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ON THE CONNECTION IN FINSLER SPACE

S. S. CHERN*

(*Dedicated to the memory of Jacques-Louis Lions*)

Abstract

A simple derivation of the Connection in Finsler space.

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The study of Riemann-Finsler geometry has recently been enhanced by the publication of a substantial book[2]. In this book we made essential use of a connection introduced in 1948[3]. The connection is a natural generalization of the Levi-Civita connection in the Riemannian case and seems to be the right analytical basis of the subject. We have given a derivation of it. According to Anastesie it coincides with the one introduced by Rund, who kindly gave an exposition of the paper in his book.

The aim of this paper is to give a short derivation of the connection. We will also show how it gives a solution of the local congruence, i.e., a complete system of local invariants which ensures that two Finsler structures differ by a change of coordinates.

§1. A Simple Equivalence Problem

Problem. Given in R^n with the coordinates x^i n Pfaffian forms ω^i, linearly independent, and in R^n with the coordinates x^{*^i} also n linearly independent Pfaffian forms ω^{*^i}, $1 \le i \le n$. Find the conditions that there exists a coordinate transformation

$$x^{*^i} = x^{*^i}(x^1, \cdots, x^n), \tag{1.1}$$

such that

$$\omega^{*^i} = \omega^i. \tag{1.2}$$

(Our Latin subscripts and supscripts have the range 1, $\cdots$, n.)

The idea is to construct invariants under the transformation (1.1). We have, since the ω's are linearly independent,

$$d\omega^i = c^i_{jk}\omega^j \wedge \omega^k, \tag{1.3}$$

where we can suppose

$$c^i_{jk} + c^i_{kj} = 0. \tag{1.4}$$

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*Nankai Institute of Mathematics, Tianjin 300071, China.

With the condition (1.4), the c^i_{jk} are completely determined. If the corresponding quantities in R^{*^n} are denoted with asterisks, we have

$$c^i_{jk} = c^{*i}_{jk}. \tag{1.5}$$

Differentiating, we have

$$dc^i_{jk} = dc^{*i}_{jk} \tag{1.6}$$

so that

$$c^i_{jkl} = c^{*i}_{jkl} \tag{1.7}$$

if

$$dc^i_{jk} = c^i_{jkl}\omega^l \tag{1.8}$$

and similar equations with asterisks. Continuing this process, we get a sequence of functions,

$$c^i_{jk},\ c^i_{jkl},\ c^i_{jklp},\ \cdots, \tag{1.9}$$

which are equal to the corresponding functions with asterisks. The solution of our problem is thus given by the following theorem:

Theorem 1.1. *The transformation* (1.1) *has the invariant functions* (1.9). *If one of the functions is a constant, the corresponding function with asterisk must be equal to the same constant. If some of the functions are independent and another one is a function of them, the same must be true with the functions with asterisks, by the same functional relation.*

§2. The Connection in a Riemann-Finsler Space

Let M be a manifold and TM its tangent bundle. By SM we mean the manifold of its rays, i.e., the set of non-zero tangent vectors differing by a positive factor. If $n = \dim M$, then $\dim TM = 2n$ and $\dim SM = 2n-1$. We use the local coordinates x^i of M, then TM has the local coordinates $x^i,\ y^i$, if the vector is $y^i\frac{\partial}{\partial x^i}$, and SM has the same local coordinates, y^i being then homogeneous coordinates, up to a positive factor. In this section we will agree on the following ranges of indices:

$$1 \le i, j, k, \cdots \le n; \qquad 1 \le \alpha, \beta, \gamma, \cdots \le n-1. \tag{2.1}$$

A Riemann-Finsler metric on M is given by the function

$$ds = F(x^1, \cdots, x^n, dx^1, \cdots, dx^n), \tag{2.2}$$

where $F(x, y)$ is supposed to be smooth and positively homogeneous in the second variable, i.e.,

$$F(x, \lambda y) = \lambda F(x, y), \qquad \lambda > 0. \tag{2.3}$$

We introduce the quantities

$$g_{ij} = \frac{\partial^2}{\partial y^i \partial y^j}\Big(\frac{1}{2}F^2\Big), \tag{2.4}$$

which are functions on SM, and we make the regularity hypothesis that the matrix (g_{ij}) is positive definite (or more generally non-singular). The quadratic differential form $Q = g_{ij}(x, y)dx^i dx^j$ will be called the Riemann form.

The projection π pulls TM back:

$$\begin{array}{ccc} \pi^*TM & \longrightarrow & TM \\ \downarrow & & \downarrow \\ SM & \underset{\pi}{\longrightarrow} & M \end{array} \tag{2.5}$$

and we will use the bundle at the left-hand side. In this bundle the g_{ij} in (2.4), being homogeneous of degree 0 in y^k and therefore functions on SM, define an inner product. SM has the distinguished one-form

$$H = \frac{\partial f}{\partial y^i} dx^i. \tag{2.6}$$

It will be called the Hilbert form.

