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OPERATOR-SPLITTING METHODS FOR THE SIMULATION OF BINGHAM VISCO-PLASTIC FLOW

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(*Dedicated to the memory of Jacques-Louis Lions*)

Abstract

This article discusses computational methods for the numerical simulation of unsteady Bingham visco-plastic flow. These methods are based on time-discretization by operator-splitting and take advantage of a characterization of the solutions involving some kind of Lagrange multipliers. The full discretization is achieved by combining the above operator-splitting methods with finite element approximations, the advection being treated by a wave-like equation "equivalent" formulation easier to implement than the method of characteristics or high order upwinding methods. The authors illustrate the methodology discussed in this article with the results of numerical experiments concerning the simulation of wall driven cavity Bingham flow in two dimensions.

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§1. Introduction

From the early seventies to his untimely death in 2001, Jacques-Louis Lions was always highly interested, not to say intrigued, in the system of equations and inequalities modeling Bingham visco-plastic flow (one of the success stories of the Variational Inequality Theory). Evidences of this interest can be found in the Chapters 6 of [1] and [2], which still are (to the best of our knowledge) the fundamental references concerning the mathematical properties of the variational inequalities modeling Bingham visco-plastic flow. These facts would have justified by themselves a Bingham flow related article dedicated to J. L. Lions. Actually, J. L. Lions was always concerned with the relevance of mathematics to applications and from that point of view we have been witnessing during these last years a surge of interest in Bingham visco-plastic fluids. It is very likely that this interest is motivated by the fact that material as diverse as fresh concrete, tortilla dough, fruits in syrup, blood in the capillaries, some muds used in drilling technologies, toothpastes, $\cdots$, have a Bingham medium behavior.

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In Section 2, we shall provide the Bingham flow model, and a multiplier characterization of the solutions, very useful from a computational point of view. The system of partial differential equations and inequalities modeling Bingham flow will be time-discretized in Section 3, using an operator splitting scheme. The finite element approximation will be discussed in Section 4, and the solution of the subproblems encountered at each time step in Section 5. Finally, the results of numerical experiments will be presented in Section 6.

§2. On the Modeling of Bingham Visco-Plastic Flow

Let Ω be a bounded domain of $\mathbb{R}^d$ ($d = 2$ or 3 in applications); we denote by Γ, the boundary of Ω. The isothermal flow of an incompressible Bingham visco-plastic medium, during the time interval $(0, T)$, is modeled by the following system of equations (clearly of the Navier-Stokes type):

$$\rho\Big[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\Big] = \nabla\cdot\boldsymbol{\sigma} + \mathbf{f} \quad \text{in } \Omega\times(0,T), \tag{2.1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega\times(0,T), \tag{2.2}$$

$$\boldsymbol{\sigma} = -p\mathbf{I} + \sqrt{2}g\frac{\mathbf{D}(\mathbf{u})}{|\mathbf{D}(\mathbf{u})|} + 2\mu\mathbf{D}(\mathbf{u}), \tag{2.3}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \text{ (with } \nabla\cdot\mathbf{u}_0 = 0); \tag{2.4}$$

for simplicity, we shall consider only Dirichlet boundary conditions, namely,

$$\mathbf{u} = \mathbf{u}_\Gamma \ \text{ on } \ \Gamma\times(0,T), \ \text{ with } \int_\Gamma \mathbf{u}_\Gamma(t)\cdot\mathbf{n}\,d\Gamma = 0, \text{ a.e. on } (0,T). \tag{2.5}$$

In system (2.1)–(2.5):

• ρ (resp., μ and g) is the density (resp., viscosity and plasticity yield) of the Bingham medium; we have $\rho > 0$, $\mu > 0$ and $g > 0$.

• $\mathbf{f}$ is a density of external forces.

• $\mathbf{D}(\mathbf{v}) = (\nabla\mathbf{v} + (\nabla\mathbf{v})^t)/2$ $(= (D_{ij}(\mathbf{v}))_{1\le i,j\le d})$, $\forall\mathbf{v}\in(H^1(\Omega))^d$, and $|\mathbf{D}(\mathbf{v})|$ is the Frobenius norm of tensor $\mathbf{D}(\mathbf{v})$, i.e.,

$$|\mathbf{D}(\mathbf{v})| = \Big(\sum_{1\le i,j\le d}|D_{ij}(\mathbf{v})|^2\Big)^{1/2}.$$

We clearly have trace $\mathbf{D}(\mathbf{v}) = 0$ if $\nabla\cdot\mathbf{v} = 0$.

We observe that if $g = 0$, system (2.1)–(2.5) reduces to the Navier-Stokes equations modeling isothermal incompressible Newtonian viscous fluid flow. Having said all that, if $g > 0$, the above model makes no sense on the set

$$Q_0 = \{\{x,t\}|\{x,t\}\in\Omega\times(0,T),\ \mathbf{D}(\mathbf{u})(x,t) = \mathbf{0}\}.$$

Following Duvaut and Lions[1 and 2, Chapter 6] we eliminate the above difficulty by considering, instead of the (doubly) nonlinear equations (2.1)–(2.5), the following variational inequality model:

Find $\{\mathbf{u}(t), p(t)\}\in(H^1(\Omega))^d\times L^2(\Omega)$ such that a.e. on $(0,T)$ we have

$$\begin{cases} \rho\int_\Omega \dfrac{\partial\mathbf{u}}{\partial t}(t)\cdot(\mathbf{v}-\mathbf{u}(t))dx + \rho\int_\Omega(\mathbf{u}(t)\cdot\nabla)\mathbf{u}(t)\cdot(\mathbf{v}-\mathbf{u}(t))dx \\ +\mu\int_\Omega\nabla\mathbf{u}(t):\nabla(\mathbf{v}-\mathbf{u}(t))dx + \sqrt{2}g(j(\mathbf{v}) - j(\mathbf{u}(t))) \\ -\int_\Omega p(t)\nabla\cdot(\mathbf{v}-\mathbf{u}(t))dx \ge \int_\Omega\mathbf{f}(t)\cdot(\mathbf{v}-\mathbf{u}(t))dx,\ \forall\mathbf{v}\in V_{\mathbf{u}_\Gamma(t)}, \end{cases} \tag{2.6}$$

$$\nabla\cdot\mathbf{u}(t) = 0 \quad \text{in } \Omega, \tag{2.7}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \tag{2.8}$$

$$\mathbf{u}(t) = \mathbf{u}_\Gamma(t) \quad \text{on } \Gamma, \tag{2.9}$$

with, in system (2.6)–(2.9),

$$j(\mathbf{v}) = \int_\Omega |\mathbf{D}(\mathbf{v})| dx, \ \forall \mathbf{v} \in (H^1(\Omega))^d, \tag{2.10}$$

$$V_{\mathbf{u}_\Gamma}(t) = \{\mathbf{v} | \mathbf{v} \in (H^1(\Omega))^d, \ \mathbf{v} = \mathbf{u}_\Gamma(t) \ \text{ on } \Gamma\}, \tag{2.11}$$

and $\mathbf{S} : \mathbf{T} = \sum_{i=1}^{d} \sum_{j=1}^{d} s_{ij} t_{ij}, \ \forall \mathbf{S} = (s_{ij}), \ \mathbf{T} = (t_{ij})$.

Let us be honest, formulation (2.6)–(2.9) is definitely an improvement compared to formulation (2.1)–(2.5), in the sense that we shall be able to derive from it computational methods "which work" (if $d = 2$, at least), however it is still partly formal. The rigorous formulation is more complicated and is thoroughly discussed in [1 and 2, Chapter 6, Section 3]; it is assumed there that $\mathbf{u}_\Gamma = \mathbf{0}$ on $\Gamma \times (0, T)$, and $\mathbf{u}_0 = \mathbf{0}$ if $d = 3$. If the above assumptions hold, it is shown in the above references that for $d = 2$, the time dependent variational inequality modeling the Bingham flow (a simple variant of problem (2.6)–(2.9)) has a unique solution, while uniqueness is still an open problem if $d = 3$ (as it is for the "ordinary" Navier-Stokes equations). Suppose that $d = 2$ and $\dfrac{\partial \mathbf{f}}{\partial t} = \mathbf{0}$; it is worthwhile emphasizing the fact that the uniqueness of the time dependent solution does not imply a similar property for the corresponding steady state flow problem.

