligned}$$

This proves Theorem 1.1.

Since the minimizer may not be unique, we consider several extensions of the above theorem. We have the following variant.

Theorem 2.3. *No assumption is made on the uniqueness of the minimizer. On the other hand, consider for a given energy E, a minimizer $\mu \in P(E)$. Let $F' \subset F$ be the subset of initial data uniformly bounded in $L^\infty(\Omega)$ by a fixed constant k. Then for all $\epsilon > 0$ there exists $\delta > 0$ such that, for the solutions $\omega(t)$ of the Euler equation* (1.4) *with $\omega(0) \in F'$, the assertion*

$$\|\omega(0) - \mu\|_{L^2(\Omega)} \leq \delta \quad \Rightarrow \quad \sup_{t \in R} \inf_{\nu \in \mathcal{M}(E)} \|\omega(t) - \nu\|_{L^2(\Omega)} \leq \epsilon \qquad (2.19)$$

holds.

Proof. The proof follows the preceding one and leads with no major modification to the extraction of a subsequence such that $\omega_n(t_n)$ converges weakly in $L^2(\Omega)$ to a minimizer $\nu \in \mathcal{M}(E)$. Since the uniqueness of this minimizer is not assumed, the relation (2.18) is no longer valid and the convexity of the entropy H (with the uniform boundedness) is used instead. In fact, by a Taylor expansion we have

$$\begin{aligned}&\int_\Omega \big(H(\omega_n(x,t_n)-H(\nu(x,t)\big)dx\\ &=\int_\Omega (1+\log\nu(x,t))\big(\omega_n(x,t_n)-\nu(x,t)\big)dx+\int_\Omega \frac{1}{2\overline{(\omega(x,t))}}\big(\omega_n(x,t_n)-\nu(x,t)\big)^2dx\end{aligned} \tag{2.20}$$

with $\overline{\omega(x,t)}$ between $\omega_n(x,t_n)$ and $\nu(x)$. Using the uniform boundedness of $\omega_n(x,0)$ and the weak L^2 convergence of $\omega_n(t_n)$, we have for $0<\alpha$ small enough the inequality

$$\alpha\lim\int_\Omega \big(\omega_n(x,t_n)-\nu(x,t)\big)^2dx\le\lim\int_\Omega \big(H(\omega_n(x,t_n)-H(\nu(x,t)\big)dx=0, \tag{2.21}$$

which proves the strong convergence.

If the domain is simply connected, there is, as recalled above in Theorem 2.2, a one-to-one correspondence between the solutions of the mean field equation for $\beta>-8\pi$ and the minimizers with energy $E>E_c$. This does not seem to exclude in general the existence of other minimizers for the same energy which solve a mean field equation with $\beta<-8\pi$. In accordance with this observation we give the following theorem.

Theorem 2.4. *Assume that the open set Ω is simply connected and consider a solution Ψ^* of the mean field equation*

$$-\Delta\Psi^*=\frac{e^{-\beta\Psi^*}}{\int_\Omega e^{-\beta\Psi^*}dx}\,,\ \Psi^*=0\quad on\quad \partial\Omega. \tag{2.22}$$

Denote by $\mu=-\Delta\Psi^*$ the corresponding vorticity with energy $E(\mu)$ and consider the same set of initial data F' as in Theorem 2.3. Then for all $\epsilon>0$ there exists $\delta>0$ such that, for the solutions $\omega(t)$ of the Euler equation (1.4) with $\omega(0)\in F'$, the assertion

$$\|\omega(0)-\mu\|_{L^2(\Omega)}\le\delta\Rightarrow\sup_{t\in R}\|\omega(t)-\mu\|_{L^2(\Omega)}\le\epsilon \tag{2.23}$$

holds.

Proof. According to Theorem 3.2 ii) of [4], one has

$$\mathcal{M}(E(\mu))=\{\mu\}\cup\mathcal{M}^*, \tag{2.24}$$

where any element $\nu\in\mathcal{M}^*$ is a solution of the mean field equation with a temperature $\beta(\nu)<-8\pi$. Observe by contradiction that the $L^2(\Omega)$-distance d between μ and $\mathcal{M}^*$ is strictly positive. By Theorem 2.3, for all $\epsilon>0$ there exists $\delta>0$ such that the relation

$$\|\omega(0)-\mu\|_{L^2(\Omega)}\le\delta \tag{2.25}$$

implies the relation

$$\sup_{t\in R}\inf_{\nu\in\mathcal{M}(E(\mu))}\|\omega(t)-\nu\|_{L^2(\Omega)}\le\epsilon\,. \tag{2.26}$$

We observe that for $\omega(0)\in F'$ we have $\omega(.)\in C(I\!R_t;L^2(\Omega))$. Choosing $\epsilon<\frac{d}{2}$ and $\delta<\frac{d}{2}$ and using the triangle inequality, we conclude that

$$\sup_{t\in R}\inf_{\nu\in\mathcal{M}(E(\mu))}\|\omega(t)-\nu\|_{L^2(\Omega)}=\sup_{t\in R}\|\omega(t)-\mu\|_{L^2(\Omega)}. \tag{2.27}$$

This completes the proof.

The previous stability results involve, as in most of the contributions in the subject (cf. for instance [1, 13, 19]), the L^2 norm of the vorticity. Finally we conclude this section by proving, using the notion of entropic convergence, a weaker result under a weaker hypothesis.