Lemma 2.1. *Under the regularity hypothesis, the Hilbert form satisfies*

$$H \wedge (dH)^{n-1} \neq 0, \tag{2.7}$$

and hence define a contact structure on SM.

For proof, refer to [2, p. 272].

In the bundle at the left-hand side of (2.5) we take a frame field e_i and let ω^i be the coframe field dual to e_i.

A connection D is by definition the absolute differential

$$De_i = \omega_i^j e_j. \tag{2.8}$$

Then the tensor $\sum \omega^i \otimes e_i$ is independent of the choice of e_i and the invariant condition

$$D(\omega^i \otimes e_i) = 0 \tag{2.9}$$

is called the vanishing of torsion. This condition becomes, when written explicitly,

$$d\omega^i = \omega^j \wedge \omega_j^i. \tag{2.10}$$

We wish to introduce a torsionless connection in the bundle at the left column of (2.5). Analytically this is to determine the forms ω_j^i so that (2.10) are satisfied. We will make use of the local coordinates x^i, y^j described above and choose an orthonormal frame xe_i such that e_n is the unit vector along the vector $y^i \frac{\partial}{\partial x^i}$. On SM, ω^i, ω_n^α form a base of the exterior algebra of differential forms.

We suppose our connection to preserve the length of e_n and the orthogonality of e_n and e_α. The connection forms therefore satisfy the conditions

$$\omega_{nn} = 0, \qquad \omega_{\alpha n} + \omega_{n\alpha} = 0. \tag{2.11}$$

Here and later we use the Kronecker indices δ_{ij} to raise or lower indices. Notice that in the connection forms ω_j^i the second index is an upper index.

We complete the Hilbert form into a coframe

$$\omega^i = v_k^i dx^k, \tag{2.12}$$

with

$$\omega^n = H, \ \text{ i.e., } \ v_i^n = \frac{\partial F}{\partial y^i}, \tag{2.13}$$

$$y^k v_k^\alpha = 0, \tag{2.14}$$

i.e., $\langle e_n, \omega^\alpha \rangle = 0$. Let (u_i^k) be the inverse matrix of (v_i^k), so that

$$u_i^k v_k^j = v_i^k u_k^j = \delta_i^j. \tag{2.15}$$

Then

$$u_n^k = \frac{y^k}{F} \tag{2.16}$$

and we have

$$d\omega^n = \frac{\partial^2 F}{\partial x^i \partial y^k} dx^i \wedge dx^k + \frac{\partial^2 F}{\partial y^i \partial y^k} dy^i \wedge dx^k$$
$$= \frac{\partial^2 F}{\partial x^i \partial y^k} u_p^i u_q^k \omega^p \wedge \omega^q + \frac{\partial^2 F}{\partial y^i \partial y^k} dy^i \wedge u_q^k \omega^q.$$

Since $\frac{\partial F}{\partial y^i}$ is homogeneous of degree zero in y^k, $\frac{\partial^2 F}{\partial y^k \partial y^i} y^k = 0$ by Euler's theorem, and we can write

$$d\omega^n = \omega^\alpha \wedge \omega_\alpha^n, \tag{2.17}$$

where

$$\omega_\alpha^n = -u_\alpha^k \frac{\partial^2 F}{\partial y^j \partial y^k} dy^j + \frac{1}{F} u_\alpha^j \Big(\frac{\partial F}{\partial x^j} - \frac{\partial^2 F}{\partial x^j \partial y^k} y^k \Big) \omega^n + u_\alpha^j u_\beta^k \frac{\partial^2 F}{\partial x^j \partial y^k} \omega^\beta + \lambda_{\alpha\beta} \omega^\beta, \tag{2.18}$$

where $\lambda_{\alpha\beta} = \lambda_{\beta\alpha}$ are to be determined.

On the other hand, we have

$$d\omega^\alpha = dv_k^\alpha \wedge dx^k = -v_k^\alpha du_i^k \wedge \omega^i = -v_k^\alpha du_\beta^k \wedge \omega^\beta - v_k^\alpha d\Big(\frac{y^k}{F}\Big)\omega^n. \tag{2.19}$$

We now study the equations (2.11). The first equation can clearly be satisfied. For the existence of ω_n^α satisfying the second equation of (2.11) and

$$d\omega^\alpha = \omega^\beta \wedge \omega_\beta^\alpha + \omega^n \wedge \omega_n^\alpha, \tag{2.20}$$

it is necessary and sufficient that $-\frac{1}{F} v_k^\alpha$ is equal to the coefficient of dy^k in the expression (2.18) for ω_α^n. This gives

$$u_\alpha^j G_{jk} = \delta_{\alpha\beta} v_k^\beta, \tag{2.21}$$

where

$$G_{jk} = F\ F_{jk}, \qquad F_{jk} = \frac{\partial^2 F}{\partial y^i \partial y^k} \tag{2.22}$$

are functions on SM.