For those readers who are already experts at solving the "ordinary" Navier-Stokes equations the main difficulty with model (2.6)–(2.9) is clearly the non-differentiable functional $j(\cdot)$. A simple way to overcome the above difficulty is to approximate $j(\cdot)$ by regularization, i.e., to replace it by a differentiable functional such as $j_\epsilon(\cdot)$ defined by

$$j_\epsilon(\mathbf{v}) = \int_\Omega \sqrt{\epsilon^2 + |\mathbf{D}(\mathbf{v})|^2} dx, \ \forall \mathbf{v} \in (H^1(\Omega))^d. \tag{2.12}$$

Since, $\forall \mathbf{v} \in (H^1(\Omega))^d$, we have

$$|j_\epsilon(\mathbf{v}) - j(\mathbf{v})| = \epsilon^2 \int_\Omega \frac{dx}{\sqrt{\epsilon^2 + |\mathbf{D}(\mathbf{v})|^2} + |\mathbf{D}(\mathbf{v})|} \le \epsilon |\Omega|, \tag{2.13}$$

where $|\Omega| = \text{meas.}(\Omega)$, $j_\epsilon(\cdot)$ is clearly an approximation of $j(\cdot)$. Concerning the differentiability of $j_\epsilon(\cdot)$ one can show that the differential $j'_\epsilon(\mathbf{v})$ of $j_\epsilon(\cdot)$ at $\mathbf{v} \in (H^1(\Omega))^d$ verifies:

$$\langle j'_\epsilon(\mathbf{v}), \mathbf{w} \rangle = \int_\Omega \frac{\mathbf{D}(\mathbf{v}) : \mathbf{D}(\mathbf{w})}{\sqrt{\epsilon^2 + |\mathbf{D}(\mathbf{v})|^2}} dx, \ \forall \mathbf{v} \in (H^1(\Omega))^d, \ \forall \mathbf{w} \in (H_0^1(\Omega))^d, \tag{2.14}$$

where, in (2.14), $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $(H^{-1}(\Omega))^d$ and $(H_0^1(\Omega))^d$. Taking advantage of (2.14) it is tempting to "approximate" problem (2.6)–(2.9) (and indeed (2.1)–(2.5)) by

$$\begin{cases} \rho \int_\Omega \dfrac{\partial \mathbf{u}_\epsilon}{\partial t}(t) \cdot (\mathbf{v} - \mathbf{u}_\epsilon(t)) dx + \rho \int_\Omega (\mathbf{u}_\epsilon(t) \cdot \nabla) \mathbf{u}_\epsilon(t) \cdot (\mathbf{v} - \mathbf{u}_\epsilon(t)) dx \\ +\mu \int_\Omega \nabla \mathbf{u}_\epsilon(t) : \nabla(\mathbf{v} - \mathbf{u}_\epsilon(t)) dx + \sqrt{2} g (j_\epsilon(\mathbf{v}) - j_\epsilon(\mathbf{u}_\epsilon(t)) \\ - \int_\Omega p_\epsilon(t) \nabla \cdot (\mathbf{v} - \mathbf{u}_\epsilon(t)) dx \ge \int_\Omega \mathbf{f}(t) \cdot (\mathbf{v} - \mathbf{u}_\epsilon(t)) dx, \ \forall \mathbf{v} \in \mathbf{V}_{\mathbf{u}_\Gamma}(t), \end{cases} \tag{2.15}$$

$$\nabla \cdot \mathbf{u}_\epsilon(t) = 0 \text{ in } \Omega, \tag{2.16}$$

$$\mathbf{u}_\epsilon(0) = \mathbf{u}_0, \tag{2.17}$$

$$\mathbf{u}_\epsilon(t) = \mathbf{u}_\Gamma(t) \text{ on } \Gamma. \tag{2.18}$$

Replacing, in (2.15), $\mathbf{v}$ by $\mathbf{u}_\epsilon(t) + \theta \mathbf{w}$ with $\theta > 0$ and $\mathbf{w} \in (H_0^1(\Omega))^d$, dividing by θ, and taking (2.14) into account, we obtain at the limit as $\theta \to 0_+$ that $\{\mathbf{u}_\epsilon, p_\epsilon\}$ is solution of the following nonlinear variational problem:

$$\begin{cases} \rho \int_\Omega \dfrac{\partial \mathbf{u}_\epsilon}{\partial t}(t) \cdot \mathbf{w} dx + \rho \int_\Omega (\mathbf{u}_\epsilon(t) \cdot \nabla)\mathbf{u}_\epsilon(t) \cdot \mathbf{w} dx \\ +\mu \int_\Omega \nabla \mathbf{u}_\epsilon(t) : \nabla \mathbf{w} dx + \sqrt{2} g \int_\Omega \dfrac{\mathbf{D}(\mathbf{u}_\epsilon(t)) : \mathbf{D}(\mathbf{w})}{\sqrt{\epsilon^2 + |\mathbf{D}(\mathbf{u}_\epsilon(t)|^2}} dx \\ - \int_\Omega p_\epsilon(t) \nabla \cdot \mathbf{w} dx = \int_\Omega \mathbf{f}(t) \cdot \mathbf{w} dx, \ \forall \mathbf{w} \in (H_0^1(\Omega))^d, \end{cases} \tag{2.19}$$

$$\nabla \cdot \mathbf{u}_\epsilon(t) = 0 \text{ in } \Omega, \tag{2.20}$$

$$\mathbf{u}_\epsilon(0) = \mathbf{u}_0, \tag{2.21}$$

$$\mathbf{u}_\epsilon(t) = \mathbf{u}_\Gamma(t) \text{ on } \Gamma. \tag{2.22}$$

Since tensor $(\epsilon^2 + |\mathbf{D}(u_\epsilon)|^2)^{-1/2}\mathbf{D}(\mathbf{u}_\epsilon)$ is symmetric, we clearly have

$$(\epsilon^2+|\mathbf{D}(\mathbf{u}_\epsilon)|^2)^{-1/2}\mathbf{D}(\mathbf{u}_\epsilon) : \mathbf{D}(\mathbf{w}) = (\epsilon^2+|\mathbf{D}(\mathbf{u}_\epsilon)|^2)^{-1/2}\mathbf{D}(\mathbf{u}_\epsilon) : \nabla \mathbf{w}, \ \forall \mathbf{w} \in (H_0^1(\Omega))^d. \tag{2.23}$$

Combining relations (2.19) and (2.23) implies that $\{\mathbf{u}_\epsilon, p_\epsilon\}$ verifies:

$$\rho \Big[\frac{\partial \mathbf{u}_\epsilon}{\partial t} + (\mathbf{u}_\epsilon \cdot \nabla)\mathbf{u}_\epsilon \Big] - \mu \Delta \mathbf{u}_\epsilon - \sqrt{2} g \nabla \cdot \frac{\mathbf{D}(\mathbf{u}_\epsilon)}{\sqrt{\epsilon^2 + |\mathbf{D}(\mathbf{u}_\epsilon)|^2}} + \nabla p_\epsilon = \mathbf{f} \text{ in } \Omega \times (0,T), \tag{2.24}$$

$$\nabla \cdot \mathbf{u}_\epsilon = 0 \text{ in } \Omega \times (0,T), \tag{2.25}$$

$$\mathbf{u}_\epsilon(0) = \mathbf{u}_0, \tag{2.26}$$

$$\mathbf{u}_\epsilon = \mathbf{u}_\Gamma \text{ on } \Gamma \times (0,T), \tag{2.27}$$

a regularized variant of problem (2.1)–(2.5) that could have been obtained directly. From a computational point of view, the situation looks good since we have replaced the variational inequality problem (2.6)–(2.9) by (2.24)–(2.27), which looks like a "not too complicated" variant of the usual Navier-Stokes equations. However, a closer inspection shows that the second derivative of $j_\epsilon(\cdot)$ at $\mathbf{v}$ is given by

$$\begin{cases} \langle j_\epsilon''(\mathbf{v})\mathbf{w}, \mathbf{z} \rangle = \int_\Omega \dfrac{(\epsilon^2 + |\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{w}) : \mathbf{D}(\mathbf{z}) - (\mathbf{D}(\mathbf{v}) : \mathbf{D}(\mathbf{w}))(\mathbf{D}(\mathbf{v}) : \mathbf{D}(\mathbf{z}))}{(\epsilon^2 + |\mathbf{D}(\mathbf{v})|^2)^{3/2}} dx, \\ \forall \mathbf{v} \in (H^1(\Omega))^d, \ \forall \{\mathbf{w}, \mathbf{z}\} \in (H_0^1(\Omega))^d \times (H_0^1(\Omega))^d, \end{cases} \tag{2.28}$$

which implies that close to those $\mathbf{v}$ such that $\mathbf{D}(\mathbf{v})$ is "small" we have

$$\|j''(\mathbf{v})\| \simeq 1/\epsilon. \tag{2.29}$$

The situation is quite clear now: For $\mathbf{u}_\epsilon$ to be a good approximation of the solution $\mathbf{u}$ of problem (2.6)–(2.9), we have to use small ϵ's; on the other hand, relation (2.29) shows that we can expect problem (2.19)–(2.22), (2.24)–(2.27) to be badly conditioned for those situations where the rigid set

$$Q_0 = \{\{x,t\} | \{x,t\} \in \Omega \times (0,T), \ \mathbf{D}(\mathbf{u})(x,t) = \mathbf{0}\}$$

is large, implying that derivative based iterative methods such as Newton's, quasi-Newton's, and conjugate gradient will perform poorly. Fortunately for the practitioner, there exists an elegant way to overcome the computational difficulties associated to the non-differentiability of functional $j(\cdot)$, and make the solution of problem (2.6)–(2.9) almost as simple as that of the usual Navier-Stokes equations. This simplification is a direct consequence of Theorem 9.1 in [1 and 2, Chapter 6, Section 9]. When applied to problem (2.6)–(2.9), the Duvaut and Lions' results can be formulated as follows:

Theorem 2.1. *Let* $\{\mathbf{u}, p\}$ *be a solution of problem* (2.6)–(2.9)*; there exists then a tensor-*

valued function $\boldsymbol{\lambda}$ $(= (\lambda_{ij})_{1\le i,j\le d})$, *not necessarily unique, such that*

$$\boldsymbol{\lambda} \in (L^\infty(\Omega\times(0,T)))^{d\times d},\ \boldsymbol{\lambda} = \boldsymbol{\lambda}^t, \tag{2.30}$$

$$|\boldsymbol{\lambda}| \le 1 \ \textit{a.e. in}\ \Omega\times(0,T), \tag{2.31}$$

$$\boldsymbol{\lambda} : \mathbf{D}(\mathbf{u}) = |\mathbf{D}(\mathbf{u})| \ \textit{a.e.}\quad \textit{in}\ \Omega\times(0,T), \tag{2.32}$$

$$\rho\left[\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] - \mu\Delta\mathbf{u} - \sqrt{2}g\nabla\cdot\boldsymbol{\lambda} + \nabla p = \mathbf{f}\ \textit{in}\ \Omega\times(0,T), \tag{2.33}$$

$$\nabla\cdot\mathbf{u} = 0\ \textit{in}\ \Omega\times(0,T), \tag{2.34}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \tag{2.35}$$

$$\mathbf{u} = \mathbf{u}_\Gamma\ \textit{on}\ \Gamma\times(0,T), \tag{2.36}$$

with $|\boldsymbol{\lambda}| = \Big(\sum_{1\le i,j\le d}\lambda_{ij}^2\Big)^{1/2}$ *in* (2.31). *Conversely, if a triple* $\{\mathbf{u}, p, \boldsymbol{\lambda}\}$ *verifies relations* (2.30)–(2.36)*, then* $\{\mathbf{u}, p\}$ *is a solution of problem* (2.6)–(2.9).

Proof. (i) Relations (2.30)–(2.36) imply (2.6)–(2.9): Observe that the symmetry of $\boldsymbol{\lambda}$ implies that

$$\boldsymbol{\lambda} : \nabla\mathbf{v} = \boldsymbol{\lambda} : \mathbf{D}(\mathbf{v}),\ \forall\mathbf{v}\in(H^1(\Omega))^d. \tag{2.37}$$

Multiplying both sides of relation (2.33) by $\mathbf{v}-\mathbf{u}(t)$, with $\mathbf{v}\in V_{\mathbf{u}_\Gamma}(t)$, integrating by parts, and taking relation (2.37) into account, we obtain

$$\begin{cases} \rho\int_\Omega \dfrac{\partial\mathbf{u}}{\partial t}(t)\cdot(\mathbf{v}-\mathbf{u}(t))dx + \rho\int_\Omega(\mathbf{u}(t)\cdot\nabla)\mathbf{u}(t)\cdot(\mathbf{v}-\mathbf{u}(t))dx \\ +\mu\int_\Omega\nabla\mathbf{u}(t):\nabla(\mathbf{v}-\mathbf{u}(t))dx - \int_\Omega p(t)\nabla\cdot(\mathbf{v}-\mathbf{u}(t))dx \\ +\sqrt{2}g\Big(\int_\Omega\boldsymbol{\lambda}(t):\mathbf{D}(\mathbf{v})dx - \int_\Omega\boldsymbol{\lambda}(t):\mathbf{D}(\mathbf{u}(t))dx\Big) \\ = \int_\Omega\mathbf{f}(t)\cdot(\mathbf{v}-\mathbf{u}(t))dx, \forall\mathbf{v}\in V_{\mathbf{u}_\Gamma}(t). \end{cases} \tag{2.38}$$

From (2.31) and (2.32), we clearly have

$$\int_\Omega\boldsymbol{\lambda}(t):\mathbf{D}(\mathbf{v})dx \le \int_\Omega|\boldsymbol{\lambda}(t)||\mathbf{D}(\mathbf{v})|dx \le \int_\Omega|\mathbf{D}(\mathbf{v})|dx,\ \forall\mathbf{v}\in(H^1(\Omega))^d,$$

$$\int_\Omega\boldsymbol{\lambda}(t):\mathbf{D}(\mathbf{u}(t))dx = \int_\Omega|\mathbf{D}(\mathbf{u}(t))|dx,$$

which, combined with relation (2.38), imply relation (2.6). We have thus shown that (2.30)–(2.36) implies (2.6)–(2.9).

(ii) Relations (2.6)–(2.9) imply (2.30)–(2.36): If $\mathbf{u}_\Gamma = \mathbf{0}$ on $\Gamma\times(0,T)$, the implication (2.6)–(2.9) $\Rightarrow$ (2.30)–(2.36) is a relatively simple consequence of the Hahn-Banach theorem and of the fact that $j(\theta\mathbf{v}) = \theta j(\mathbf{v}),\ \forall\theta\ge 0,\ \forall\mathbf{v}\in(H^1(\Omega))^d$; we shall say no more sending the interested reader to [1 and 2, Chapter 6, Section 9] for the details of the proof. If $\mathbf{u}_\Gamma \ne \mathbf{0}$, the above result still holds, but is more complicated to prove.

Remark 2.1. It is shown in the above references that trace$(\boldsymbol{\lambda}) = 0$; the main reasons we did not mention this property earlier are that:

(i) Relation trace$(\boldsymbol{\lambda}) = 0$ is not necessary to prove the reciprocal implication (2.30)–(2.36) $\Rightarrow$ (2.6)–(2.9).

(ii) It plays no role from a computational point of view.

On the other hand, what will play an important computational role is the fact that relations (2.31) and (2.32) imply

$$\boldsymbol{\lambda}(t) = P_\Lambda(\boldsymbol{\lambda}(t) + r\sqrt{2}g\mathbf{D}(\mathbf{u}(t))),\quad \forall r>0,\quad \text{a.e.}\quad \text{on}\quad (0,T), \tag{2.39}$$

where, in (2.39), Λ is the closed convex set of $(L^2(\Omega))^{d\times d}$ (and $(L^\infty(\Omega))^{d\times d}$) defined by

$$\Lambda = \{\mathbf{q}|\mathbf{q} = (q_{ij})_{1\le i,j\le d}\in(L^2(\Omega))^{d\times d},\ |\mathbf{q}(x)|\le 1\ \text{a.e.}\quad \text{on}\ \Omega\}, \tag{2.40}$$

and $P_\Lambda : (L^2(\Omega))^{d\times d} \to \Lambda$ is the orthogonal-projection operator defined by

$$P_\Lambda(\mathbf{q})(x) = \begin{cases} \mathbf{q}(x) & \text{if } |\mathbf{q}(x)| \le 1, \\ \mathbf{q}(x)/|\mathbf{q}(x)| & \text{if } |\mathbf{q}(x)| > 1, \end{cases} \tag{2.41}$$

a.e. on Ω, $\forall \mathbf{q} \in (L^2(\Omega))^{d\times d}$. We observe that operator P_Λ is symmetry preserving.