Theorem 2.5. *The family of initial data F is defined as in Theorem* 1.1. *For some $p > 1, F_p \subset F$ denotes the subset of initial data uniformly bounded in $L^p(\Omega)$; that is, there exists $k < \infty$ such that*

$$\forall \omega(0) \in F_p\,, \ \int_\Omega |\omega(0,x)|^p dx \le k. \tag{2.28}$$

Then for all (A, E) and $\epsilon > 0$ there exists $\delta > 0$ such that

$$\Big|\int_\Omega \omega(0,x)dx - A\Big| + \Big|\frac{1}{2}\int_\Omega |(\mathrm{curl}^{-1}\omega(0))(x)|^2 dx - E\Big| + \Big|\int_\Omega H(\omega(0,x))dx - S(A,E)\Big| \le \delta \tag{2.29}$$

implies

$$\sup_{t\in R} \inf_{\nu\in\mathcal{M}(A,E)} \|\mathrm{curl}^{-1}\omega(t) - \mathrm{curl}^{-1}\nu\|_{L^2(\Omega)} \le \epsilon\,. \tag{2.30}$$

Proof. By (2.25) we extract from any sequence $\omega_n(t_n)$ a subsequence, still denoted by $\omega_n(t_n)$, which converges weakly in $L^p(\Omega)$ to a function $\nu \in L^p(\Omega)$. The inequality (2.29) and the compactness of the Sobolev imbedding $W^{1,p}(\Omega) \subset L^2(\Omega)$ imply the following limits:

$$A = \lim \int_\Omega \omega_n(t_n, x)dx = \int_\Omega \nu(x)dx\,, \tag{2.31}$$

$$E = \lim \frac{1}{2}\int_\Omega |(\mathrm{curl}^{-1}\omega_n(t_n))(x)|^2 dx = \frac{1}{2}\int_\Omega |\mathrm{curl}^{-1}\nu|^2 dx\,. \tag{2.32}$$

Then by the entropic convergence we have

$$\int_\Omega H(\nu)dx \le \liminf \int_\Omega H(\omega_n(t_n))dx = \lim \int_\Omega H(\omega_n(0))dx = S(A,E)\,. \tag{2.33}$$

Therefore ν is a minimizer, the equality (2.32) implies the $L^2(\Omega)$ strong convergence of the sequence $\mathrm{curl}^{-1}\omega_n(t_n))$, and the proof is complete.

§3. From Linear to Nonlinear Instability

This section is devoted to the proof of Theorem 1.2. First we recall some facts about the Liapunov exponent. The classical Liapunov exponent σ for the flow $X_0(t,x)$ induced by u_0,

$$\frac{\partial X_0}{\partial t} = u_0(X_0), \quad X_0(0,x) = x\,, \tag{3.1}$$

is defined by

$$\sigma = \sup_x \lim_{t\to\infty} \frac{1}{t} \log\Big|\frac{\partial X_0}{\partial x}\Big|, \tag{3.2}$$

where $\frac{\partial X_0}{\partial x}$ denotes the 2×2 matrix $(\partial X_0^i/\partial x^j)$.

Proposition 3.1. *The Liapunov exponent can also be defined as*

$$\sigma = \lim_{t\to\infty} \frac{1}{t} \sup_x \log\Big|\frac{\partial X_0}{\partial x}\Big|\,. \tag{3.3}$$

This fact might exist in the theory of dynamical systems but in the absence of a reference a short proof is given. Clearly

$$\lim_{t\to\infty} \frac{1}{t} \sup_x \log\Big|\frac{\partial X_0}{\partial x}\Big| \ge \sup_x \lim_{t\to\infty} \frac{1}{t} \log\Big|\frac{\partial X_0}{\partial x}\Big|\,. \tag{3.4}$$

To prove the converse we follow the argument of [7]. By definition of σ , for every pair $\epsilon > 0$ and $x \in \Omega$, there exists a "time" $T[x] > 0$ such that

$$t \ge T[x] \Rightarrow \Big|\frac{\partial X_0(t,x)}{\partial x}\Big| < e^{t(\sigma+\epsilon)}\,.$$

By the continuity of $\partial_x X_0$ and the boundary condition, for all $x \in \overline{\Omega}$ there is a neighborhood $B_x \subset \overline{\Omega}$ such that

$$\Big|\Big(\frac{\partial X_0}{\partial x}\Big)(T[x], x)\Big| < e^{T[x](\sigma+\epsilon)}\,.$$

We introduce a finite covering of $\overline{\Omega}$ by such open sets: $\overline{\Omega} = B_{x_1} \cup B_{x_2} \cup \ldots \cup B_{x_N}$ with $N < \infty$. Denote $B_i = B_{x_i}, T_i = T[x_i]$ for $1 \le i \le N$. Given $x \in \overline{\Omega}$, choose i_1 such that $x \in B_{i_1}$ and then choose the sequence

$$\begin{aligned}
&y_1 = X_0(T_{i_1}, x) \in B_{i_2},\\
&y_2 = X_0(T_{i_1} + T_{i_2}, x) \in B_{i_3},\\
&\quad\cdots\cdots\\
&y_1 = X_0(T_{i_1} + \ldots + T_{i_k}, x) \in B_{i_{k+1}} \quad (1 \le k < \infty)\,.
\end{aligned}$$

Now

$$y_2 = X_0(T_{i_1} + T_{i_2}, x) = X_0(T_{i_2}, y_1) = X_0(T_{i_2}, X_0(T_{i_1}, x))\,. \tag{3.5}$$

