Chin. Ann. of Math.
23B:2(2002),165-180.

REGULARITY THEORY FOR SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS WITH NEUMANN BOUNDARY CONDITIONS

A. BENSOUSSAN* J. FREHSE**

(*Dedicated to the memory of Jacques-Louis Lions*)

Abstract

The objective of this paper is to consider the theory of regularity of systems of partial differential equations with Neumann boundary conditions. It complements previous works of the authors for the Dirichlet case. This type of problem is motivated by stochastic differential games. The Neumann case corresponds to stochastic differential equations with reflection on boundary of the domain.

Keywords Regularity theory, Neumann boundary conditions, Dirichlet problem
2000 MR Subject Classification 35J55, 35B65
Chinese Library Classification O175.25 **Document Code** A
Article ID 0252-9599(2002)02-0165-16

§1. Introduction

We consider here a system of nonlinear P.D.E. with Neumann boundary conditions. It is the counterpart of the Dirichlet problem considered by the authors in several publications (see in particular the book [1]). Although a natural counterpart of the Dirichlet case, the results of this paper are presented here in a simplified and self-contained manner.

§2. Setting of the Problem

2.1. Notation and Assumptions

We consider a smooth, bounded domain Ω of R^n. In particular, the boundary will be representable by local charts, so that in a sense the Neumann problem can be reduced to the Dirichlet problem. We consider the operator

$$A = -\operatorname{div} a\,D, \tag{2.1}$$

where $a(x) \equiv a_{ij}(x)$ satisfies

$$\begin{aligned} &a_{ij}(x) = a_{ji}(x) \quad \text{is Lipschitz continuous on } \bar{\Omega}, \\ &a_{ji}(x)\xi_i\xi_j \geq \alpha_0|\xi|^2, \quad \forall \xi \in R^n, \alpha_0 > 0. \end{aligned} \tag{2.2}$$

Manuscript received December 27, 2001.

*University Paris Dauphine and CNES (Centre National d'Etudes Spatiales), 2, Place Maurice Quentin, 75001, Paris, France.
**Institüt für Angewandte Mathematik, Universität Bonn, Bonn, Germany.

We now consider "Hamiltonians", namely functions $H_\nu(x,p), \nu = 1,\cdots,N$, $p = p_1,\cdots,p_N$, with $p_\nu \in R^n$, with the following properties

$$H_\nu(x,p) \quad \text{are Caratheodory functions,} \tag{2.3}$$

$$\sum_\nu H_\nu(x,p) \geq -\lambda - \lambda^0 \Big| \sum_\nu p_\nu \Big|^2, \tag{2.4}$$

$$H_\nu(x,p) \leq \lambda_\nu + \lambda_\nu^0 |p_\nu|^2, \tag{2.5}$$

$$H_\nu(x,p) = Q(x,p)p_\nu + H_\nu^0(x,p), \tag{2.6}$$

with

$$|Q(x,p)| \leq k + K|p|, \tag{2.7}$$

$$|H_\nu^0(x,p)| \leq k_\nu + K_\nu \sum_{\mu \leq \nu} |p_\mu|^2. \tag{2.8}$$

Let α be a positive constant, we are interested in the system

$$\begin{aligned} Au_\nu + \alpha u_\nu &= H_\nu(x, Du), \\ \frac{\partial u_\nu}{\partial n_A}\Big|_{\partial\Omega} &= 0, \end{aligned} \tag{2.9}$$

which is written in the variational form

$$\int_\Omega a Du_\nu Dv dx + \alpha \int_\Omega u_\nu v dx = \int_\Omega H_\nu(x, Du) v dx, \quad \forall v \in H^1(\Omega) \cap L^\infty(\Omega). \tag{2.10}$$

2.2. Statement of the Main Result

Theorem 2.1. *We assume* (2.2) *to* (2.8). *Then there exists a solution of* (2.10) *which belongs to* $\big(W^{2,s}(\Omega)\big)^N, \forall 2 \leq s < \infty$.

Remark 2.1. The assumptions are subject to some flexibility, in the sense that we can combine in a linear manner the equations, to achieve the desired structure (see [1] for details).

Remark 2.2. The assumptions (2.6), (2.7) and (2.8) are restrictive only for $\nu = 1,\cdots,N-1$. A general quadratic growth can be assumed by $H_N(x,p)$, and it is always possible to define $H_N^0(x,p)$ by the relation

$$H_N^0(x,p) = H_N(x,p) - Q(x,p)p_N$$

and (2.8) will be verified.

§3. The $H^1(\Omega) \cap L^\infty(\Omega)$ Theory

3.1. Approximation

The problem (2.10) makes sense for solutions in $H^1(\Omega) \cap L^\infty(\Omega)$. This is what we are looking for to begin with. We consider as an approximation the problem with Hamiltonians

$$H_\nu^\varepsilon(x,p) = \frac{H_\nu(x,p)}{1 + \varepsilon|H(x,p)|}, \tag{3.1}$$

where $H(x,p)$ stands for the vector $H_1,\cdots,H_N$. Note that all the assumptions (2.3)–(2.8) are satisfied for the hamiltonian $H_\nu^\varepsilon(x,p)$ with the same constants, provided we define

$$Q^\varepsilon(x,p) = \frac{Q(x,p)}{1 + \varepsilon|H(x,p)|}, \tag{3.2}$$

$$H_\nu^{0,\varepsilon}(x,p) = \frac{H_\nu^0(x,p)}{1 + \varepsilon|H(x,p)|}. \tag{3.3}$$

Morevover one has

$$|H^\varepsilon(x,p)| \leq \frac{1}{\varepsilon}, \quad \forall x, p. \tag{3.4}$$

Therefore, by application of Schauder's fixed point theorem, one can obtain the existence of a solution $u^\varepsilon = (u^\varepsilon_\nu)$ in $\big(W^{2,s}(\Omega)\big)^N$ of the problem

$$\int_\Omega aDu^\varepsilon_\nu Dvdx + \alpha \int_\Omega u^\varepsilon_\nu vdx = \int_\Omega H^\varepsilon_\nu(x, Du^\varepsilon)vdx, \quad \forall v \in H^1(\Omega) \cap L^\infty(\Omega). \tag{3.5}$$

3.2. L^∞ Estimate

Since the estimate are exactly the same for (3.5) and (2.10), we shall to simplify the notation consider only (2.10), with a priori estimates, assuming a solution in $\big(W^{2,s}(\Omega)\big)$ exists. Let us write

$$\tilde{u} = \sum_\nu u_\nu,$$

and summing up (2.10), we get

$$\int_\Omega aD\tilde{u}_\nu Dvdx + \alpha \int_\Omega \tilde{u}_\nu vdx = \int_\Omega \sum_\nu H_\nu(x, Du)vdx. \tag{3.6}$$