§3. Time-Discretization of Problem (2.6)–(2.9) by Operator Splitting

There are many ways to time-discretize problem (2.6)–(2.9) by operator splitting. Among the many possible schemes, we shall discuss only one, of the Marchuk-Yanenko type; this scheme reads as follows (with, as usual, $t^{n+\alpha} = (n+\alpha)\Delta t$) :

$$\mathbf{u}^0 = \mathbf{u}_0, \tag{3.1}$$

then, for $n \ge 0$, $\mathbf{u}^n$ being known, we compute $\{\mathbf{u}^{n+1/3}, p^{n+1}\}$, $\mathbf{u}^{n+2/3}$ and $\mathbf{u}^{n+1}$ as follows:

Solve the generalized Stokes problem

$$\begin{cases} \rho \dfrac{\mathbf{u}^{n+1/3} - \mathbf{u}^n}{\Delta t} - \dfrac{\mu}{2}\Delta \mathbf{u}^{n+1/3} + \nabla p^{n+1} = \mathbf{f}^{n+1} \ (= \mathbf{f}(t^{n+1})) \ \text{ in } \Omega, \\ \nabla \cdot \mathbf{u}^{n+1/3} = 0 \ \text{ in } \Omega, \\ \mathbf{u}^{n+1/3} = \mathbf{u}_\Gamma^{n+1} \ (= \mathbf{u}_\Gamma(t^{n+1})) \ \text{ on } \Gamma, \end{cases} \tag{3.2}$$

then the transport problem

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}^{n+1/3} \cdot \nabla)\mathbf{u} = \mathbf{0} \ \text{ in } \Omega \times (t^n, t^{n+1}), \\ \mathbf{u}(t^n) = \mathbf{u}^{n+1/3}, \\ \mathbf{u} = \mathbf{u}_\Gamma^{n+1} \ \text{ on } \Gamma_-^{n+1} \times (t^n, t^{n+1}), \end{cases} \tag{3.3.1}$$

and set

$$\mathbf{u}^{n+2/3} = \mathbf{u}(t^{n+1}); \tag{3.3.2}$$

finally, solve the elliptic variational inequality

$$\begin{cases} \mathbf{u}^{n+1} \in (H^1(\Omega))^d, \ \mathbf{u}^{n+1} = \mathbf{u}_\Gamma^{n+1} \ \text{ on } \Gamma, \\ \rho \int_\Omega \dfrac{\mathbf{u}^{n+1} - \mathbf{u}^{n+2/3}}{\Delta t} \cdot (\mathbf{v} - \mathbf{u}^{n+1}) dx + \dfrac{\mu}{2} \int_\Omega \nabla \mathbf{u}^{n+1} : \nabla(\mathbf{v} - \mathbf{u}^{n+1}) dx \\ +g\sqrt{2}(j(\mathbf{v}) - j(\mathbf{u}^{n+1})) \ge 0, \ \forall \mathbf{v} \in (H^1(\Omega))^d, \ \mathbf{v} = \mathbf{u}_\Gamma^{n+1} \ \text{ on } \Gamma; \end{cases} \tag{3.4}$$

in (3.3.1), we have $\Gamma_-^{n+1} = \{x | x \in \Gamma, \ (\mathbf{u}_\Gamma^{n+1} \cdot \mathbf{n})(x) < 0\}$. Closely related operator splitting techniques have been used in [3] for the simulation of Bingham flow in two-dimensional square cavities.

Remark 3.1. It follows from, e.g., [4, Chapters 1 and 2] that the variational inequality problem (3.4) has a unique solution, characterized by the existence of a $d \times d$ tensor-valued function $\boldsymbol{\lambda}^{n+1}$ such that:

$$\rho \frac{\mathbf{u}^{n+1} - \mathbf{u}^{n+2/3}}{\Delta t} - \frac{\mu}{2}\Delta \mathbf{u}^{n+1} - g\sqrt{2}\nabla \cdot \boldsymbol{\lambda}^{n+1} = \mathbf{0} \ \text{ in } \Omega, \tag{3.5}$$

$$\mathbf{u}^{n+1} = \mathbf{u}_\Gamma^{n+1} \ \text{ on } \Gamma, \tag{3.6}$$

$$\boldsymbol{\lambda}^{n+1} \in (L^\infty(\Omega))^{d\times d}, \ \boldsymbol{\lambda}^{n+1} = (\boldsymbol{\lambda}^{n+1})^t, \tag{3.7}$$

$$|\boldsymbol{\lambda}^{n+1}(x)| \le 1 \ \text{ a.e. on } \Omega, \tag{3.8}$$

$$\boldsymbol{\lambda}^{n+1}(x) : \mathbf{D}(\mathbf{u}^{n+1})(x) = |\mathbf{D}(\mathbf{u}^{n+1})(x)| \ \text{ a.e. on } \Omega. \tag{3.9}$$

The multiplier $\boldsymbol{\lambda}^{n+1}$ is not necessarily unique.

§4. On the Finite Element Approximation of Problem (2.6)–(2.9)

In this section (assuming that Ω is a bounded polygonal domain of $\mathbb{R}^2$) we are going to space-approximate problem (2.6)–(2.9) by a variant of the Bercovier-Pironneau finite element method discussed in, e.g., [4, Chapter 7]. The fundamental discrete spaces are thus:

$$V_h = \{\mathbf{v}_h | \mathbf{v}_h \in (C^0(\overline{\Omega}))^2,\ \mathbf{v}_h|_T \in (P_1)^2,\ \forall T \in \mathcal{T}_{h/2}\}, \tag{4.1}$$

$$V_{0h} = \{\mathbf{v}_h | \mathbf{v}_h \in V_h,\ \mathbf{v}_h = \mathbf{0} \ \text{ on } \ \Gamma\}\ (= V_h \cap (H_0^1(\Omega))^2), \tag{4.2}$$

$$P_h = \{q_h | q_h \in C^0(\overline{\Omega}),\ q_h|_T \in P_1,\ \forall T \in \mathcal{T}_h\}. \tag{4.3}$$

In (4.1), P_1 is the space of the polynomials in two variables of degree ≤ 1. The continuous in time approximation of problem (2.6)–(2.9), associated to the above finite element spaces, is defined as follows:

For $t \in (0, T)$ find $\{\mathbf{u}_h(t), p_h(t)\} \in V_h \times P_h$ such that

$$\begin{cases} \rho \int_\Omega \left[\dfrac{\partial \mathbf{u}_h}{\partial t}(t) + (\mathbf{u}_h(t) \cdot \nabla)\mathbf{u}_h(t)\right] \cdot (\mathbf{v}_h - \mathbf{u}_h(t))dx \\ +\mu \int_\Omega \nabla \mathbf{u}_h(t) : \nabla(\mathbf{v}_h - \mathbf{u}_h(t))dx - \int_\Omega p_h(t)\nabla \cdot (\mathbf{v}_h - \mathbf{u}_h(t))dx \\ +g\sqrt{2}(j(\mathbf{v}_h) - j(\mathbf{u}_h(t))) \geq \int_\Omega \mathbf{f}_h(t) \cdot (\mathbf{v}_h - \mathbf{u}_h(t))dx, \\ \forall \mathbf{v}_h \in V_h,\ \mathbf{v}_h = \mathbf{u}_{\Gamma h}(t) \ \text{ on } \ \Gamma, \end{cases} \tag{4.4}$$

$$\int_\Omega \nabla \cdot \mathbf{u}_h(t) q_h dx = 0,\ \forall q_h \in P_h, \tag{4.5}$$

$$\mathbf{u}_h(t) = \mathbf{u}_{0h}, \tag{4.6}$$

$$\mathbf{u}_h(t) = \mathbf{u}_{\Gamma h}(t) \ \text{ on } \ \Gamma; \tag{4.7}$$

in (4.4)–(4.7):

- $\mathbf{f}_h$ is an approximation of $\mathbf{f}$.
- $\mathbf{u}_{\Gamma h}$ is an approximation of $\mathbf{u}_\Gamma$ so that

$$\begin{cases} \int_\Gamma \mathbf{u}_{\Gamma h}(t) \cdot \mathbf{n} d\Gamma = 0,\ \forall t \in (0, T), \\ \mathbf{u}_{\Gamma h}(t) \in \gamma V_h = \{\boldsymbol{\mu}_h | \boldsymbol{\mu}_h = \mathbf{v}_h|_\Gamma,\ \mathbf{v}_h \in V_h\}. \end{cases}$$

- $\mathbf{u}_{0h}$ is an approximation of $\mathbf{u}_0$ so that $\mathbf{u}_{0h} \in V_h$, $\mathbf{u}_{0h} = \mathbf{u}_{\Gamma h}(0)$ on Γ.
- It is easy to compute $j(v_h)$, $\forall \mathbf{v}_h \in V_h$, since (4.1) implies that, $\forall T \in \mathcal{T}_{h/2}$, we have $\mathbf{D}(\mathbf{v}_h|_T) \in \mathbb{R}^4$ and therefore $|\mathbf{D}(\mathbf{v}_h|_T)| \in \mathbb{R}$, which implies in turn that $j(v_h) = \int_\Omega |\mathbf{D}(\mathbf{v}_h)| dx = \sum_{T \in \mathcal{T}_{h/2}} \text{meas.}(T)|\mathbf{D}(\mathbf{v}_h|_T)|$, $\forall \mathbf{v}_h \in V_h$. There is thus no need for numerical integration to compute $j(v_h)$. The convergence, as $h \to 0$, of $\{\mathbf{u}_h, p_h\}_h$ to its continuous counterpart $\{\mathbf{u}, p\}$ is discussed in, e.g., [5], [6 and 7, Chapter 6].