Therefore

$$\frac{\partial y_2}{\partial x} = \Big(\frac{\partial X_0}{\partial x}\Big)(T_{i_2}, y_1) \cdot \Big(\frac{\partial X_0}{\partial x}\Big)(T_{i_1}, x), \tag{3.6}$$

so that

$$\Big|\frac{\partial[X_0(T_{i_1+i_2}, x)]}{\partial x}\Big| \le e^{[T_{i_1}+T_{i_2}](\sigma+\epsilon)}\,. \tag{3.7}$$

Similarly for any x and k we have the inequality

$$\Big|\frac{\partial[X_0(T_{i_1+\ldots+i_k}, x)]}{\partial x}\Big| \le e^{[T_{i_1}+\ldots+T_{i_k}](\sigma+\epsilon)}\,. \tag{3.8}$$

Now, given t and x, we choose

$$S_k = T_{i_1} + \ldots + T_{i_k} \le t < T_{i_1} + \ldots + T_{i_k} + T_{i_k+1}\,, \tag{3.9}$$

and denote by X_0' the derivative of

$$X_0(t, x) = X_0(t - S_k, X_0(S_k, x))$$

with respect to x. By the chain rule and estimate (3.8) we obtain, for any pair $(t, x) \in [0, \infty) \times \overline{\Omega}$,

$$\begin{aligned}
\Big|\frac{\partial[X_0(t, x)]}{\partial x}\Big| &\le \sup_{y\in\overline{\Omega}} |X_0'(t - S_k, y)| \cdot |X_0'(S_k, x)|\\
&\le \Big[\sup_{y\in\overline{\Omega},\, 0\le s\le \max_i T_i} |X_0'(s, y)|\Big] \cdot e^{(\sigma+\epsilon)S_k}\\
&\le Ce^{(\sigma+\epsilon)S_k} \le C'e^{(\sigma+\epsilon)t},
\end{aligned} \tag{3.10}$$

which implies (3.3).

The following key lemma states that if a velocity field $v(t, x)$ is close enough to $u_0(x)$ in C^1, then their corresponding flows are close together in a sufficiently short time interval where $\eta e^{(t-s)\mu}$ is small.

Lemma 3.1. *Let $u_0(x)$ be a steady C^1 solution. Let $X_0(t, s, x) = X_0(t - s, x)$ be its classical flow with Liapunov exponent σ. Let $v(t, x) \in C^1$ be another vector field defined on $\mathbb{R}_t \times \bar{\Omega}$ that is incompressible and tangent to the boundary*

$$\nabla \cdot v = 0 \text{ in } \Omega \text{ and } v \cdot \vec{n} = 0 \text{ on } \partial\Omega\,. \tag{3.11}$$

Denote by $X(t,s,x)$ its flow

$$\frac{\partial X}{\partial t} = v(t,X),\ X(s,s,x) = x.$$

Let $0<\epsilon$ and $\mu > \sigma+\epsilon$. Then there exist positive constants C_1, C_2 and θ_0 with the following property.

For any positive constant η the estimate

$$\|v(t,\cdot) - u_0(\cdot)\|_{C^1(\Omega)} \le \eta e^{(t-s)\mu} \tag{3.12}$$

for $s \le t \le s+S_\eta$ with

$$S_\eta \equiv \frac{1}{\mu} \ln \frac{\theta_0}{\eta} \tag{3.13}$$

implies for $s \le t \le s+S_\eta$ the a priori estimates

$$|X(t,s,x) - X_0(t,s,x)| \le C_1 \eta e^{(t-s)\mu}, \tag{3.14}$$

$$\Big|\frac{\partial (X(t,s,x) - X_0(t,s,x))}{\partial x}\Big| \le C_2 \theta_0 e^{(t-s)(\sigma+\epsilon)}. \tag{3.15}$$

Proof. When there is no risk of confusion, the arguments (t,s,x) in X and X_0 will be omitted. The difference $X - X_0$ satisfies

$$\frac{\partial (X-X_0)}{\partial t} = v(t,X) - u_0(X_0);\ \ (X-X_0)(s,s,x) = 0.$$

Thus

$$\begin{aligned}&\Big[\frac{\partial}{\partial t} - \frac{\partial u_0}{\partial X}(X_0)\Big](X-X_0)\\ &= \Big[-\frac{\partial u_0}{\partial X}(X_0)\Big](X-X_0) + [u_0(X) - u_0(X_0)] + [v(t,X) - u_0(X)] \equiv g.\end{aligned} \tag{3.16}$$

By the Taylor expansion and (3.12), we have

$$|g| \le \frac{1}{2}\Big| -\frac{\partial^2 u_0}{\partial X^2}(\bar{X})(X-X_0)^2\Big| + \eta e^{(t-s)\mu} \le C|X-X_0|^2 + \eta e^{(t-s)\mu}.$$