Consider the function

$$E = \exp \gamma(\tilde{u} + L)^-, \tag{3.7}$$

where $\gamma, L > 0$ will be defined later. We test (3.6) with $v = 1 - E$, which is a negative function in $H^1 \cap L^\infty$. We get

$$\gamma \int_\Omega aD\tilde{u}.D\tilde{u}E\mathbf{1}_{\{\tilde{u}+L<0\}}dx + \alpha \int_\Omega \tilde{u}(1-E)dx = \int_\Omega \sum_\nu H_\nu(x,u)(1-E)dx$$

and using the assumption (2.4), together with the fact that $1 - E < 0$, yields

$$\leq \int_\Omega (-\lambda - \lambda^0|D\tilde{u}|^2)(1-E)dx.$$

Therefore, also using (2.2) we get

$$\alpha_0\gamma \int_\Omega |D\tilde{u}|^2 E\mathbf{1}_{\{\tilde{u}+L<0\}}dx + \alpha \int_\Omega (\tilde{u}+L)(1-E)dx$$
$$\leq \int_\Omega (\alpha L - \lambda)(1-E)dx + \lambda^0 \int_\Omega |D\tilde{u}|^2 E\mathbf{1}_{\{\tilde{u}+L<0\}}dx.$$

Choosing

$$\gamma = \frac{\lambda^0}{\alpha_0}, \quad L = \frac{\lambda}{\alpha}$$

yields

$$\int_\Omega (\tilde{u}+L)(1-E)dx \leq 0.$$

Since clearly $(\tilde{u}+L)(1-E) \geq 0$, we deduce

$$(\tilde{u}+L)(1-E) = 0 \quad \text{a.e., hence} \quad \tilde{u}+L \geq 0 \quad \text{a.e.}$$

Therefore we have shown

$$\tilde{u} \geq -\frac{\lambda}{\alpha} \quad \text{a.e.} \tag{3.8}$$

We next consider in (2.10) the test function $E_\nu - 1$, with

$$E_\nu = \exp \gamma_\nu (u_\nu - l_\nu)^+$$

for convenient positve constants γ_ν, l_ν. Note that $E_\nu - 1 \geq 0$. Making use of the assumption (2.5), it follows that

$$\begin{aligned}&\gamma_\nu \int_\Omega a D u_\nu E_\nu \mathbf{1}_{\{u_\nu > l_\nu\}} dx + \alpha \int_\Omega u_\nu (E_\nu - 1) dx \\ \leq &\int_\Omega \lambda_\nu (E_\nu - 1) dx + \lambda_\nu^0 \int_\Omega |D u_\nu|^2 E_\nu \mathbf{1}_{\{u_\nu > l_\nu\}} dx.\end{aligned}$$

Thus taking

$$\gamma_\nu = \frac{\lambda_\nu^0}{\alpha_0}, \quad l_\nu = \frac{\lambda_\nu}{\alpha},$$

we deduce again

$$(u_\nu - l_\nu)(E_\nu - 1) = 0 \quad \text{a.e.}$$

which implies

$$u_\nu \leq \frac{\lambda_\nu}{\alpha}. \tag{3.9}$$

Continuing (3.8) and (3.9) we obtain

$$u_\nu \geq \frac{-\Big(\lambda + \sum\limits_{\mu \neq \nu} \lambda_\mu\Big)}{\alpha}. \tag{3.10}$$

So we have proven

$$\| u_\nu \|_\infty \leq \frac{\zeta_\nu}{\alpha} = \frac{1}{\alpha} \text{Max}\Big(\lambda_\nu, \lambda + \sum_{\mu \neq \nu} \lambda_\mu\Big). \tag{3.11}$$

Remark 3.1. The estimate (3.11) is the same as for the solution of the Dirichlet problem. This is due to the presence of the zero order term αu_ν, and to the fact that for the Dirichlet as well as for the Neumann problem, the maximum and the minimum of the function u_ν do not take place at the boundary of the domain.

3.3. H^1 Estimate

The H^1 estimate is done by using the special structure (2.6), (2.7), (2.8). Recall that

$$|u_\nu(x)| \leq \frac{\zeta_\nu}{\alpha}. \tag{3.12}$$

We introduce the function

$$\beta(s) = e^s - s - 1, \quad F = \prod_{\nu=1}^N \exp \beta(\gamma_\nu u_\nu),$$

where γ_ν is a positive constant to be defined later. We test (2.10) with $v = F \gamma_\nu \beta'(\gamma_\nu u_\nu) \in H^1 \cap L^\infty$. So we get (as in the Dirichlet case)

$$\begin{aligned}&\sum_\nu \int_\Omega \gamma_\nu^2 a D u_\nu . D u_\nu e^{\gamma_\nu u_\nu} F dx + \int_\Omega a \frac{DF.DF}{F} dx \\ = &\int_\Omega Q D F dx + \int_\Omega \sum_\nu \gamma_\nu (H_\nu^0(Du) - \alpha u_\nu) F (e^{\gamma_\nu u_\nu} - 1) dx,\end{aligned}$$

hence also

$$\sum_\nu \int_\Omega \gamma_\nu^2 aDu_\nu.Du_\nu e^{\gamma_\nu u_\nu} F dx$$
$$\leq \int_\Omega F\frac{a^{-1}Q.Q}{4}dx + \int_\Omega \sum_\nu \gamma_\nu (H_\nu^0(Du) - \alpha u_\nu) F(e^{\gamma_\nu u_\nu} - 1)dx. \tag{3.13}$$

Introduce next the function

$$X = \prod_{\nu=1}^N (\exp\beta(\gamma_\nu u_\nu) + \exp\beta(-\gamma_\nu u_\nu)) \tag{3.14}$$

and the related quantities

$$X_\nu = X\frac{e^{\gamma_\nu u_\nu}\exp\beta(\gamma_\nu u_\nu) + e^{-\gamma_\nu u_\nu}\exp\beta(-\gamma_\nu u_\nu)}{\exp\beta(\gamma_\nu u_\nu) + \exp\beta(-\gamma_\nu u_\nu)},$$
$$\tilde{X}_\nu = X\frac{(e^{\gamma_\nu u_\nu} - 1)\exp\beta(\gamma_\nu u_\nu) - (e^{-\gamma_\nu u_\nu} - 1)\exp\beta(-\gamma_\nu u_\nu)}{\exp\beta(\gamma_\nu u_\nu) + \exp\beta(-\gamma_\nu u_\nu)} \tag{3.15}$$

and we have the inequalities

$$2^N \leq X \leq X_\nu \leq Xe^{\frac{\gamma_\nu \zeta_\nu}{\alpha}}, \quad |\tilde{X}_\nu| \leq X_\nu.$$

Applying the relation (3.13) with γ_ν changed into $-\gamma_\nu$, and summing up the 2^N relations obtained in this way, we get the inequality

$$\sum_\nu \int_\Omega \gamma_\nu^2 aDu_\nu Du_\nu X_\nu dx \leq \int_\Omega X\frac{a^{-1}Q.Q}{4}dx + \int_\Omega \sum_\nu \gamma_\nu (H_\nu^0(Du) - \alpha u_\nu)\tilde{X}_\nu dx \tag{3.16}$$

and using the assumption (2.8) yields

$$\leq \int_\Omega X\frac{a^{-1}Q.Q}{4}dx + \sum_\nu \int_\Omega \gamma_\nu (k_\nu + \zeta_\nu + K_\nu |Du_\nu|^2) X_\nu dx$$
$$+ \sum_\nu \int_\Omega |Du_\nu|^2 \sum_{\mu>\nu} \gamma_\mu K_\mu X_\mu dx.$$