Notice that

$$v_i^\beta u_\alpha^j u_\beta^k G_{jk} = v_i^l u_l^k u_\alpha^j G_{jk} - v_i^n u_n^k u_\alpha^j G_{jk} = u_\alpha^j G_{ji},$$

since $u_n^k G_{jk} = \frac{1}{F} y^k (F\, F_{jk}) = 0$. Hence (2.21) can be rewritten

$$u_\alpha^j u_\beta^k G_{jk} = \delta_{\alpha\beta}. \tag{2.23}$$

It can also be written

$$\delta_{\alpha\beta} v_i^\alpha v_j^\beta = G_{ij}. \tag{2.24}$$

In forms the last equation becomes $\sum_\alpha {\omega^\alpha}^2 = Q - H^2$. Comparing (2.19) and (2.20), we get

$$\omega_\beta^\alpha = v_k^\alpha du_\beta^k - \delta^{\alpha\gamma} \Big(u_\gamma^j u_\beta^k \frac{\partial^2 F}{\partial x^j \partial y^k} + \lambda_{\beta\gamma} \Big) \omega^n + \mu_{\beta\gamma}^\alpha \omega^\gamma,$$

where $\mu_{\beta\gamma}^\alpha = \mu_{\gamma\beta}^\alpha$, but are otherwise arbitrary.

It remains to determine ω_β^α. We find

$$\delta_{\alpha\sigma} \omega_\rho^\alpha + \delta_{\alpha\rho} \omega_\sigma^\alpha = -\, dG_{ij} u_\rho^i u_\sigma^j - u_\rho^i u_\sigma^j \Big(\frac{\partial^2 F}{\partial x^i \partial y^j} + \frac{\partial^2 F}{\partial x^j \partial y^i} \Big) \omega^n$$
$$- 2\lambda_{\rho\sigma} \omega^n + (\delta_{\alpha\sigma} \mu_{\rho\gamma}^\alpha + \delta_{\alpha\rho} \mu_{\sigma\gamma}^\alpha) \omega^\gamma. \tag{2.25}$$

Since G_{ij} are functions on SM, we can write

$$dG_{ij} = G^{\alpha}_{ij}\omega^k_{\alpha} + G_{ijk}\omega^k. \tag{2.26}$$

We choose $\lambda_{\rho\sigma},\ \mu^{\alpha}_{\rho\sigma}$ so that the following equation holds

$$\omega_{\rho\sigma} + \omega_{\sigma\rho} = H^{\alpha}_{\rho\sigma}\omega^n_{\alpha}. \tag{2.27}$$

This determines $\lambda_{\rho\sigma},\ \mu^{\alpha}_{\rho\sigma}$ completely by

$$\begin{aligned} \lambda_{\rho\sigma} &= -\frac{1}{2}u^i_{\rho}u^j_{\sigma}\Big(G_{ijn} + \frac{\partial^2 F}{\partial x^i \partial y^j} + \frac{\partial^2 F}{\partial x^j \partial y^i}\Big), \\ \mu^{\alpha}_{\rho\sigma} &= \frac{1}{2}\delta^{\alpha\beta}(\xi_{\rho\beta\alpha} + \xi_{\alpha\beta\rho} - \xi_{\rho\sigma\beta}), \\ \xi_{\rho\sigma\beta} &= G_{ij\beta}u^i_{\rho}u^j_{\sigma}. \end{aligned} \tag{2.28}$$

Thus all the ω^i_j are determined. We state the result as the theorem:

Theorem 2.1. *Given the Riemann-Finsler metric, there is a uniquely defined connection in the bundle $\pi^*TM \to SM \ni (x, y)$ characterized by the conditions:*

(1) *It is torsionless;*

(2) *The length of the vector y and the property of a vector $\perp$ to y are preserved;*

(3) *Relative to orthonormal frames the conditions*

$$\omega_{\alpha\beta} + \omega_{\beta\alpha} = H_{\alpha\beta\gamma}\omega_{\gamma n} \tag{2.27a}$$

are satisfied.

§3. Cartan Tensor

Condition (2.27a) in Theorem 2.1 in the last section means that the inner product is preserved by the parallelism defined by the connection when $\omega_{\alpha n} = 0$, i.e., when the parallelism preserves the vector y. We wish to calculate the function $H_{\alpha\beta\gamma}$ in terms of F.