Let

$$T^* = \sup\{t_1 :\ |X(t_1) - X_0(t_1)| \le C_1 \eta e^{(t_1 - s)\mu}\} \tag{3.17}$$

with C_1 to be determined. In the interval $[s, T^*]$, we have from (3.2) and (3.16) the estimate

$$\begin{aligned}|X(t) - X_0(t)| &\le C_\epsilon \int_s^t e^{(\sigma+\epsilon)(t-\tau)}[C_1 \eta e^{(\tau-s)\mu}]^2 d\tau + C'_\epsilon \eta \int_s^t e^{(\sigma+\epsilon)(t-\tau)} e^{(\tau-s)\mu} d\tau\\ &\le C_\epsilon e^{(\sigma+\epsilon)(t-s)} C_1^2 \eta^2 \int_0^{t-s} e^{(\sigma+\epsilon)\rho} e^{2\mu\rho} d\rho + C''_\epsilon \eta e^{\mu(t-s)}\\ &\le C_1^2 C'_\epsilon [\eta e^{\mu(t-s)}]^2 + C''_\epsilon [\eta e^{\mu(t-s)}].\end{aligned}$$

Putting $t = T^*$, we have from (3.17)

$$C_1[\eta e^{\mu(T^*-s)}] = |X(T^*) - X_0(T^*)| \le C_1^2 C'_\epsilon [\eta e^{\mu(T^*-s)}]^2 + C''_\epsilon [\eta e^{\mu(T^*-s)}].$$

Hence

$$\eta e^{\mu(T^*-s)} \ge \frac{1}{C_1 C'_\epsilon} - \frac{C''_\epsilon}{C_1^2 C'_\epsilon} > \theta_0$$

if we choose

$$C_1 = 2C''_\epsilon \quad \text{and} \quad 0 < \theta_0 < \frac{1}{4C'_1 C''_\epsilon}.$$

By (3.13) we have

$$S_\eta < T^* - s$$

and we deduce from (3.17) that (3.14) is valid.

Next we define $Y = \frac{\partial}{\partial x}(X - X_0)$, where $\frac{\partial}{\partial x} = \frac{\partial}{\partial x_1}$ or $\frac{\partial}{\partial x_2}$. It satisfies

$$\begin{aligned}\frac{\partial Y}{\partial t} &= \frac{\partial}{\partial x}[v(t, X(t,s,x)) - u_0(X_0(t,s,x))] = \frac{\partial v}{\partial X}\frac{\partial X}{\partial x} - \frac{\partial u_0}{\partial X}\frac{\partial X_0}{\partial x} \\ &= \frac{\partial u_0}{\partial X}Y + \frac{\partial[v-u_0]}{\partial X}\Big\{\frac{\partial X_0}{\partial x} + Y\Big\} \equiv \frac{\partial u_0}{\partial X}Y + h. \end{aligned} \tag{3.18}$$

By assumption (3.12) and (3.14), for $0 \le t - s \le S_\eta$,

$$\begin{aligned}\frac{\partial[v-u_0]}{\partial X} &= \frac{\partial v(t,X)}{\partial X} - \frac{\partial u_0(X_0)}{\partial X} \\ &= \Big\{\frac{\partial v(t,X)}{\partial X} - \frac{\partial u_0(X)}{\partial X}\Big\} + \Big\{\frac{\partial u_0(X)}{\partial X} - \frac{\partial u_0(X_0)}{\partial X}\Big\} = O(\theta_0).\end{aligned}$$

Therefore by (3.13) we have

$$|h| \le C\theta_0[1 + |Y|].$$

By (3.18) we can now estimate

$$\begin{aligned}|Y(t)| &\le C\int_s^t e^{(t-\tau)(\sigma+\epsilon/2)}|h(\tau)|d\tau \le C\theta_0\int_s^t e^{(t-\tau)(\sigma+\epsilon/2)}[1 + |Y(\tau)|]d\tau \\ &\le C\theta_0 e^{(t-s)(\sigma+\epsilon/2)}\{1 + \int_s^t e^{(s-\tau)(\sigma+\epsilon/2)}|Y(\tau)|d\tau\}.\end{aligned}$$

It thus follows, for $t - s \le S_\eta$ and θ_0 sufficiently small, that

$$|Y(t)| \le C\theta_0 e^{(t-s)(\sigma+\epsilon)}.$$

Lemma 3.2. *Given u_0 and v as in Lemma 3.1, let $\tilde\omega$ solve the linear equation*

$$(\partial_t + v\cdot\nabla)\tilde\omega = 0.$$

Then for any $1 \le p \le \infty$ and

$$0 \le t \le S_\eta = \frac{1}{\mu}\log\frac{\theta_0}{\eta} \tag{3.19}$$

($0 \le t < \infty$ if $v \equiv u_0$), we have

$$\|\tilde\omega(t)\|_{W^{1,p}} \le C_\epsilon e^{(\sigma+\epsilon)t}\|\tilde\omega(0)\|_{W^{1,p}}\ .$$

Proof. Denoting $\Gamma(x) = \tilde\omega(0,x)$, we have

$$\tilde\omega(t,x) = \Gamma(X(t,0,x)).$$

Clearly $\|\tilde\omega(t)\|_{L^p} = \|\Gamma\|_{L^p}$ and

$$\Big|\frac{\partial\tilde\omega}{\partial x}\Big| = \Big|\frac{\partial\Gamma}{\partial X}\cdot\frac{\partial X}{\partial x}\Big| \le Ce^{(\sigma+\epsilon)t}\Big|\frac{\partial\Gamma}{\partial X}\Big|$$

by Lemma 3.1. Hence

$$\int_\Omega\Big|\frac{\partial\tilde\omega}{\partial x}\Big|^p dx \le Ce^{p(\sigma+\epsilon)t}\int_\Omega\Big|\frac{\partial\Gamma}{\partial X}\Big|^p dx$$

since the Jacobian equals one. This proves the desired estimate on the first derivatives.