Use (2.7) to note that

$$\frac{a^{-1}Q.Q}{4} \leq \frac{1}{2\alpha_0}(k^2 + K^2|Du|^2),$$

hence it follows from (3.16) also

$$\alpha_0 \sum_\nu \int_\Omega \gamma_\nu^2 |Du\nu|^2 X_\nu dx \leq \int_\Omega \left(\frac{1}{2\alpha_0}k^2 X + \sum_\nu \gamma_\nu (k_\nu + \zeta_\nu) X_\nu\right) dx$$
$$+ \int_\Omega \sum_\nu |Du_\nu|^2 \left(\frac{K^2}{2\alpha_0}X_\nu + \gamma_\nu K_\nu X_\nu + \sum_{\mu>\nu} \gamma_\mu K_\mu X_\mu\right) dx,$$

but also

$$X_\mu \leq X_\nu e^{\frac{\gamma_\mu \zeta_\mu}{\alpha}}, \quad \forall \mu, \nu,$$

hence

$$\sum_\nu \int_\Omega |Du_\nu|^2 X_\nu \left(\alpha_0\gamma_\nu^2 - \frac{K^2}{2\alpha_0} - \gamma_\nu K_\nu - \sum_{\mu>\nu}\gamma_\mu K_\mu e^{\frac{\gamma_\mu\zeta_\mu}{\alpha}}\right) dx$$
$$\le \int_\Omega \left(\frac{1}{2\alpha_0}k^2 X + \sum_\nu \gamma_\nu(k_\nu+\zeta_\nu)X_\nu\right) dx. \tag{3.17}$$

Choosing the constants γ_ν so that

$$\alpha_0\gamma_\nu^2 - \frac{K^2}{2\alpha_0} - \gamma_\nu K_\nu - \sum_{\mu>\nu}\gamma_\mu K_\mu e^{\frac{\gamma_\mu\zeta_\mu}{\alpha}} > 0, \quad \forall\nu \tag{3.18}$$

and recalling that $X_\nu \ge 2^N$, we obtain

$$\int_\Omega |Du|^2 dx \le K_0. \tag{3.19}$$

3.4. Convergence

Since the preceding estimates hold for u^ε, we have also

$$\| u^\varepsilon \|_\infty \le \zeta, \tag{3.20}$$
$$\| u^\varepsilon \|_{(H^1(\Omega))^N} \le K_1. \tag{3.21}$$

So we can extract a subsequence, still denoted by u^ε, such that

$$u_\nu^\varepsilon \to u_\nu \text{ in } H^1(\Omega) \text{ weakly and } L^\infty(\Omega) \text{ weak star, also a.e.} \tag{3.22}$$

We want to prove that

$$u_\nu^\varepsilon \to u_\nu \text{ in } H^1(\Omega) \quad \text{strongly.} \tag{3.23}$$

We perform a calculation close to that of Section 3.3, with this time

$$F = \prod_{\nu=1}^N \exp\beta(\gamma_\nu(u_\nu^\varepsilon - u_\nu)),$$

and test (3.5) with $v = F\gamma_\nu\beta'(\gamma_\nu(u_\nu^\varepsilon - u_\nu))$. We obtain

$$\sum_\nu \int_\Omega \gamma_\nu^2 aD(u_\nu^\varepsilon - u_\nu).D(u_\nu^\varepsilon - u_\nu)e^{\gamma_\nu(u_\nu^\varepsilon-u_\nu)}F dx + \int_\Omega a\frac{DF.DF}{F}dx$$
$$= \int_\Omega \left(Q^\varepsilon - \gamma_\nu aDu_\nu(e^{\gamma_\nu(u_\nu^\varepsilon-u_\nu)}-1)\right) DF dx$$
$$+ \int_\Omega F\left(H_\nu^{0,\varepsilon} - cu_\nu^\varepsilon + Q^\varepsilon Du_\nu\right)\gamma_\nu(e^{\gamma_\nu(u_\nu^\varepsilon-u_\nu)}-1)dx$$
$$- \int_\Omega FaDu_\nu.D(u_\nu^\varepsilon - u_\nu)\gamma_\nu^2 e^{\gamma_\nu(u_\nu^\varepsilon-u_\nu)}dx$$
$$= \mathrm{I} + \mathrm{II} + \mathrm{III}. \tag{3.24}$$

Introduce the function

$$h(s) = \sum_\nu \gamma_\nu^2(e^{|\gamma_\nu s_\nu|}-1)^2, \tag{3.25}$$

where s represents the vector $(s_1,\cdots,s_N)$. Note

$$|a(x)| \le M, \quad |Q^\varepsilon| \le k + K|Du| + K|Du^\varepsilon - Du|.$$

Moreover

$$DF = F\sum_{\nu=1}^N \gamma_\nu(e^{\gamma_\nu(u_\nu^\varepsilon-u_\nu)}-1)D(u_\nu^\varepsilon - u_\nu),$$

hence

$$|DF| \le F|D(u^\varepsilon - u)|h^{1/2}(u^\varepsilon - u). \tag{3.26}$$

Therefore considering the integrals on the right hand side of (3.24), we have

$$\begin{aligned}|\mathrm{I}| \le &\int_\Omega (K|Du^\varepsilon - Du| + M|Du|h^{1/2}(u^\varepsilon - u))|DF|dx\\ &+ \int_\Omega F(k + K|Du|)|D(u^\varepsilon - u)|h^{1/2}(u^\varepsilon - u)dx,\end{aligned} \tag{3.27}$$

so

$$\begin{aligned}&\int_\Omega a\frac{DF.DF}{F}dx\\ &\ge \mathrm{I} - \int_\Omega F\Big[\Big(\frac{K^2}{2\alpha_0} + \frac{1}{2}\Big)|D(u^\varepsilon - u)|^2 + h(u^\varepsilon - u)\Big(k^2 + K^2 + \frac{M^2}{2\alpha_0}\Big)|Du|^2\Big]dx,\end{aligned} \tag{3.28}$$

$$\begin{aligned}&\sum_\nu \int_\Omega \gamma_\nu^2 aD(u_\nu^\varepsilon - u_\nu)e^{\gamma_\nu(u_\nu^\varepsilon - u_\nu)}Fdx\\ &\le \int_\Omega F\Big[\Big(\frac{K^2}{2\alpha_0} + \frac{1}{2}\Big)|D(u^\varepsilon - u)|^2\\ &\quad + h(u^\varepsilon - u)\Big(k^2 + \Big(K^2 + \frac{M^2}{2\alpha_0}\Big)|Du|^2\Big)\Big]dx + \mathrm{II} + \mathrm{III}.\end{aligned} \tag{3.29}$$