§5. Solution of the Subproblems Encountered at Each Time Step of Scheme (3.1)–(3.4)

5.1. Solution of the Generalized Stokes Subproblems (3.2)

Combining scheme (3.1)–(3.4) with the finite element spaces described in Section 4 leads to the following approximation of the generalized Stokes problem (3.2):

Find $\{u_h^{n+1/3}, p_h^{n+1}\} \in V_h \times P_h$ such that

$$\begin{cases} \rho \int_\Omega \dfrac{\mathbf{u}_h^{n+1/3} - \mathbf{u}_h^n}{\Delta t} \cdot \mathbf{v}_h dx + \dfrac{\mu}{2} \int_\Omega \nabla \mathbf{u}_h^{n+1/3} : \nabla \mathbf{v}_h dx - \int_\Omega p_h^{n+1} \nabla \cdot \mathbf{v}_h dx \\ = \int_\Omega \mathbf{f}_h^{n+1} \cdot \mathbf{v}_h dx,\ \forall \mathbf{v}_h \in V_{0h}, \end{cases} \tag{5.1}$$

$$\int_\Omega \nabla\cdot \mathbf{u}_h^{n+1/3} q_h dx = 0,\ \forall q_h \in P_h, \tag{5.2}$$

$$\mathbf{u}_h^{n+1/3} = \mathbf{u}_{\Gamma h}^{n+1} \quad \text{on} \quad \Gamma. \tag{5.3}$$

The approximate generalized Stokes problem (5.1)–(5.3) is clearly of the Bercovier-Pironneau type; it can be solved using the discrete analogues of the preconditioned conjugate gradient algorithms discussed in, e.g., [8]–[10].

5.2. Solution of the Transport Sub-Problems (3.3)

To solve the transport problem (3.3) we shall combine the finite element spaces described in Section 4 to the wave-like equation approach advocated in [11–13]; we obtain then the following discrete wave-like equation problem:

Find $\mathbf{u}_h(t) \in V_h$, such that, $\forall t \in (t^n, t^{n+1})$,

$$\begin{cases} \int_\Omega \dfrac{\partial^2 \mathbf{u}_h}{\partial t^2}(t)\cdot \mathbf{v}_h dx + \int_\Omega (\mathbf{u}_h^{n+1/3}\cdot\nabla)\mathbf{u}_h(t)\cdot(\mathbf{u}_h^{n+1/3}\cdot\nabla)\mathbf{v}_h dx \\ +\displaystyle\int_{\Gamma\setminus\Gamma_-^{n+1}} \mathbf{u}_h^{n+1/3}\cdot\mathbf{n}\dfrac{\partial \mathbf{u}_h}{\partial t}(t)\cdot\mathbf{v}_h d\Gamma = 0, \forall \mathbf{v}_h \in V_{0h}^{-,n+1}, \end{cases} \tag{5.4}$$

$$\mathbf{u}_h(t^n) = \mathbf{u}_h^{n+1/3}, \tag{5.5}$$

$$\begin{cases} \dfrac{\partial \mathbf{u}_h}{\partial t}(t^n) \in V_{0h}^{-,n+1}, \\ \int_\Omega \dfrac{\partial \mathbf{u}_h}{\partial t}(t^n)\cdot\mathbf{v}_h dx = -\int_\Omega (\mathbf{u}_h^{n+1/3}\cdot\nabla)\mathbf{u}_h^{n+1/3}\cdot\mathbf{v}_h dx,\ \forall \mathbf{v}_h \in V_{0h}^{-,n+1}, \end{cases} \tag{5.6}$$

$$\mathbf{u}_h(t) = \mathbf{u}_{\Gamma h}^{n+1} \quad \text{on} \quad \Gamma_-^{n+1}, \tag{5.7}$$

with, in (5.4)–(5.7),

$$\Gamma_-^{n+1} = \{x | x\in\Gamma,\ (\mathbf{u}_h^{n+1/3}\cdot\mathbf{n})(x) < 0\},$$
$$V_{0h}^{-,n+1} = \{\mathbf{v}_h | \mathbf{v}_h \in V_h,\ \mathbf{v}_h = \mathbf{0} \quad \text{on} \quad \Gamma_-^{n+1}\}.$$

The solution of discrete wave-like equation problems such as (5.4)–(5.7) has been addressed in [11]–[13].

5.3. Solution of the Elliptic Variational Inequalities (3.4)

We approximate problem (3.4) by the following discrete elliptic variational inequality

$$\begin{cases} \mathbf{u}_h^{n+1} \in V_h,\ \mathbf{u}_h^{n+1} = \mathbf{u}_{\Gamma h}^{n+1} \quad \text{on} \quad \Gamma, \\ \rho\int_\Omega \dfrac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n+2/3}}{\Delta t}\cdot(\mathbf{v}_h - \mathbf{u}_h^{n+1})dx + \dfrac{\mu}{2}\int_\Omega \nabla\mathbf{u}_h^{n+1} : \nabla(\mathbf{v}_h - \mathbf{u}_h^{n+1})dx \\ +g\sqrt{2}(j(\mathbf{v}_h) - j(\mathbf{u}_h^{n+1})) \ge 0,\ \forall \mathbf{v}_h \in V_h,\ \mathbf{v}_h = \mathbf{u}_{\Gamma h}^{n+1} \quad \text{on} \quad \Gamma. \end{cases} \tag{5.8}$$

Problem (5.8) has a unique solution. To solve the above problem we are going to take advantage of its equivalence with:

$$\mathbf{u}_h^{n+1} \in V_h,\ \mathbf{u}_h^{n+1} = \mathbf{u}_{\Gamma h}^{n+1} \ \text{on}\ \Gamma,\ \boldsymbol{\lambda}_h^{n+1} \in L_h,\ \boldsymbol{\lambda}_h^{n+1} = (\boldsymbol{\lambda}_h^{n+1})^t, \tag{5.9}$$

$$\begin{cases} \rho\int_\Omega \dfrac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n+2/3}}{\Delta t}\cdot\mathbf{v}_h dx + \dfrac{\mu}{2}\int_\Omega \nabla\mathbf{u}_h^{n+1} : \nabla\mathbf{v}_h dx \\ +g\sqrt{2}\int_\Omega \boldsymbol{\lambda}_h^{n+1} : \mathbf{D}(\mathbf{v}_h)dx = 0,\ \forall \mathbf{v}_h \in V_{0h}, \end{cases} \tag{5.10}$$

$$|\boldsymbol{\lambda}_h^{n+1}| \le 1 \quad \text{a.e.} \quad \text{in}\ \Omega,\ \boldsymbol{\lambda}_h^{n+1} : \mathbf{D}(\mathbf{u}_h^{n+1}) = |\mathbf{D}(\mathbf{u}_h^{n+1})| \quad \text{a.e.} \quad \text{in}\ \Omega, \tag{5.11}$$

where, in (5.9), space L_h is defined by

$$L_h = \{\mathbf{q}_h | \mathbf{q}_h \in (L^\infty(\Omega))^4,\ \mathbf{q}_h|_T \in \mathbb{R}^4,\ \forall T \in \mathcal{T}_{h/2}\}; \tag{5.12}$$

we have thus $\nabla\mathbf{v}_h$ and $\mathbf{D}(\mathbf{v}_h)$ belonging to L_h, $\forall\mathbf{v}_h \in V_h$. It follows from the symmetry of $\boldsymbol{\lambda}_h^{n+1}$ that

$$\int_\Omega \boldsymbol{\lambda}_h^{n+1} : \mathbf{D}(\mathbf{v}_h)dx = \int_\Omega \boldsymbol{\lambda}_h^{n+1} : \nabla\mathbf{v}_h dx,\ \forall\mathbf{v}_h \in V_h, \tag{5.13}$$

and from relations (5.11) that

$$\boldsymbol{\lambda}_h^{n+1} = P_{\Lambda_h}(\boldsymbol{\lambda}_h^{n+1} + rg\sqrt{2}\mathbf{D}(\mathbf{u}_h^{n+1})),\ \forall r \geq 0, \tag{5.14}$$

where $\Lambda_h = \Lambda \cap L_h$, i.e.,

$$\Lambda_h = \{\mathbf{q}_h | \mathbf{q}_h \in L_h,\ |(\mathbf{q}_h|_T)| \leq 1,\ \forall T \in \mathcal{T}_{h/2}\}, \tag{5.15}$$

and where the orthogonal-projection operator from L_h onto Λ_h verifies

$$P_{\Lambda_h}(\mathbf{q}_h)|_T = \begin{cases} \mathbf{q}_h|_T \text{ if } |(\mathbf{q}_h|_T)| \leq 1, \\ \mathbf{q}_h|_T/|(\mathbf{q}_h|_T)| \text{ if } |(\mathbf{q}_h|_T)| > 1. \end{cases} \tag{5.16}$$

Denote by Λ_h^σ the (closed convex) subset of Λ_h defined by

$$\Lambda_h^\sigma = \{\mathbf{q}_h | \mathbf{q}_h \in \Lambda_h,\ \mathbf{q}_h = \mathbf{q}_h^t\}; \tag{5.17}$$

it is an easy exercise to show that

$$P_{\Lambda_h^\sigma}(\mathbf{q}_h) = P_{\Lambda_h}\Big(\frac{\mathbf{q}_h + \mathbf{q}_h^t}{2}\Big),\ \forall\mathbf{q}_h \in L_h. \tag{5.18}$$