Now we prove that the eigenfunctions are smooth.

Lemma 3.3. *Any eigenfunction $\phi_\lambda \in L^2(\Omega)$ corresponding to an eigenvalue λ with $\operatorname{Re}\lambda > \sigma$ belongs to $W^{1,p}(\Omega)$ for all $p < \infty$.*

Proof. The eigenfunction satisfies

$$(\lambda + u_0 \cdot \nabla)\phi_\lambda + \mathrm{curl}^{-1}\phi_\lambda \cdot \nabla\omega_0 = 0, \tag{3.20}$$

from which we deduce the relation

$$\phi_\lambda = \int_0^\infty e^{-\lambda\tau} e^{-\tau u_0 \cdot \nabla}\big(\mathrm{curl}^{-1}\phi_\lambda \cdot \nabla\omega_0\big)d\tau. \tag{3.21}$$

By assumption, ϕ_λ is in $L^2(\Omega)$ so that $\mathrm{curl}^{-1}\phi_\lambda \cdot \nabla\omega_0$ is in $H^1(\Omega)$. By Lemma 3.1 in the simple case $v = u_0$ and $0 \le t < \infty$, we have

$$e^{-tu_0\cdot\nabla} : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$$

with norm $O(e^{(\sigma+\epsilon)t})$ for any $\epsilon > 0$. Therefore the assumption $\operatorname{Re}\lambda > \sigma$ implies that ϕ_λ is in $H^1(\Omega)$. Now this implies that $\mathrm{curl}^{-1}\phi_\lambda \cdot \nabla\omega_0$ belongs to $H^2(\Omega) \subset W^{1,p}(\Omega)$ for $p < \infty$. Using once again the estimate

$$\|e^{-tu_0\cdot\nabla}\|_{W^{1,p}} \le O(e^{(\sigma+\epsilon)t}) \text{ and } \operatorname{Re}\lambda > \sigma, \tag{3.22}$$

we conclude the proof of the lemma.

The main ingredient in the proof of instability is a bootstrap lemma (Theorem 3.2). Before giving this lemma it is worthwhile to compare it with classical results on the $2D$ Euler equation. It is known that for smooth initial data the solution of the Euler equation remains as smooth as the data. For instance, for any $W^{1,p}$ norm we have the crude estimate

$$\|\omega(t)\|_{W^{1,p}} \le \|\omega(0)\|_{W^{1,p}} \ \exp\Big(C\int_0^t \|\nabla u(s)\|_{L^\infty(\Omega)}ds\Big). \tag{3.23}$$

However due to the fact that curl^{-1} is not continuous from L^∞ to $W^{1,\infty}$, time dependent estimates on $\|\nabla u(t)\|_{L^\infty(\Omega)}$ are subtle[20]. The same observation applies to the difference $\tilde\omega(t) = \omega(t) - \omega_0$ where ω_0 is a stationary solution and $\omega(t)$ a perturbation. The equation for $\tilde\omega$ is

$$\partial_t\tilde\omega + u\nabla\tilde\omega + \mathrm{curl}^{-1}(\tilde\omega)\cdot\nabla\omega_0 = 0\,. \tag{3.24}$$

Applying the operator D (derivative with respect to the first or the second spatial variable) to the equation (3.24), multiplying this equation by $(D\tilde\omega)^{p-1}$, integrating over Ω and using the Gronwall lemma, we deduce for $\tilde\omega$ the inequality

$$\|\tilde\omega(t)\|_{W^{1,p}} \le \|\tilde\omega(0)\|_{W^{1,p}}\exp\Big(C\int_0^t \big(\|\nabla\tilde u(s)\|_{L^\infty(\Omega)} + 1\big)ds\Big). \tag{3.25}$$

By the same argument involving a nonlinear Gronwall lemma,

$$\|\nabla\tilde u(t)\|_{L^\infty(\Omega)}$$

can be estimated, but only for a short time which depends on the initial data. More precisely, for

$$e^{Ct}\|\tilde\omega(0)\|_{W^{1,p}} \le D, \tag{3.26}$$

we have

$$\|\nabla\tilde u(t)\|_{L^\infty(\Omega)} \le \frac{E\|\tilde\omega(0)\|_{W^{1,p}}}{D - e^{Ct}\|\tilde\omega(0)\|_{W^{1,p}}} \tag{3.27}$$

with convenient constants C, D and E. Combining the formulas (3.25) and (3.27), we easily obtain the following result.

Theorem 3.1. *Given any stationary solution ω_0 of the 2D Euler equation in a bounded domain Ω, there exist positive constants C, C' and θ which depend only on Ω and ω_0 such that for any perturbation*

$$\tilde{\omega}(t) = \omega(t) - \omega_0$$

we have the implication

$$|t| \le \frac{1}{C} \log\Big(\frac{\theta}{\|\tilde{\omega}(0)\|_{W^{1,p}}}\Big) \quad \Rightarrow \quad \|\tilde{\omega}(t)\|_{W^{1,p}} \le \|\tilde{\omega}(0)\|_{W^{1,p}} e^{C't} \,. \tag{3.28}$$

In contrast to the sharp, but global, Wolibner estimate[20], the above estimates are only local in time and the constants are not "sharp". The bootstrap lemma states that under convenient hypotheses we can sharpen the constants in (3.28).