By (2.25) and (2.27) we have

$$H^{\alpha}_{\rho\sigma} = -G^{\alpha}_{ij}u^i_{\rho}u^j_{\sigma}. \tag{3.1}$$

Comparing the coefficients of dy^k in (2.26), we get $G^{\alpha}_{ij}u^k_{\alpha}F_{kl} = -F\,F_{ijl} - F_l F_{ij}$. Here and later subscripts of F mean partial differentiation with respect to the corresponding y^i. It following that

$$G^{\alpha}_{ij}y^j u^k_{\alpha}F_{kl} = F\,F_{il} \quad \text{or} \quad (G^{\alpha}_{ij}y^j u^k_{\alpha} - \delta^k_i F)F_{kl} = 0.$$

Since the matrix (F_{kl}) is of rank $n-1$, this holds only when $G^{\alpha}_{ij}y^j u^k_{\alpha} - \delta^k_i F = p_i y^k$. Multiplication of this equation by F_k and subsequent summation give $p_i = -F_i$. Hence

$$G^{\alpha}_{ij}y^j u^k_{\alpha} = -F_i y^k + \delta^k_i F, \tag{3.2}$$

$$G^{\alpha}_{ij}y^j = v^{\alpha}F. \tag{3.3}$$

With the help of this relation we find

$$\begin{aligned} v^{\rho}_l v^{\sigma}_m H^{\alpha}_{\rho\sigma} &= -G^{\alpha}_{ij}\Big(\delta^i_l - \frac{y^i}{F}F_l\Big)\Big(\delta^i_m - \frac{y^i}{F}F_m\Big) = -G^{\alpha}_{lm} + v^{\alpha}_l F_m + v^{\alpha}_m F_l, \\ v^{\rho}_l v^{\alpha}_m H^{\alpha}_{\rho\sigma}u^k_{\alpha}F_{kj} &= \Big(\frac{1}{2}F^2\Big)_{lmj}. \end{aligned}$$

Multiplying this by u^j_β, we get $v^\rho_l v^\sigma_m H_{\rho\alpha\beta} = F\big(\frac{1}{2}F^2\big)_{lmj} u^j_\beta$, which gives

$$H_{\rho\sigma\alpha} = F\Big(\frac{1}{2}F^2\Big)_{ijk} u^i_\rho u^j_\sigma u^k_\alpha, \tag{3.4}$$

$$F\Big(\frac{1}{2}F^2\Big)_{ijk} = H_{\rho\sigma\alpha} v^\rho_i v^\sigma_j v^\alpha_k. \tag{3.5}$$

$H_{\rho\sigma\alpha}$ is usually called the Cartan tensor. For a Riemannian metric it is zero and our connection reduces to the connection of Levi-Civita.

§4. Equivalence Theorem

The following theorem is immediate.

Theorem 4.1. *Consider the bundle* $\pi^*TM \to SM$ *at the left-hand side of* (2.5). *Let*

$$P \to SM \tag{4.1}$$

be its principal bundle of orthonormal frames. Then $\dim P = \frac{1}{2}n(n+1)$ *and in it are the forms* ω^i, ω_{ij}, *which are* $\frac{1}{2}n(n+1)$ *in number and are linearly independent. If the corresponding entities in* M^* *are denoted by asterisks, the two Riemann-Finsler structures differ by a coordinate transformation if and only if there is a coordinate transformation from* P *to* P^* *such that* ${\omega^*}^i = \omega^i$, $\omega^*_{ij} = \omega_{ij}$.

This reduces the equivalence problem to the problem solved in §1.

In the principal bundle P the forms ω^i, ω^j_i constitute a basis of $\wedge(T^*P)$, the exterior algebra of its cotangent bundle. By our Theorem 1.1 the local invariants of our Finsler structure are obtained through the exterior derivatives of ω^i, ω^j_i. The exterior derivatives $d\omega^i$ are given by (2.10). To find $d\omega^j_i$ we differentiate (2.10), obtaining

$$\omega^j \wedge (d\omega^i_j - \omega^k_j \wedge \omega^i_k) = 0.$$

It follows that

$$d\omega^i_j = \omega^k_j \wedge \omega^i_k + R^i_{jkl}\omega^k \wedge \omega^l + P^i_{jk\alpha}\omega^k \wedge \omega^\alpha_k, \tag{4.2}$$

where we suppose

$$R^i_{jkl} + R^i_{jlk} = 0, \tag{4.3}$$

and we have

$$R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0, \tag{4.4}$$

$$P^i_{jk\alpha} = P^i_{kj\alpha}. \tag{4.5}$$

The R^i_{jkl} from the Riemann curvature tensor.

From the Riemann curvature one defines by contraction the Ricci curvature. The Ricci curvature is a scalar function on SM and is the most important local invariant in Finsler geometry. For details refer to [2, p. 190].

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