Introducing the quantities analogous to (3.14), (3.15), namely

$$X^\varepsilon = \prod_{\nu=1}^N \left(\exp\beta(\gamma_\nu(u_\nu^\varepsilon - u_\nu)) + \exp\beta(-\gamma_\nu(u_\nu^\varepsilon - u_\nu))\right), \tag{3.30}$$

$$\begin{aligned}X_\nu^\varepsilon &= X\frac{e^{\gamma_\nu(u_\nu^\varepsilon - u_\nu)}\exp\beta(\gamma_\nu(u_\nu^\varepsilon - u_\nu)) + e^{-\gamma_\nu(u_\nu^\varepsilon - u_\nu)}\exp\beta(-\gamma_\nu(u_\nu^\varepsilon - u_\nu))}{\exp\beta(\gamma_\nu(u_\nu^\varepsilon - u_\nu)) + \exp\beta(-\gamma_\nu(u_\nu^\varepsilon - u_\nu))},\\ \tilde{X}_\nu^\varepsilon &= X\frac{(e^{\gamma_\nu(u_\nu^\varepsilon - u_\nu)} - 1)\exp\beta(\gamma_\nu(u_\nu^\varepsilon - u_\nu)) - (e^{-\gamma_\nu(u_\nu^\varepsilon - u_\nu)} - 1)\exp\beta(-\gamma_\nu(u_\nu^\varepsilon - u_\nu))}{\exp\beta(\gamma_\nu(u_\nu^\varepsilon - u_\nu)) + \exp\beta(-\gamma_\nu(u_\nu^\varepsilon - u_\nu))},\end{aligned} \tag{3.31}$$

we have of course

$$2^N \le X^\varepsilon \le X_\nu^\varepsilon \le X^\varepsilon e^{2\frac{\gamma_\nu\zeta_\nu}{\alpha}}, \quad |\tilde{X}_\nu^\varepsilon| \le X_\nu \tag{3.32}$$

and also

$$|\tilde{X}_\nu^\varepsilon| \le (e^{|\gamma_\nu(u_\nu^\varepsilon - u_\nu)|} - 1)X^\varepsilon. \tag{3.33}$$

From (3.30), we deduce, writing the 2^N inequalities corresponding to all changes of γ_ν into $-\gamma_\nu$,

$$\begin{aligned}&\sum_\nu \int_\Omega \gamma_\nu^2 aD(u_\nu^\varepsilon - u_\nu)D(u_\nu^\varepsilon - u_\nu)X_\nu^\varepsilon dx\\ &\le \int_\Omega X^\varepsilon\Big[\Big(\frac{K^2}{2\alpha_0} + \frac{1}{2}\Big)|D(u^\varepsilon - u)|^2 + h(u^\varepsilon - u)\Big(k^2 + \Big(K^2 + \frac{M^2}{2\alpha_0}\Big)|Du|^2\Big)\Big]dx\\ &\quad + \sum_\nu \int_\Omega (H_\nu^{0,\varepsilon} - cu_\nu^\varepsilon + Q^\varepsilon Du_\nu)\gamma_\nu\tilde{X}_\nu^\varepsilon dx - \sum_\nu \int_\Omega \gamma_\nu^2 X_\nu^\varepsilon D(u_\nu^\varepsilon - u_\nu)Du_\nu dx\\ &= \mathrm{I}' + \mathrm{II}' + \mathrm{III}'.\end{aligned} \tag{3.34}$$

We first check that

$$|H_\nu^{0,\varepsilon} - cu_\nu^\varepsilon| \le k_\nu + \zeta_\nu + 2K_\nu|Du|^2 + 2K_\nu \sum_{\mu\le\nu} |D(u_\mu^\varepsilon - u_\mu)|^2,$$

hence using (3.33) and (3.34) we have

$$\begin{aligned}
&\Big|\sum_\nu (H_\nu^{0,\varepsilon} - cu_\nu^\varepsilon)\gamma_\nu \tilde{X}_\nu^\varepsilon\Big| \\
&\le X^\varepsilon \gamma_\nu (k_\nu + \zeta_\nu + 2K_\nu|Du|^2)(e^{|\gamma_\nu(u_\nu^\varepsilon - u_\nu)|} - 1) + 2K_\nu\gamma_\nu X_\nu^\varepsilon \sum_{\mu\le\nu} |D(u_\mu^\varepsilon - u_\mu)|^2 \\
&\le X^\varepsilon h^{1/2}(u^\varepsilon - u)\Big(\Big(\sum_\nu (k_\nu+\zeta_\nu)^2\Big)^{1/2} + 2\Big(\sum_\nu K_\nu^2\Big)^{1/2}|Du|^2\Big) \\
&\quad + 2\sum_\nu k_\nu\gamma_\nu X_\nu^\varepsilon |D(u_\nu^\varepsilon - u_\nu)|^2 + 2\sum_\nu |D(u_\nu^\varepsilon - u_\nu)|^2 \sum_{\mu>\nu}\gamma_\mu K_\mu X\mu^\varepsilon
\end{aligned}$$

and

$$\Big|\sum_\nu Q^\varepsilon Du_\nu \gamma_\nu \tilde{X}_\nu^\varepsilon\Big| \le X^\varepsilon(k + K|Du^\varepsilon|)|Du|h^{1/2}(u^\varepsilon - u).$$

Collecting results, we deduce from (3.36)

$$\begin{aligned}
&\sum_\nu \int_\Omega |D(u_\nu^\varepsilon - u_\nu)|^2\Big(\alpha_0\gamma_\nu^2 - \frac{K^2}{2\alpha_0} - \frac{1}{2} - 2K_\nu\gamma_\nu - 2\sum_{\mu>\nu}\gamma_\mu K_\mu e^{\frac{2\zeta_\mu\gamma_\mu}{\alpha}}\Big)X_\nu^\varepsilon dx \\
&\le \int_\Omega X^\varepsilon h(u^\varepsilon - u)\Big(k^2 + \Big(K^2 + \frac{M}{2\alpha_0}\Big)|Du|^2\Big)dx \\
&\quad + \int_\Omega X^\varepsilon h^{1/2}(u^\varepsilon - u)\Big(\Big(\sum_\nu (k_\nu+\zeta_\nu)^2\Big)^{1/2} + 2\Big(\sum_\nu K_\nu^2\Big)^{1/2}|Du|^2\Big)dx \\
&\quad + \int_\Omega X^\varepsilon h^{1/2}(u^\varepsilon - u)|Du|(k + K|Du^\varepsilon|)dx - \sum_\nu \int_\Omega \gamma_\nu^2 X_\nu^2 D(u_\nu^\varepsilon - u_\nu)Du_\nu dx
\end{aligned} \tag{3.35}$$

as $\varepsilon \to 0$, $h(u^\varepsilon - u) \to 0$ pointwise, $X^\varepsilon \to 2^N$, $X_\nu^\varepsilon \to 2^N$. In view of the weak convergence in L^2 of Du_ν^ε to Du_ν, and the L^∞ bounds, it is easy to convince oneself that the right hand side of (3.37) tends to 0. Choosing the γ_ν so that

$$\alpha_0\gamma_\nu^2 - \frac{K^2}{2\alpha_0} - \frac{1}{2} - 2K_\nu\gamma_\nu - 2\sum_{\mu>\nu}\gamma_\mu K_\mu e^{\frac{2\zeta_\mu\gamma_\mu}{\alpha}} > 0$$

and recalling the $X_\nu^\varepsilon \ge 2^N$, we deduce immediately from (3.37) that

$$\int_\Omega |D(u^\varepsilon - u)|^2 dx \to 0.$$

Hence we have proven (3.23). We may assume that

$$Du^\varepsilon \to Du \quad \text{a.e.},$$

thus

$$H_\nu^\varepsilon(x, Du^\varepsilon) \to H_\nu(x, Du) \quad \text{a.e.}$$

Moreover since

$$|H^\varepsilon(x, Du^\varepsilon)| \le C(1 + |Du^\varepsilon|^2),$$

the convergence of Du^ε in L^2 implies the equi-integrability of $H^\varepsilon_\nu(x, Du^\varepsilon)$. Hence from Vitali's theorem

$$H^\varepsilon_\nu(x, Du^\varepsilon) \to H_\nu(x, Du) \text{ in } L^1.$$

Therefore we can pass to the limit in (3.5), showing that u is an $H^1(\Omega) \cap L^\infty(\Omega)$ solution of (2.10).