Theorem 3.2 (Bootstrap Lemma). *Denote by u_0 a steady flow with classical Liapunov exponent σ as in (3.2). Let μ be any real number strictly greater than σ. Let $T > 0$ and $p > 2$. Then there exist positive constants θ, C_1, C_2, and C_3 with the following property. Let $\omega(t,x) \in C(I\!R_t; W^{1,p})$ be any solution of the nonlinear Euler equation which satisfies the initial estimate*

$$\|\omega(0) - \omega_0\|_{W^{1,p}(\Omega)} \le C_1 \delta \tag{3.29}$$

and the L^2 estimate

$$\|\omega(t) - \omega_0\|_{L^2(\Omega)} \le C_2 \delta e^{\mu t} \ \ in\ [0,T]. \tag{3.30}$$

Let

$$T_\delta = \frac{1}{\mu} \log \frac{\theta}{\delta} \,.$$

Then in the time interval $0 \le t \le \min\{T, T_\delta\}$, the solution also satisfies the $W^{1,p}$ estimate

$$\|\omega(t) - \omega_0\|_{W^{1,p}(\Omega)} \le C_3 \delta e^{\mu t}. \tag{3.31}$$

Proof. Introduce the notation

$$\tilde{\omega}(t) = \omega(t) - \omega_0 \ \text{ and } \tilde{u}(t) = u(t) - u_0 = \text{curl}^{-1}\tilde{\omega}(t) \,.$$

Given $\eta > 0$, let

$$S_\eta = \frac{1}{\mu} \log \frac{\theta_0}{\eta} \qquad \text{and} \qquad S = \sup\{t : \ \|\tilde{u}(t)\|_{C^1} \le \eta e^{\mu t}\}. \tag{3.32}$$

Observe that $(\tilde{\omega}, \tilde{u})$ solves the equation

$$(\partial_t + u \cdot \nabla)\tilde{\omega} = -\tilde{u} \cdot \nabla \omega_0 \tag{3.33}$$

with initial data

$$\|\tilde{\omega}(0)\|_{W^{1,p}(\Omega)} \le C_1 \delta \,. \tag{3.34}$$

By the Duhamel Principle and Lemma 3.2, this implies, for $0 \le t \le \min\{T, S_\eta, S\}$, the estimate

$$\|\tilde{\omega}(t)\|_{W^{1,p}} \le C\delta e^{(\sigma+\epsilon)t} + C \int_0^t e^{(\sigma+\epsilon)(t-\tau)} \|\tilde{u}(\tau) \cdot \nabla\omega_0\|_{W^{1,p}} d\tau. \tag{3.35}$$

The norm on the right is estimated by

$$\|\tilde{u}(\tau) \cdot \nabla\omega_0\|_{W^{1,p}} \le C\|\tilde{u}\|_{W^{1,p}} \le C\|\tilde{\omega}\|_{L^p} \le \gamma\|\tilde{\omega}\|_{W^{1,p}} + C_\gamma\|\tilde{\omega}\|_{L^2} \le \gamma\|\tilde{\omega}\|_{W^{1,p}} + C_\gamma \delta e^{\mu\tau}$$

by (3.30), where γ is arbitrarily small. This is placed into the integral inequality (3.35) as

$$\|\tilde{\omega}(t)\|_{W^{1,p}} \le C\delta e^{(\sigma+\epsilon t} + C_\gamma \delta e^{\mu t} + C\gamma \int_0^t e^{(\sigma+\epsilon)(t-\tau)} \|\tilde{\omega}(\tau)\|_{W^{1,p}} \ d\tau.$$

Multiplying by $e^{-\mu t}$, we obtain

$$e^{-\mu t}\|\tilde{\omega}(t)\|_{W^{1,p}} \le C\delta + C\gamma\{\sup_{0\le\tau\le t} e^{-\mu\tau}\|\tilde{\omega}(\tau)\|_{W^{1,p}}\}\int_0^t e^{(\sigma+\epsilon-\mu)s}ds$$
$$\le C\delta + C\gamma\{\sup_{0\le\tau\le t} e^{-\mu\tau}\|\tilde{\omega}(\tau)\|_{W^{1,p}}\}.$$

Hence for $\gamma = 1/2C$ and for $0 \le t \le \min\{T, S_\eta, S\}$, we have

$$e^{-\mu t}\|\tilde{\omega}(t)\|_{W^{1,p}} \le 2C\delta. \tag{3.36}$$

Thus, since $p > 2$, we have for $0\le t \le \min\{T, S_\eta, S\}$,

$$\|\tilde{u}(t)\|_{C^1} \le C\|\tilde{u}(t)\|_{W^{2,p}} \le C\|\tilde{\omega}(t)\|_{W^{1,p}} \le C_0\delta e^{\mu t} \tag{3.37}$$

for some constant C_0. Now we choose $\eta = 2C_0\delta$, and $\theta = \theta_0/2C_0$. It follows from the definition of S that $S > S_\eta$. Hence

$$\|\tilde{\omega}(t)\|_{W^{1,p}} \le 2C\delta e^{\mu t} \tag{3.38}$$

for $0\le t \le \min\{T, S_\eta\}$. Noticing that $S_\eta = T_\delta$, we deduce (3.31).