Combining relation (5.18) with (5.14) yields

$$\boldsymbol{\lambda}_h^{n+1} = P_{\Lambda_h^\sigma}(\boldsymbol{\lambda}_h^{n+1} + rg\sqrt{2}\nabla\mathbf{u}_h^{n+1}), \quad \forall r \geq 0. \tag{5.19}$$

We have thus shown that problem (5.8), (5.9)–(5.11) is equivalent to

$$\mathbf{u}_h^{n+1} \in V_h,\ \mathbf{u}_h^{n+1} = \mathbf{u}_{\Gamma h}^{n+1} \text{ on } \Gamma,\ \boldsymbol{\lambda}_h^{n+1} \in L_h, \tag{5.20}$$

$$\begin{cases} \rho \int_\Omega \dfrac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n+2/3}}{\Delta t} \cdot \mathbf{v}_h dx + \dfrac{\mu}{2}\int_\Omega \nabla\mathbf{u}_h^{n+1} : \nabla\mathbf{v}_h dx \\ +g\sqrt{2}\int_\Omega \boldsymbol{\lambda}_h^{n+1} : \nabla\mathbf{v}_h dx = 0,\ \forall\mathbf{v}_h \in V_{0h}, \end{cases} \tag{5.21}$$

$$\boldsymbol{\lambda}_h^{n+1} = P_{\Lambda_h^\sigma}(\boldsymbol{\lambda}_h^{n+1} + rg\sqrt{2}\nabla\mathbf{u}_h^{n+1}),\ \forall r \geq 0. \tag{5.22}$$

Following, e.g., [6], [7], and [14], we shall use the following iterative method à la Uzawa to solve problem (5.8):

$$\boldsymbol{\lambda}_h^{n+1,0} \text{ is given in } \Lambda_h^\sigma; \tag{5.23}$$

then, for $k \geq 0$, assuming that $\boldsymbol{\lambda}_h^{n+1,k} \in \Lambda_h^\sigma$ is known, solve

$$\begin{cases} \mathbf{u}_h^{n+1,k} \in V_h,\ \mathbf{u}_h^{n+1,k} = \mathbf{u}_{\Gamma h}^{n+1} \text{ on } \Gamma, \\ \rho\int_\Omega \mathbf{u}_h^{n+1,k} \cdot \mathbf{v}_h dx + \dfrac{\mu\Delta t}{2}\int_\Omega \nabla\mathbf{u}_h^{n+1,k} : \nabla\mathbf{v}_h dx \\ = \rho\int_\Omega \mathbf{u}_h^n \cdot \mathbf{v}_h dx - g\sqrt{2}\Delta t\int_\Omega \boldsymbol{\lambda}_h^{n+1,k} : \nabla\mathbf{v}_h dx,\ \forall\mathbf{v}_h \in V_{0h}, \end{cases} \tag{5.24}$$

and compute

$$\boldsymbol{\lambda}_h^{n+1,k+1} = P_{\Lambda_h^\sigma}(\boldsymbol{\lambda}_h^{n+1,k} + rg\sqrt{2}\nabla\mathbf{u}_h^{n+1,k}). \tag{5.25}$$

Concerning the convergence of algorithm (5.23)–(5.25), we have the following

Theorem 5.1. *Suppose that*

$$0 < r < \frac{\mu}{2g^2}; \tag{5.26}$$

we have then, $\forall\boldsymbol{\lambda}_h^{n+1,0} \in \Lambda_h^\sigma$,

$$\lim_{k\to+\infty}\{\mathbf{u}_h^{n+1,k}, \boldsymbol{\lambda}_h^{n+1,k}\} = \{\mathbf{u}_h^{n+1}, \boldsymbol{\lambda}_h^{n+1,*}\}, \tag{5.27}$$

where, in (5.27)*, the pair* $\{\mathbf{u}_h^{n+1}, \boldsymbol{\lambda}_h^{n+1,*}\}$ *is a solution of problem* (5.9)–(5.11)*,* $\mathbf{u}_h^{n+1}$ *being then the unique solution of problem* (5.8)*.*

Proof. Proving the convergence of $\{\mathbf{u}_h^{n+1,k}\}_{k\ge 0}$ is fairly easy:

Suppose that $\mathbf{q}_h \in L_h$; we shall denote by $\|\mathbf{q}_h\|_0$ the $L^2(\Omega)$-norm of $\mathbf{q}_h$ defined by $\|\mathbf{q}_h\|_0 = \Big(\int_\Omega |\mathbf{q}_h|^2 dx\Big)^{1/2}$; operator $P_{\Lambda_h^\sigma}$ is a contraction for the above norm. Next, we denote by $\overline{\mathbf{u}}_h^{n+1,k}$ and $\overline{\boldsymbol{\lambda}}_h^{n+1,k}$ the differences $\mathbf{u}_h^{n+1,k} - \mathbf{u}_h^{n+1}$ and $\boldsymbol{\lambda}_h^{n+1,k} - \boldsymbol{\lambda}_h^{n+1}$, where $\{\mathbf{u}_h^{n+1}, \boldsymbol{\lambda}_h^{n+1}\} \in V_h \times \Lambda_h^\sigma$ is a solution of problem (5.9)–(5.11). By subtraction, we clearly obtain

$$\begin{cases} \overline{\mathbf{u}}_h^{n+1,k} \in V_{0h}, \\ \rho \int_\Omega \overline{\mathbf{u}}_h^{n+1,k} \cdot \mathbf{v}_h dx + \mu \dfrac{\Delta t}{2} \int_\Omega \nabla \overline{\mathbf{u}}_h^{n+1,k} : \nabla \mathbf{v}_h dx \\ = -g\sqrt{2}\Delta t \int_\Omega \overline{\boldsymbol{\lambda}}_h^{n+1,k} : \nabla \mathbf{v}_h dx, \ \forall \mathbf{v}_h \in V_{0h}, \end{cases} \tag{5.28}$$

$$\|\overline{\boldsymbol{\lambda}}_h^{n+1,k+1}\|_0 \le \|\overline{\boldsymbol{\lambda}}_h^{n+1,k} + rg\sqrt{2}\nabla \overline{\mathbf{u}}_h^{n+1,k}\|_0, \ \forall r \ge 0. \tag{5.29}$$

Taking $\mathbf{v}_h = \overline{\mathbf{u}}_h^{n+1,k}$ in (5.28) and combining with (5.29) we obtain

$$\begin{cases} \quad \|\overline{\boldsymbol{\lambda}}_h^{n+1,k}\|_0^2 - \|\overline{\boldsymbol{\lambda}}_h^{n+1,k+1}\|_0^2 \\ \ge -2rg\sqrt{2}\int_\Omega \overline{\boldsymbol{\lambda}}_h^{n+1,k} : \nabla \overline{\mathbf{u}}_h^{n+1,k} dx - 2r^2 g^2 \|\nabla \overline{\mathbf{u}}_h^{n+1,k}\|_0^2 \\ \ge r\mu(\frac{2\rho}{\mu\Delta t}\|\overline{\mathbf{u}}_h^{n+1,k}\|_{(L^2(\Omega))^2}^2 + \|\nabla \overline{\mathbf{u}}_h^{n+1,k}\|_0^2) - 2r^2 g^2 \|\nabla \overline{\mathbf{u}}_h^{n+1,k}\|_0^2 \\ \ge r(\mu - 2rg^2)\Big(\dfrac{2\rho}{\mu\Delta t}\|\overline{\mathbf{u}}_h^{n+1,k}\|_{(L^2(\Omega))^2}^2 + \|\nabla \overline{\mathbf{u}}_h^{n+1,k}\|_0^2\Big). \end{cases} \tag{5.30}$$

Suppose that inequalities (5.26) hold; it follows then from (5.30) that the sequence $\{\|\overline{\boldsymbol{\lambda}}_h^{n+1,k}\|_0\}_{k\ge 0}$ is decreasing. Since it is bounded from below by 0, it converges to some limit, implying that

$$\lim_{k\to+\infty} \big(\|\overline{\boldsymbol{\lambda}}_h^{n+1,k}\|_0^2 - \|\overline{\boldsymbol{\lambda}}_h^{n+1,k+1}\|_0^2\big) = 0; \tag{5.31}$$

since (5.26) implies $r(\mu - 2rg^2) > 0$, combining (5.30) with (5.31) shows that $\lim\limits_{k\to+\infty} \overline{\mathbf{u}}_h^{n+1,k} = \mathbf{0}$, i.e., $\lim\limits_{k\to+\infty} \mathbf{u}_h^{n+1,k} = \mathbf{u}_h^{n+1}$. To prove the convergence of $\{\boldsymbol{\lambda}_h^{n+1,k}\}_{k\ge 0}$ we should proceed as in, e.g., [7, Appendix 2, Section 3].