§4. Regularity

4.1. An Inequality

We associate to u_ν a constant c_ν which is arbitrary, provided that

$$|c_\nu| \le \frac{\zeta_\nu}{\alpha}, \tag{4.1}$$

and consider as in Section 3.3, the function

$$F = \prod_{\nu=1}^N \exp\beta(\gamma_\nu(u_\nu - c_\nu)).$$

Let ψ be in $C^1(\bar\Omega), \psi \ge 0$, and test (2.10) with

$$v = F\gamma_\nu\beta'(\gamma_\nu(u_\nu - c_\nu))\psi.$$

We obtain

$$\sum_\nu \int_\Omega \gamma_\nu^2 aDu_\nu Du_\nu e^{\gamma_\nu(u_\nu-c_\nu)} F\psi dx + \int_\Omega a\frac{DF.DF}{F}\psi dx + \int_\Omega aDF.D\psi dx$$
$$= \int_\Omega Q.DF\psi dx + \sum_\nu \int_\Omega \gamma_\nu(H^0_\nu(Du) - \alpha u_\nu)F(e^{\gamma_\nu(u_\nu-c_\nu)} - 1)\psi dx,$$

hence instead of (3.13)

$$\sum_\nu \int_\Omega \gamma_\nu^2 aDu_\nu Du_\nu e^{\gamma_\nu(u_\nu-c_\nu)} F\psi dx + \int_\Omega aDF.D\psi dx$$
$$\le \int_\Omega F\frac{a^{-1}Q.Q}{4}\psi dx + \sum_\nu \int_\Omega \gamma_\nu(H^0_\nu(Du) - \alpha u_\nu)F(e^{\gamma_\nu(u_\nu-c_\nu)} - 1)\psi dx. \tag{4.2}$$

Introduce the function

$$X = \prod_{\nu=1}^N (\exp\beta(\gamma_\nu(u_\nu - c_\nu)) + \exp\beta(-\gamma_\nu(u_\nu - c_\nu))) \tag{4.3}$$

and the related quantities

$$X_\nu = X\frac{e^{\gamma_\nu(u_\nu-c_\nu)}\exp\beta(\gamma_\nu(u_\nu - c_\nu)) + e^{-\gamma_\nu(u_\nu-c_\nu)}\exp\beta(-\gamma_\nu(u_\nu - c_\nu))}{\exp\beta(\gamma_\nu(u_\nu - c_\nu)) + \exp\beta(-\gamma_\nu(u_\nu - c_\nu))},$$
$$\tilde X_\nu = X\frac{(e^{\gamma_\nu(u_\nu-c_\nu)} - 1)\exp\beta(\gamma_\nu(u_\nu - c_\nu)) - (e^{-\gamma_\nu(u_\nu-c_\nu)} - 1)\exp\beta(-\gamma_\nu(u_\nu - c_\nu))}{\exp\beta(\gamma_\nu(u_\nu - c_\nu)) + \exp\beta(-\gamma_\nu(u_\nu - c_\nu))} \tag{4.4}$$

again with 2^N relations with the possible choices of γ_ν and $-\gamma_\nu$, and adding up, we get

$$\sum_\nu \int_\Omega \gamma_\nu^2 aDu_\nu Du_\nu X_\nu\psi dx + \int_\Omega aDX.D\psi dx$$
$$\le \int_\Omega X\frac{a^{-1}Q.Q}{4}\psi dx + \sum_\nu \int_\Omega \gamma_\nu(H^0_\nu(Du) - \alpha u_\nu))\tilde X_\nu\psi dx.$$

Performing as in Section 3.3 for all terms where ψ arises, and picking the constants γ_ν as there, we obtain

$$k_0 \int_\Omega |Du|^2 \psi dx + \int_\Omega aDX.D\psi dx \le K_0 \int_\Omega \psi dx. \tag{4.5}$$

Note that

$$DX = \sum_\nu \gamma_\nu \tilde{X}_\nu Du_\nu \tag{4.6}$$

and thus

$$|DX| \le C|u-c||Du|, \tag{4.7}$$

where c stands for the vector of constants $(c_1, \cdots, c_N)$.

4.2. $W^{1,p}$ Regularity for $2 \le p<p+\varepsilon$

Consider balls $B_R(x_0)$ of center x_0 and radius R, such that $|\Omega \cap B_R(x_0)| > 0$. By the smoothness of the boundary, we have (denote $\Gamma = \partial\Omega$)

$$|\Omega \cap B_R(x_0)| \ge c_0 R^n \quad |(R^n - \Omega) \cap B_R(x_0)| \ge c_1 R^n, \text{ if } x_0 \in \Gamma. \tag{4.8}$$

To the function u_ν and to a ball $B-R(x_0)$ we associate the constant

$$c_\nu^R = \begin{cases} \frac{1}{|B_{2R}|} \int_{B_{2R(x_0)}} u_\nu dx, & \text{if } B_{2R}(x_0) \subset \Omega, \\ \frac{1}{|B_{4R}(x_0') \cap \Omega|} \int_{B_{4R}(x_0') \cap \Omega} u_\nu dx, & \text{if } B_{2R}(x_0) \cap (R^n - \Omega) \ne \emptyset, \\ \text{where } x_0' \in \Gamma \cap B_{2R}(x_0). \end{cases} \tag{4.9}$$

We shall use the Poincaré's inequality

$$\Big(\int_{\Omega \cap B_{2R(x_0)}} |u_\nu - u_\nu^R|^\lambda dx \Big)^{\frac{1}{\lambda}} \le cR^{n\left(\frac{1}{\lambda} - \frac{1}{\mu}\right)+1} \Big(\int_{\Omega \cap B_{6R(x_0)}} |Du_\nu|^\mu dx \Big)^{\frac{1}{\mu}} \tag{4.10}$$

with $\lambda, \mu \ge 1, n\left(\frac{1}{\lambda} - \frac{1}{\mu}\right) + 1 \ge 0$. Let τ be a cut off function

$$\tau = \begin{cases} 1 \text{ on } B_1(0), \\ 0 \text{ outside } B_2(0) \end{cases}$$

and $0 \le \tau \le 1, \tau \in C^\infty$. We denote

$$\tau_R(x) = \tau\Big(\frac{x - x_0}{R}\Big).$$

We take in (4.5) $\psi = \tau_R^2$, and $c_\nu = c_\nu^R$. So from (4.7) we have

$$|DX| \le C|u - c^R||Du|$$

and thus we deduce from (4.7) the inequality

$$\int_{B_R(x_0) \cap \Omega} |Du|^2 dx \le C \int_{B_{2R}(x_0) \cap \Omega} |Du| \frac{|u - c^R|}{R} dx + CR^n. \tag{4.11}$$

From Hölder's inequality and Poincaré's inequality, we have

$$\begin{aligned} \int_{B_{2R} \cap \Omega} |Du| \frac{|u - c^R|}{R} dx &\le \frac{C}{R} \Big(\int_{B_{2R} \cap \Omega} |Du|^{\frac{2n}{n+1}} dx \Big)^{\frac{n+1}{2n}} \Big(\int_{B_{2R} \cap \Omega} |u - c^R|^{\frac{2n}{n-1}} dx \Big)^{\frac{n-1}{2n}} \\ &\le \frac{C}{R} \Big(\int_{B_{6R} \cap \Omega} |Du|^{\frac{2n}{n+1}} dx \Big)^{\frac{n+1}{2n}}. \end{aligned}$$

So set $z = |Du|^{\frac{2n}{n+1}} \mathbf{1}_\Omega$, then we have the inequality

$$⨍_{B_R} z^{\frac{n+1}{n}} dx \le C \Big(⨍_{B_{6R}} z dx \Big)^{\frac{n+1}{n}} + C, \tag{4.12}$$

where

$$\fint_{B_R} = \frac{1}{|B_R|}\int_{B_R}.$$

This is the reverse Hölder's inequality, implying Gehring's result, namely $z^{\frac{n+1}{n}} + \varepsilon$ is integrable for some positive ε, hence $u \in W^{1,p}(\Omega)$, for $2 \leq p < p + \varepsilon'$.