Proof of the Instability Theorem 1.2. We now return to the nonlinear equation

$$(\partial_t + u_0\cdot\nabla)\tilde{\omega} + (\tilde{u}\cdot\nabla)\omega_0 = -(\tilde{u}\cdot\nabla)\tilde{\omega} \tag{3.39}$$

satisfied by the perturbation $\tilde{\omega} = \omega - \omega_0$. It takes the Duhamel form

$$\tilde{\omega}(t) = e^{tA_0}\tilde{\omega}(0) - \int_0^t e^{(t-\tau)A_0}(\tilde{u}\cdot\nabla)\tilde{\omega}(\tau)d\tau\,. \tag{3.40}$$

Assuming that $\Lambda > \sigma$, by compactness (see (1.15)) there is at least one eigenvalue λ such that $\mathrm{Re}\,\lambda = \Lambda$. If $\lambda = \Lambda$ is a real eigenvalue, we can choose $\tilde{\omega}(0,x) = \delta\phi_\lambda(x)$, where δ is small and ϕ_λ is the eigenfunction

$$A_0\phi_\lambda = \lambda\phi_\lambda. \tag{3.41}$$

However for the sake of generality the case where Λ is not an eigenvalue is considered below and then we choose for initial data the function $\tilde{\omega}(0,x) = \delta\mathcal{I}m\phi_\lambda(x)$. Taking the L^2 norm in (3.40) gives

$$\|\tilde{\omega}(t) - \delta e^{tA_0}\mathcal{I}m\phi_\lambda\|_{L^2} \le C_\nu\int_0^t e^{(t-\tau)\nu}\ \|(\tilde{u}\cdot\nabla)\tilde{\omega}\|_{L^2}d\tau \tag{3.42}$$

for any $\Lambda < \nu < 2\Lambda$. Define

$$T = \sup\Big\{s : \|\tilde{\omega}(t) - \delta e^{tA_0}\mathcal{I}m\phi_\lambda\|_{L^2} \le \frac{\delta}{2}\|e^{tA_0}\mathcal{I}m\phi_\lambda\|_{L^2}\,,\quad \forall t\in[0,s]\Big\}. \tag{3.43}$$

Then for $0 \le t \le T$ we have

$$\|\tilde{\omega}(t)\|_{L^2} \le C_2\delta e^{\mu t}, \tag{3.44}$$

where $\mu = \mathrm{Re}\,\lambda = \Lambda$. Hence from Theorem 3.2 with $\mu = \Lambda > \sigma$, we crudely estimate the nonlinear term as

$$\|(\tilde{u}\cdot\nabla)\tilde{\omega}\|_{L^2} \le \|\tilde{u}\|_{L^\infty}\|\nabla\tilde{\omega}\|_{L^2} \le C\|\tilde{\omega}\|^2_{W^{1,p}} \le C\{C_3\delta e^{\mu t}\}^2 \tag{3.45}$$

provided $t \le \min\{T_\delta, T\}$. We shall prove that $T_\delta \le T$ for δ small. If not, notice that

$$\|e^{tA_0}\delta\mathcal{I}m\phi_\lambda\|_{L^2} \ge c_0\delta e^{\mu t}. \tag{3.46}$$

Therefore we have for $0 \le t \le \min\{T_\delta, T\}$ the estimate

$$\begin{aligned}\|\tilde{\omega}(t) - \delta e^{tA_0}\mathcal{I}m\phi_\lambda\|_{L^2} &\le C\int_0^t e^{(t-\tau)\nu}C_3\{\delta e^{\mu\tau}\}^2 d\tau \le C\{\delta e^{\mu t}\}^2 \\ &\le \Big\{\frac{C}{c_0}\delta e^{\mu t}\Big\}\|\delta e^{tA_0}\mathcal{I}m\phi_\lambda\|_{L^2} \le C'\theta\|\delta e^{tA_0}\mathcal{I}m\phi_\lambda\|_{L^2} \\ &\le \frac{\delta}{4}\|e^{tA_0}\mathcal{I}m\phi_\lambda\|_{L^2} \end{aligned} \tag{3.47}$$

by choosing θ small. If $T < T_\delta$, we choose $t = T$ above to obtain

$$\|\tilde{\omega}(T) - \delta e^{TA_0}\mathcal{I}m\phi_\lambda\|_{L^2} \le \frac{\delta}{4}\|e^{tA_0}\mathcal{I}m\phi_\lambda\|_{L^2}\,, \tag{3.48}$$

which contradicts the definition of T. Therefore $T_\delta \le T$ and we can put $t = T_\delta$ to obtain

$$\|\tilde{\omega}(T_\delta)\|_{L^2} \ge \frac{3}{4}\|e^{T_\delta A_0}\delta\,\mathcal{I}m\phi_\lambda\|_{L^2} \ge \frac{3}{4}c_0\theta \equiv \epsilon_0 > 0. \tag{3.49}$$

§4. Conclusion and Acknowledgments

It has become increasingly apparent that the notions of stability and instability are dependent on the norms. Therefore our purpose has been to analyze both of them in the same norm.

The analysis of the stability could be also done for other types of minimizers. Consider a convex function $\omega \to G(\omega)$ and the minimizer of the functional

$$\int_\omega G(\omega)dx$$

under the constraints of mass equal to 1 and energy given. Formally such a minimizer is a solution of a generalized mean field equation

$$-\Delta\Psi = C\ (G')^{-1}(-\beta\Psi), \tag{4.1}$$

which ought to be studied in the same way as the standard mean field equation. In this direction some related stability results can be found in [19].