Remark 5.1. Actually, the upper bound in (5.26) is pessimistic. Indeed, we can easily show (from relation (5.30)) that the convergence result (5.27) still holds if r verifies

$$0 < r < \Big(1 + \frac{2\rho}{\mu\Delta t \beta_h^M}\Big)\frac{\mu}{2g^2}, \tag{5.32}$$

where, in (5.32), β_h^M is the largest eigenvalue of the following discrete eigenvalue/eigenfunction problem:

$$\begin{cases} \{\mathbf{w}_h, \ \beta\} \in V_{0h} \times \mathbb{R}_+, \\ \int_\Omega \nabla \mathbf{w}_h : \nabla \mathbf{v}_h dx = \beta \int_\Omega \mathbf{w}_h \cdot \mathbf{v}_h dx, \ \forall \mathbf{v}_h \in V_{0h}, \\ \int_\Omega |\mathbf{w}_h|^2 dx = 1. \end{cases} \tag{5.33}$$

We recall that $\beta_h^M = O(h^{-2})$.

§6. Numerical Experiments

The numerical simulation of Bingham flow has not motivated as many publications as the solution of the “ordinary” Navier-Stokes equations. Besides [5], relevant publications are, e.g., [15], [16], [17], [18], [19, Chapter 6] and [3]; some of the results reported in the above references have been obtained using a stream-function formulation. The test problems considered here (and, actually, the methodology to solve them) are closely related to those in [3]. These test problems are all particular cases of the following problem:

Find $\{\mathbf{u}(t), p(t)\} \in (H^1(\Omega))^2 \times L^2(\Omega)$ such that a.e. on $(0, T)$ we have

$$\begin{cases} \int_\Omega \frac{\partial \mathbf{u}}{\partial t}(t) \cdot (\mathbf{v} - \mathbf{u}(t))dx + \int_\Omega (\mathbf{u}(t) \cdot \nabla)\mathbf{u}(t) \cdot (\mathbf{v} - \mathbf{u}(t))dx \\ +\mu \int_\Omega \nabla \mathbf{u}(t) : \nabla(\mathbf{v} - \mathbf{u}(t))dx + \sqrt{2}g\Big(j(\mathbf{v}) - j(\mathbf{u}(t))\Big) \\ -\int_\Omega p(t)\nabla \cdot (\mathbf{v} - \mathbf{u}(t))dx \geq 0, \ \forall \mathbf{v} \in V_{\mathbf{u}_\Gamma}, \end{cases} \tag{6.1}$$

$$\nabla \cdot \mathbf{u}(t) = 0 \ \text{ in } \ \Omega, \tag{6.2}$$

$$\mathbf{u}(0) = \mathbf{0}, \tag{6.3}$$

$$\mathbf{u}(t) = \mathbf{u}_\Gamma \ \text{ on } \ \Gamma, \tag{6.4}$$

with $j(\mathbf{v}) = \int_\Omega |\mathbf{D}(\mathbf{v})|dx, \ \forall \mathbf{v} \in (H^1(\Omega))^2$. In system (6.1)–(6.4), we have

- $\Omega = (0,1)^2, \ \Gamma = \partial\Omega$.
- $\Gamma_N = \{x | x = \{x_1, x_2\}, \ x_2 = 1, \ 0 < x_1 < 1\}$, and

$$\mathbf{u}_\Gamma(x) = \begin{cases} \mathbf{0} \ \text{ if } \ x \in \Gamma \setminus \Gamma_N, \\ 16U\{x_1^2(1-x_1)^2, 0\} \ \text{ if } \ x \in \Gamma_N, \end{cases}$$

with $U > 0$.

- $V_{\mathbf{u}_\Gamma} = \{\mathbf{v} | \mathbf{v} \in (H^1(\Omega))^2, \ \mathbf{v} = \mathbf{u}_\Gamma \ \textit{on} \ \Gamma\}$.

For the time-discretization of problem (6.1)–(6.4), we have employed the Marchuk-Yanenko scheme (3.1)–(3.4). For the space discretization we have used a 128×128 uniform grid to define the finite element spaces $V_h, \ V_{0h}$ and P_h (see relations (4.1)–(4.3)); from these spaces we proceeded as in Sections 4 and 5 to approximate problem (6.1)–(6.4) and compute its solutions. We have, in particular, used $r = \mu/rg^2$ when computing $\{\mathbf{u}_h^{n+1}, \boldsymbol{\lambda}_h^{n+1}\}$ by algorithm (5.23)–(5.25).

First Test Problem. It is the particular case of problem (6.1)–(6.4) corresponding to $U = 1/16, \ \mu = 1$ and $g = .1$; for the time discretization we have used $\Delta t = 10^{-3}$. Recalling that $\mathbf{u}(0) = \mathbf{0}$, we have shown in Fig. 6.1(a) the variation of the computed kinetic energy; it is clear from the above figure that "we" converge quickly to a steady state solution. The streamlines of the computed solution at $t = 2.39$ are shown in Fig. 6.1(b). The rigidity (black) and plastic (white) regions have been visualized in Fig. 6.1(c). The rigidity region (3-connected here) is the one where $\mathbf{D}(\mathbf{u}) = \mathbf{0}$; it is also the region where $|\boldsymbol{\lambda}(x)| < 1$, as shown in Fig. 6.1(d) where the graph of $|\boldsymbol{\lambda}|$ has been visualized. To conclude this presentation of the results associated to this first test problem, let us report on the following numerical experiment: The parameters $\mathbf{u}_0, \ \mu, \ g, \ U$ being as above, we solved problem (6.1)–(6.4) up to $t = 1.2$; let us denote by $\mathbf{u}(1.2)$ the velocity field at $t = 1.2$. At $t = 1.2$, we froze the motion of the upper wall implying that for $t > 1.2$ the Bingham flow is still modeled by relations (6.1), (6.2) completed by the boundary condition

$$\mathbf{u}(t) = \mathbf{0} \ \text{ on } \ \Gamma, \ \text{if } t > 1.2,$$

with $\mathbf{u}(1.2)$ as initial condition at $t = 1.2$. In principle, due to the absence of body forces and to the immobility of the boundary, the medium should return to rest in finite time (see Remark 6.1, hereafter), i.e., we should have $\mathbf{u}(t) = \mathbf{0}, \ \forall t \geq t_c, \ t_c$ being finite. Fig. 6.1(e) shows that indeed $\|\mathbf{u}(t)\|_{(L^2(\Omega))^2}$ converges to zero very quickly as $t \to +\infty$, but finite time convergence is doubtful from the above figure. Actually convergence in finite time takes place as shown in, e.g., [7, Appendix 6], [17], [18] and [19, Chapter 6]. In the above references time discretization was achieved with a fully implicit scheme à la backward Euler. It seems that for the calculation presented here, the splitting errors associated to scheme (3.1)–(3.4) prevent convergence to zero in finite time.

Fig. 6.1(a) Variation of the computed kinetic energy ($\mu = 1,\ g = 0.1,\ U = 1/16,\ \Delta x_1 = \Delta x_2 = 1/128,\ \Delta t = 10^{-3}$)

Fig. 6.1(b) Streamlines of the computed steady state velocity field ($\mu = 1,\ g = 0.1,\ U = 1/16,\ \Delta x_1 = \Delta x_2 = 1/128,\ \Delta t = 10^{-3}$)

Fig. 6.1(c) Visualization of the computed plastic (white) and rigid (black) regions at steady state ($\mu = 1,\ g = 0.1,\ U = 1/16,\ \Delta x_1 = \Delta x_2 = 1/128,\ \Delta t = 10^{-3}$)

Fig. 6.1(d) Graph of $|\boldsymbol{\lambda}_h|$ at steady state ($\mu = 1,\ g = 0.1,\ U = 1/16,\ \Delta x_1 = \Delta x_2 = 1/128,\ \Delta t = 10^{-3}$)

Second Test Problem. This test problem is the variation of the first one obtained by taking $U = 1$ instead of $1/16$. Besides this modification, all the other physical and numerical parameters are the same. The kinetic energy variation, the streamlines, the plastic and rigid regions and the multiplier $\boldsymbol{\lambda}_h$ have been visualized in Fig. 6.2(a) to Fig. 6.2(d). The velocity of the upper wall being much larger the kinectic energy reaches much higher values than in the first test problem. Similarly, due to the higher level of stress, the plastic region is much larger than in the first case (compare Fig. 6.2(c) to Fig. 6.1(c)). We observe that in both cases, the viscous effects are so strong that the advection plays practically no role as shown by the symmetry of the computed results with respect to the line $x_1 = 0.5$.