4.3. Interior C^δ Regularity

Let Ω_1 be an open subdomain of Ω, with $\bar{\Omega_1} \subset \Omega$. We want first to prove the following estimate

$$\int_{\Omega_1} |Du|^2 |x - x_0|^{2-n} dx \leq C_{\Omega_1}, \quad \forall x_0 \in \bar{\Omega_1}. \tag{4.13}$$

We consider the Green function $G = G^{x_0}$, solution of

$$-\mathrm{div}(a(x) DG^{x_0}) = \delta(x - x_0),$$
$$G^{x_0}\big|_{\partial\Omega} = 0. \tag{4.14}$$

Note that

$$G^{x_0} \in L^q(\Omega) \cap W_0^{1,r}(\Omega), \quad 1 \leq q < \frac{n}{n-2}, \quad 1 \leq r < \frac{n}{n-1}. \tag{4.15}$$

Moreover

$$c_0 |x - x_0|^{2-n} \leq G^{x_0} \leq c_1 |x - x_0|^{2-n}, \quad \forall x \in \Omega_1, \tag{4.16}$$

where the constants c_0, c_1 depend only on Ω_1. We just take in (4.5) $\psi = G^{x_0}$[1], and from the definition of the Green function, one has

$$\int_\Omega a DX . DG^{x_0} dx = X(x_0) > 0,$$

hence

$$k_0 \int_\Omega |Du|^2 G^{x_0} dx \leq K_0 \int_\Omega G^{x_0} dx \leq C.$$

Using (4.16), the estimate (4.13) follows immediately. We want to prove C^δ regularity in Ω_1. We shall use Morrey's result, namely

$$\sup_{\substack{x,y\in\Omega_1 \\ x\neq y}} \frac{|u(x) - u(y)|}{|x - y|^\delta} \leq C \left(\sup_{\substack{x_0\in\Omega_1 \\ B_R(x_0)\subset\Omega_1}} \frac{\int_{B_R(x_0)} |Du|^2 dx}{R^{n-2+2\delta}} \right)^{1/2}. \tag{4.17}$$

So we want to show that

$$\int_{B_R(x_0)} |Du|^2 dx \leq K R^{n-2+2\delta}, \quad \forall x_0 \in \Omega_1, B_R(x_0) \subset \Omega_1; \tag{4.18}$$

of course, the constant K will depend on Ω. To check (4.18), we shall prove the inequality

$$\int_{B_R(x_0)} |Du|^2 |x - x_0|^{2-n} dx \leq C \int_{B_{\sigma R}(x_0) - B_R(x_0)} |Du|^2 |x - x_0|^{2-n} dx + C R^\beta \tag{4.19}$$

with $\sigma > 2, \beta > 0, B_{\sigma R}(x_0) \subset \Omega_1$. If (4.19) holds, then we can rely on the hole filling technique of Widman[2]. Filling the hole, we deduce from (4.19)

$$\int_{B_R(x_0)} |Du|^2 |x - x_0|^{2-n} dx \leq \theta \int_{B_{\sigma R}(x_0)} |Du|^2 |x - x_0|^{2-n} dx + C R^\beta \tag{4.20}$$

[1] This is formal, since G^{x_0} is not in C^1. One proceeds first with an approximation of the Green function. We skip this step

with $\theta < 1$. Set (x_0 being fixed in Ω_1), choosing $2\delta < \beta$,

$$\varphi(R) = \frac{\int_{B_R(x_0)} |Du|^2 |x-x_0|^{2-n} dx}{R^{2\delta}}, \quad R \le R_0,$$

where

$$R_0 = \sup\{R \mid B_R(x_0) \subset \Omega_1\},$$

we may write from (4.20)

$$\varphi(R) \le \mu\varphi(\sigma R) + C, \quad R \le \frac{R_0}{\sigma},$$

with $\mu = \theta\sigma^{2\delta} < 1$. Since $\varphi(R_0) < \infty$, as a consequence of (4.13), we get $\varphi(R) \le C, \forall R \le R_0$. So we have shown that

$$\int_{B_R(x_0)} |Du|^2 |x-x_0|^{2-n} dx \le KR^{2\delta}, \quad \forall x_0 \in \Omega_1, B_R(x_0) \subset \Omega_1,$$

which implies (4.18).

Proof of (4.13). We apply (4.5) with

$$\psi = G^{x_0}\tau_R^2, \quad x_0 \in \Omega_1, \quad B_{2R}(x_0) \subset \Omega_1, \tag{4.21}$$

$$c_\nu = c_\nu^R = \frac{1}{|B_{2R} - B_{R/2}|}\int_{B_{2R}(x_0)-B_{R/2}(x_0)} u_\nu dx. \tag{4.22}$$

We obtain first

$$k_0 \int_\Omega |Du|^2 G^{x_0}\tau_R^2 \ge C\int_{B_R(x_0)} |Du|^2 |x-x_0|^{2-n} dx, \tag{4.23}$$

$$K\int_\Omega G^{x_0}\tau_R^2 \le CR^{\frac{n}{q}}. \tag{4.24}$$

Next

$$D\psi = DG^{x_0}\tau_R^2 + 2G^{x_0}\tau_R D\tau_R.$$

Consider

$$\mathrm{I} = 2\int_\Omega aDXD\tau_R G^{x_0}\tau_R dx,$$

we have from (4.7)

$$\begin{aligned} |\mathrm{I}| &\le \int_{B_{2R}-B_R} |Du| \frac{|u-c^R|}{R} G^{x_0} dx \\ &\le C\int_{B_{2R}-B_R} |Du|^2 |x-x_0|^{2-n} dx + C\int_{B_{2R}-B_R} \frac{|u-c^R|^2}{R^2}|x-x_0|^{2-n} dx. \end{aligned}$$