However it is the mean field equation itself or its generalization given in [16, 14, 12] which seems really pertinent for the description of coherent structures. While the result of Section 2 may explain why these structures persist, it does not explain why they appear in the first place. In fact, these stability results are time-reversible, so that in order to come close to a solution of the mean field equation one has to start close to this solution. The stationary solutions do not behave like attractors but like centers of a dynamical system. However, justification of their frequent appearance could be found in one of the following possibilities.

• These solutions should be the most probable ones in terms of a convenient probability measure to be defined on the configuration space.

• They might be produced by the conjunction of several circumstances, such as the fact that the initial vorticity is bounded in L^∞ and a family of solutions converges merely weakly to a nontrivial Young measure, the fact that in mean time the solutions are limits of solutions of the Navier-Stokes equations with viscosity tending to zero, and the fact that only the ω limit set for $t \to \infty$ is important.

This contribution is part of a research program which was initiated during the visit of C. Bardos to Brown University in the spring of 2000 and he wishes to thank Brown University for its support and warm hospitality during this period.

Any specialist of the field reading this paper will guess that we had the chance to benefit from some helpful discussions with several colleagues, including S. Friedlander, G. Haller, M. Kiessling, M. Vishik and G. Wolansky and we would like to thank all of them.

And finally we would like to thank Professor Li Ta-tsien who gave us the opportunity to publish this article in this special issue dedicated to the memory of Jacques-Louis Lions. He played a fundamental role in the scientific training of two of us, having been the thesis advisor of the first author and one of the first collaborators of the third author during his postdoctoral year in Paris.

References

[1] Arnold, V. I. & Khesin, B. A., Topological methods in hydrodynamics [M], Applied Mathematical Sciences, **125**, Springer, 1997.

[2] Bardos, C., Golse, F. & Levermore, D., Fluid dynamics limits of kinetic equations II: convergence proofs for the Boltzmann equation [J], *Comm. on Pure and Appl. Math.,* **46**(1993), 667–753.

[3] Caglioti, E., Lions, P. L., Marchioro, C. & Pulvirenti, M., A special class of stationary flows for two dimensional Euler equation, a statistical mechanics description, Part I [J], *Comm. Math. Phys.,* **143**(1992), 501–525.

[4] Caglioti, E., Lions, P. L., Marchioro, C. & Pulvirenti, M., A special class of stationary flows for two dimensional Euler equation, a statistical mechanics description, Part II [J], *Comm. Math. Phys.,* **174**(1995), 229–260.

[5] Friedlander, S. & Vishik, M., Dynamo theory methods for hydrodynamic stability [J], *J. Math. Pures Appl.,* **72**(1993), 145–180.

[6] Friedlander, S., Strauss, M. & Vishik, M., Nonlinear instability in an ideal fluid [J], *Ann. I. H. P.* (*Anal Nonlin.*), **14:**2(1997), 187–209.

[7] Friedlander, S., Strauss, W. & Vishik, M., Robustness of instability for the two-dimensional Euler equation [J], *SIAM J. Math. Anal.,* **30**(1999), 1343–1354.

[8] Grenier, E., On the nonlinear instability of Euler and Prandtl equations [J], *Comm. Pure Appl. Math.,* **53**(2000), 1067–1091.

[9] Guo, Y. & Rein, G., Isotropic steady states in galactic dynamics [J], *Comm. Math. Phys.,* **219**(2001), 607–629.

[10] Guo, Y. & Strauss, W., Instability of periodic BGK equilibria [J], *Comm. Pure Appl. Math.,* **48**(1995), 861–894.

[11] Kiessling, M., Statistical mechanics of classical particles with logarithmic interaction [J], *Comm. Pure Appl. Math.,* **47**(1993), 27–56.

[12] Lynden-Bell, D., Statistical mechanics of violent relaxation in stellar systems [J], *Mont. Not. R. Astr. Soc.,* **136**(1967), 101–121.

[13] Marchioro, C. & Pulvirenti, M., Mathematical theory of incompressible nonviscous fluids [M], Applied Mathematical Sciences Vol 96, Springer Verlag, New York, Berlin, Heidelberg, 1998.

[14] Miller, J., Peter B. Weichman & Cross, C., Statistical mechanics, Euler's equation and Jupiter's red spot [J], *Phys. Rev. A,* **45:**4(1992), 2328–2358.

[15] Onsager, L., Statistical hydrodynamics [J], *Supplemento al Nuovo Cimento,* **6**(1949), 279–287.

[16] Robert, R. & Sommeria, J., Statistical equilibrium states for two dimensional flows [J], *J. Fluid Mech.* **229**(1991), 291–310.

[17] Turkington, B., Statistical equilibrium measures and coherent states in two-dimensional turbulence [J], *Comm. Pure Appl. Math.,* **52**(1999), 781–809.

[18] Vishik, M., Spectrum of small oscillations of an ideal fluid and Lyapunov exponents [J], *J. Math. Pures Appl.,* **75**(1996), 531–557.

[19] Wolansky, G. & Ghil, M., Nonlinear stability for saddle solutions of ideal flows and symmetry breaking [J], *Comm. Math. Phys.,* **192**(1998), 713–736.

[20] Wolibner, W., Un théorème sur l'existence du mouvement plan d'un fluide parfait, homogène et incompressible, pendant un temps infiniment long [J], *Mat. Z.,* **37**(1933), 698–726.

[21] Yudovich, V. I., Nonstationary flow of an ideal incompressible liquid [J], *Zh. Vych. Mat.,* **3**(1963), 1032–1066.