But, using Poincaré's inequality we have

$$\begin{aligned} \int_{B_{2R}-B_R} \frac{|u-c^R|^2}{R^2}|x-x_0|^{2-n} dx &\le \frac{C}{R^n}\int_{B_{2R}-B_R} |u-c^R|^2 dx \\ &\le \frac{C}{R^{n-2}}\int_{B_{2R}-B_{R/2}} |Du|^2 dx \\ &\le C\int_{B_{2R}-B_{R/2}} |Du|^2 |x-x_0|^{2-n} dx, \end{aligned}$$

hence

$$|\mathrm{I}| \le C\int_{B_{2R}-B_{R/2}} |Du|^2 |x-x_0|^{2-n} dx. \tag{4.25}$$

Consider next the term

$$\begin{aligned}\mathrm{II} &= \int_\Omega aDXDG^{x_0}\tau_R^2dx \\ &= \int_\Omega aDG^{x_0}D((X-2^N)\tau_R^2)dx - 2\int_\Omega aDG^{x_0}D\tau_R(X-2^N)\tau_R dx \\ &\geq -2\int_\Omega aDG^{x_0}D\tau_R(X-2^N)\tau_R dx \\ &\geq -C\int_{B_{2R}-B_R}|DG|\frac{|u-c^R|^2}{R}\tau_R dx,\end{aligned}$$

where we have used the estimate

$$|X-2^N| \leq C|u-c^R|^2. \tag{4.26}$$

We estimate II from below as follows

$$\mathrm{II} \geq -C\int_{B_{2R}-B_R} G\frac{|u-c^R|^2}{R^2}dx - C\int_{B_{2R}-B_R} G^{-1}|DG|^2|u-c^R|^2\tau_R^2dx. \tag{4.27}$$

The first term on the right hand side is estimated by the right hand side of (4.21). So there remains to estimate the term

$$\mathrm{III} = \int_{B_{2R}-B_R} G^{-1}|DG|^2|u-c^R|^2\tau_R^2dx.$$

Now, we introduce a new cut off function ξ, satisfying

$$\xi = \begin{cases} 0 & \text{for } |x| \leq \frac{1}{2}, \\ \tau & \text{for } |x| > 1, \end{cases}$$

and we set

$$\xi_R(x) = \xi\Big(\frac{x-x_0}{R}\Big),$$

thus

$$\xi_R = \tau_R \text{ on } B_{2R} - B_R.$$

We first test the Green function equation (4.14) with $G^{-1/2}|u-c^R|^2\xi_R^2$ which vanishes on the boundary of Ω, since $B_{2R} \subset \Omega_1$. It also vanishes on x_0, hence

$$\frac{1}{2}\int_\Omega aDGDGG^{-3/2}|u-c^R|^2\xi_R^2dx = \int_\Omega aD(|u-c^R|^2\xi_R^2).DGG^{-1/2}dx. \tag{4.28}$$

On the other hand, taking in (2.10) $v = (u_\nu - c_\nu^R)G^{1/2}\xi_R^2$ yields, summing up in ν,

$$\begin{aligned}&\sum_\nu \int_\Omega aDu_\nu D_\nu G^{1/2}\xi_R^2dx + \frac{1}{4}\int_\Omega aD(|u-c^R|^2\xi_R^2)DGG^{-1/2}dx \\ &- \frac{1}{2}\int_\Omega aDG.D\xi_R|u-c^R|^2G^{-1/2}\xi_Rdx + \int_\Omega aD|u-c^R|^2.D\xi_RG^{1/2}\xi_Rdx \\ =& \int_\Omega (H_\nu(x,Du)-\alpha u_\nu)(u_\nu-c_\nu^R)G^{1/2}\xi_r^2dx.\end{aligned}$$

So we deduce, taking into account the quadratic growth of H_ν,

$$\begin{aligned}&\int_\Omega aD(|u-c^R|^2\xi_R^2)DGG^{-1/2}dx \leq 2\int_\Omega aDG.D\xi_R|u-c^R|^2G^{-1/2}\xi_Rdx \\ &+ CR^{\frac{2-n}{2}}\int_{B_{2R}-B_{R/2}}|Du|^2dx + CR^{n\left(1-\frac{1}{2q}\right)} + CR^{\frac{2-n}{2}}\int_{B_{2R}-B_{R/2}}\frac{|u-c^R|^2}{R^2}dx.\end{aligned} \tag{4.29}$$

Furthermore

$$\int_\Omega aDG.D\xi_R|u-c^R|^2G^{-1/2}\xi_R dx$$
$$\le C\delta\int_\Omega aDG.DGG^{-3/2}|u-c^R|^2\xi_R^2dx+\frac{C}{\delta}R^{\frac{2-n}{2}}\int_{B_{2R}-B_{R/2}}\frac{|u-c^R|^2}{R^2}dx,\quad\forall\delta,$$

and combining this estimate in (4.29), (4.28) for δ sufficiently small yields

$$\int_\Omega aDGDGG^{-3/2}|u-c^R|^2\xi_R^2dx$$
$$\le CR^{n\left(1-\frac{1}{2q}\right)}+CR^{\frac{2-n}{2}}\int_{B_{2R}-B_{R/2}}|Du|^2dx+CR^{\frac{2-n}{2}}\int_{B_{2R}-B_{R/2}}\frac{|u-c^R|^2}{R^2}dx,$$

and using Poincaré's inequality, we obtain

$$\int_\Omega aDGDGG^{-3/2}|u-c^R|^2\xi_R^2dx\le CR^{n\left(1-\frac{1}{2q}\right)}+CR^{\frac{2-n}{2}}\int_{B_{2R}-B_{R/2}}|Du|^2dx. \tag{4.30}$$

Going back to the definition of III, and recalling that $\xi_R=\tau_R$ on $B_{2R}-B_R$, we get

$$\text{III}\le CR^{\frac{2-n}{n}}\int_\Omega G^{-3/2}aDGDG|u-c^R|^2\xi_R^2dx,$$

and from (4.30) it follows that

$$\text{III}\le CR^{2-n}\int_{B_{2R}-B_{R/2}}|Du|^2dx+CR^{1+\frac{n}{2q'}}$$
$$\le C\int_{B_{2R}-B_{R/2}}|Du|^2|x-x_0|^{2-n}dx+CR^{1+\frac{n}{2q'}}.$$

Finally from (4.27) we obtain

$$\text{II}\ge -C\int_{B_{2R}-B_{R/2}}|Du|^2|x-x_0|^{2-n}dx-CR^{1+\frac{n}{2q'}}.$$

Collecting results in the application of (4.5) and changing R into $2R$, we obtain (4.19) with $\sigma=4$, and $\beta=\frac{n}{q'}$ since $\frac{n}{q'}<1+\frac{n}{2q'}$ and $\beta<2$.

4.4. C^δ Regularity on the Boundary

By local maps representation of the boundary, the problem amounts to the following. Consider a sufficiently small ball B centered on the boundary, and a diffeomorphism ψ from B onto $D\subset R^n$, such that

$$\Omega^+=\psi(B\cap\Omega)\subset\{y\in R^n\,|\,y_n>0\},$$
$$\Gamma'=\psi(B\cap\Gamma)\subset\{y\in R^n\,|\,y_n=0\},$$

ψ,ψ^{-1} sufficiently smooth. Define next

$$\Omega^-=\{y\,|\,y_n<0,(y_1,\cdots,y_{n-1},-y_n)\in\Omega^+\},$$
$$\Omega'=\Omega^+\cup\Omega^-\cup\Gamma'.$$

If z is defined on Ω, set

$$z'(y)=z(\psi^{-1}(y)),\quad y\in\Omega^+\cup\Gamma'$$

and define z' on Ω^- by reflection, namely

$$z'(y_1,\cdots,y_{n-1},y_n)=z'(y_1,\cdots,y_{n-1},-y_n)\text{ if }y_n<0.$$

Applying the procedure to the function u_ν, we have to prove

$$u'_\nu(y) \in C^\delta(\Omega'). \tag{4.31}$$

We begin to reduce the variational problem (2.10) to a problem on $B \cap \Omega$. Consider in (2.10) functions which vanish in $\Omega - \Omega \cap B$, we obtain

$$\int_{\Omega\cap B} aDu_\nu Dv dx + \alpha \int_{\Omega\cap B} u_\nu v dx = \int_{\Omega\cap B} H_\nu(x, Du) v dx,$$
$$\forall v \in H^1(\Omega \cap B) \cap L^\infty(\Omega \cap B),\, v|_{\partial B\cap\Omega} = 0. \tag{4.32}$$

We then perform the change of coordinates $x = \psi^{-1}(y)$, denote

$$J_\psi(x) = \text{matrix}\Big(\frac{\partial \psi_i}{\partial x_j}\Big),$$

and set

$$a'(y) = \frac{J_\psi(\psi^{-1}(y)) a(\psi^{-1}(y)) J_\psi^*(\psi^{-1}(y))}{|\det J_\psi(\psi^{-1}(y))|},$$
$$H'_\nu(y,p) = \frac{H_\nu(\psi^{-1}(y), J_\psi^*(\psi^{-1}(y))p)}{|\det J_\psi(\psi^{-1}(y))|},$$
$$a'_0(y) = \frac{\alpha}{|\det J_\psi(\psi^{-1}(y))|}.$$

The variational problem (4.32) reads

$$\int_{\Omega^+} a'(y) Du'_\nu Dv' dy + \int_{\Omega^+} a'_0(y) u'_\nu v' dy = \int_{\Omega^+} H'_\nu(y, Du') v' dy,$$
$$\forall v' \in H^1(\Omega^+) \cap L^\infty(\Omega^+), \quad v'|_{\partial\Omega^+ - \Gamma'} = 0,$$
$$u'_\nu \in H^1(\Omega^+) \cap L^\infty(\Omega^+), \tag{4.33}$$

$$a'(y)\xi.\xi \geq \alpha'_0 |\xi|^2, \quad a_0(y) \geq \bar{a}_0. \tag{4.34}$$

Moreover the Hamiltonian $H'_\nu(y,p)$ verify the special structure assumption, namely

$$H'_\nu(y,p) = Q'(y,p)p_\nu + H_\nu^{'0}(y,p) \tag{4.35}$$

with

$$Q'(y,p) = \frac{J_\psi(\psi^{-1}(y)) Q(\psi^{-1}(y), p)}{|\det J_\psi(\psi^{-1}(y))|}, \quad H_\nu^{'0}(y,p) = \frac{H_\nu^0(\psi^{-1}(y), p)}{|\det J_\psi(\psi^{-1}(y))|}$$

and we have

$$|Q'(y,p)| \leq k + K|p|, \tag{4.36}$$
$$|H_\nu^{'0}(y,p) \ \leq k_\nu + K_\nu \sum_{\mu \leq \nu} |p_\nu|^2. \tag{4.37}$$

We now proceed with the reflection procedure.

Write $y = (y', y_n)$ where $y' = (y_1, \cdots, y_{n-1})$ and define for $y_n < 0$,

$$a'_{ii}(y', y_n) = a'_{ii}(y', -y_n), \quad \forall i,$$
$$a'_{ij}(y', y_n) = a'_{ij}(y', -y_n), \quad \forall i, j\ i \neq j,\ i, j \neq n,$$
$$a'_{in}(y', y_n) = -a'_{in}(y', -y_n), \quad \forall i \neq n,$$
$$a'_0(y', y_n) = a_0(y', -y_n).$$

Let also $p_\nu = (p'_\nu, p_{\nu n})$, where p'_ν stands for the components $p_{\nu 1}, \cdots, p_{\nu(n-1)}$. Then, we write for $y_n < 0$,

$$H'_\nu(y', y_n; p'_1, p_{1n}; \cdots; p'_N, p_{Nn}) = H'_\nu(y', -y_n; p'_1, -p_{1n}; \cdots; p'_N, -p_{Nn}).$$

Therefore the functions u'_ν extended by reflection appear to be solutions of the problem

$$\int_{\Omega'} a'(y) Du'_\nu Dv' dy + \int_{\Omega'} a'_0(y) u'_\nu v' dy = \int'_{\Omega} H'_\nu(y, Du') v' dy,$$
$$\forall v' \in H_0^1(\Omega') \cap L^\infty(\Omega'), \quad u'_\nu \in H^1(\Omega') \cap L^\infty(\Omega') \tag{4.38}$$

and our objective is to prove (4.31).

If we drop the prime symbol, and use x instead of y, our problem amounts to the following. Consider the variational problem

$$\int_\Omega a Du_\nu Dv dx + \int_\Omega a_0 u_\nu v dx = \int_\Omega H_\nu(x, Du) v dx,$$
$$\forall v \in H_0^1(\Omega) \cap L^\infty(\Omega), \quad u_\nu \in H^1(\Omega) \cap L^\infty(\Omega).$$

We know that the solution is $C^\delta(\Omega)$.

It is similar to the interior C^δ regularity of (2.10) considered in §4. However we have to be careful to consider only test functions, which vanish on the boundary. Nevertheless, thanks to the special structure on H_ν we derive again (4.5), for any ψ in $C^1(\bar{\Omega}), \psi \geq 0, \psi|_{\partial\Omega} = 0$. In the proof of interior C^δ regularity, we use only test functions which vanish on the boundary of Ω, hence the proof carries over, and the result follows.

4.5. End of Proof of Theorem 2.1

We know that we have a solution of (2.10) which belongs to $W^{1,p}(\Omega) \cap C^\delta(\bar{\Omega})$, $2 \leq p < 2+\varepsilon$. We can now rely on the linear theory and Miranda-Nirenberg interpolation theorem[3]. Indeed $H_\nu(x, Du)$ belongs to $L^{\frac{p_0}{2}}$ with $p_0 > 2$. Hence from the linear theory $u_\nu \in W^{2\frac{p_0}{2}}$ and since $u_\nu \in C^\delta$, we have also

$$u_\nu \in W^{1,p_1} \text{ with } \frac{1}{p_1} = \frac{1}{p_0} - \frac{\delta}{2n},$$

provided $p_0 < \frac{2n}{\delta}$, and thus $p_1 > p_0$. After a finite number of steps we get $p_i \geq \frac{2n}{\delta}$, and it follows that $u_\nu \in W^{1,p}(\Omega), \quad p > 2n$, and from the linear theory again $u_\nu \in W^{2,s}(\Omega), \quad s > n$. From Sobolev embedding theorem, $u_\nu \in W^{1,r}(\Omega), \forall s$ and thus from the linear theory again $u_\nu \in W^{2,s}(\Omega), \forall s$.

References

[1] Bensoussan, A. & Frehse, J., Nonlinear partial differential equations and applications [M], Springer-Verlag, New York, 2001.

[2] Widman, K. O., Hölder continuity of solutions of elliptic equations [J], *Manuscr. Math.,* **5**(1971), 299–308.

[3] Nirenberg, L., An extended interpolation inequality [J], *Ann. Sc. Norm. Sup. Pisa,* **cl. Sc. IV. ser. 20**(1966), 733–737.