Fig. 6.1(e) Decay of the computed kinetic energy after the sliding of the upper wall has been stopped at $t = 1.2$ ($\mu = 1,\ g = 0.1,\ U = 1/16,\ \Delta x_1 = \Delta x_2 = 1/128,\ \Delta t = 10^{-3}$)

Fig. 6.2(a) Variation of the computed kinetic energy ($\mu = 1,\ g = 0.1,\ U = 1,\ \Delta x_1 = \Delta x_2 = 1/128,\ \Delta t = 10^{-3}$)

Remark 6.1. (On the convergence to zero in finite time). Consider problem (2.6)–(2.9) and suppose that $d = 2,\ \mathbf{f} = \mathbf{0},\ \mathbf{u}_\Gamma = \mathbf{0}$ and $T = +\infty$. If the above assumptions hold, then $\mathbf{u}(t)$ converges to $\mathbf{0}$ in finite time as t increases, $\forall \mathbf{u}_0 \in (L^2(\Omega))^d$ such that $\nabla \cdot \mathbf{u}_0 = 0$ and $\mathbf{u}_0 \cdot \mathbf{n} = 0$ on Γ. To prove the above result, observe that $\mathbf{u}(t) \in (H_0^1(\Omega))^2$, a.e. $t \in (0, +\infty)$, and take $\mathbf{v} = \mathbf{0}$ and $\mathbf{v} = 2\mathbf{u}(t)$ in (2.6). We obtain then

$$\begin{cases} \dfrac{\rho}{2}\dfrac{d}{dt}\|\mathbf{u}(t)\|^2_{(L^2(\Omega))^2} + \rho \int_\Omega (\mathbf{u}(t)\cdot\nabla)\mathbf{u}(t)\cdot\mathbf{u}(t)dx + \mu\int_\Omega |\nabla\mathbf{u}(t)|^2 dx \\ +g\sqrt{2}j(\mathbf{u}(t)) - \int_\Omega p(t)\nabla\cdot\mathbf{u}(t)dx = 0, \quad \text{a.e. } \ t \in (0, +\infty), \\ \mathbf{u}(t) = \mathbf{u}_0. \end{cases}$$

Fig. 6.2(b) Streamlines of the computed steady state velocity field ($\mu = 1,\ g = 0.1,\ U = 1,\ \Delta x_1 = \Delta x_2 = 1/128,\ \Delta t = 10^{-3}$)

Fig. 6.2(c) Visualization of the computed plastic (white) and rigid (black) regions at steady state ($\mu = 1,\ g = 0.1,\ U = 1,\ \Delta x_1 = \Delta x_2 = 1/128,\ \Delta t = 10^{-3}$)

From now on, we shall denote $\|\cdot\|_{(L^2(\Omega))^2}$ by $\|\cdot\|_{0,\Omega}$; from $\nabla \cdot \mathbf{u}(t) = 0$ the above relation reduces to

$$\begin{cases} \dfrac{\rho}{2}\dfrac{d}{dt}\|\mathbf{u}(t)\|_{0,\Omega}^2 + \mu \int_\Omega |\nabla \mathbf{u}(t)|^2 dx + g\sqrt{2}j(\mathbf{u}(t)) = 0, \quad \text{a.e. } t \in (0, +\infty), \\ \mathbf{u}(t) = \mathbf{u}_0. \end{cases} \tag{6.5}$$

On the other hand, we have

$$\|\mathbf{v}\|_{0,\Omega}^2 \le \lambda_0^{-1} \int_\Omega |\nabla \mathbf{v}|^2 dx,\ \forall \mathbf{v} \in (H_0^1(\Omega))^2, \tag{6.6}$$

$$\|\mathbf{v}\|_{0,\Omega} \le \gamma j(\mathbf{v}),\ \forall \mathbf{v} \in (H_0^1(\Omega))^2, \tag{6.7}$$

where, in (6.6) and (6.7), $\lambda_0(>0)$ is the smallest eigenvalue of operator $-\Delta$ "acting" on $H_0^1(\Omega)$, and γ is a positive constant; inequality (6.7) is known as the Nirenberg-Strauss inequality and is proved in [20]. Combining relations (6.5), (6.6), and (6.7) yields

$$\begin{cases} \frac{\rho}{2}\frac{d}{dt}\|\mathbf{u}(t)\|_{0,\Omega}^2 + \mu\lambda_0\|\mathbf{u}(t)\|_{0,\Omega}^2 + g\gamma^{-1}\sqrt{2}\|\mathbf{u}(t)\|_{0,\Omega} \le 0, \quad \text{a.e. } t \in (0,+\infty), \\ \|\mathbf{u}(0)\|_{0,\Omega} = \|\mathbf{u}_0\|_{0,\Omega}. \end{cases} \tag{6.8}$$

Fig. 6.2(d) Graph of $|\boldsymbol{\lambda}_h|$ at steady state ($\mu=1,\ g=0.1,\ U=1,\ \Delta x_1=\Delta x_2=1/128,\ \Delta t=10^{-3}$)

Suppose that $\mathbf{u}(t)$ never vanishes; we have then $\|\mathbf{u}(t)\|>0,\ \forall t\ge 0$ and

$$\frac{d}{dt}\|\mathbf{u}(t)\|_{0,\Omega}^2 = 2\|\mathbf{u}(t)\|_{0,\Omega}\frac{d}{dt}\|\mathbf{u}(t)\|_{0,\Omega}. \tag{6.9}$$

Combining (6.8) and (6.9) we obtain

$$\begin{cases} \rho\frac{d}{dt}\|\mathbf{u}(t)\|_{0,\Omega} + \mu\lambda_0\|\mathbf{u}(t)\|_{0,\Omega} + g\gamma^{-1}\sqrt{2} \le 0, \quad \text{a.e. } t \in (0,+\infty), \\ \|\mathbf{u}(0)\|_{0,\Omega} = \|\mathbf{u}_0\|_{0,\Omega}. \end{cases} \tag{6.10}$$

Observe now that (6.10) is equivalent to

$$\begin{cases} \frac{d}{dt}[\|\mathbf{u}(t)\|_{0,\Omega} + g\sqrt{2}(\mu\lambda_0\gamma)^{-1}] + \frac{\mu\lambda_0}{\rho}[\|\mathbf{u}(t)\|_{0,\Omega} + g\sqrt{2}(\mu\lambda_0\gamma)^{-1}] \le 0, \quad \text{a.e. } t \in (0,+\infty), \\ \|\mathbf{u}(0)\|_{0,\Omega} = \|\mathbf{u}_0\|_{0,\Omega}. \end{cases} \tag{6.11}$$

Integrating the differential inequality in (6.11) from 0 to t we obtain

$$\|\mathbf{u}(t)\|_{0,\Omega} + g\sqrt{2}(\mu\lambda_0\gamma)^{-1} \le e^{-\frac{\mu\lambda_0}{\rho}t}[\|\mathbf{u}_0\|_{0,\Omega} + g\sqrt{2}(\mu\lambda_0\gamma)^{-1}], \quad \forall t \ge 0. \tag{6.12}$$

Since $\lim\limits_{t\to+\infty} e^{-\frac{\mu\lambda_0}{\rho}t} = 0$, relation (6.12) makes no sense as soon as $t > t_c$, with t_c defined by

$$t_c = \frac{\rho}{\lambda_0\mu}\ln\Big(1 + \frac{\lambda_0\mu\gamma}{g\sqrt{2}}\|\mathbf{u}_0\|_{0,\Omega}\Big); \tag{6.13}$$

we have then $\mathbf{u}(t) = \mathbf{0}$ if $t \ge t_c$.

The assumptions on $d,\ \mathbf{f},\ \mathbf{u}_\Gamma$ and T staying the same, suppose now that we time-discretize problem (2.6)–(2.9) by the backward Euler scheme; we obtain then

$$\mathbf{u}^0 = \mathbf{u}_0; \tag{6.14}$$

then, for $n \geq 1$, $\mathbf{u}^{n-1}$ being known, find $\{\mathbf{u}^n, p^n\} \in (H_0^1(\Omega))^d \times L^2(\Omega)$ such that

$$\begin{cases} \rho \int_\Omega \dfrac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\Delta t} \cdot (\mathbf{v} - \mathbf{u}^n) dx + \rho \int_\Omega (\mathbf{u}^n \cdot \nabla)\mathbf{u}^n \cdot (\mathbf{v} - \mathbf{u}^n) dx \\ +\mu \int_\Omega \nabla \mathbf{u}^n : \nabla(\mathbf{v} - \mathbf{u}^n) dx + g\sqrt{2}(j(\mathbf{v}) - j(\mathbf{u}^n)) - \int_\Omega p^n \nabla \cdot (\mathbf{v} - \mathbf{u}^n) dx \geq 0, \\ \forall \mathbf{v} \in (H_0^1(\Omega))^2, \end{cases} \tag{6.15}$$

$$\nabla \cdot \mathbf{u}^n = 0 \quad \text{on} \quad \Omega. \tag{6.16}$$

Assuming that problem (6.14)–(6.16) has a solution, $\forall n \geq 1$ (it is not very difficult to prove that it is, indeed, the case), take $\mathbf{v} = \mathbf{0}$ and $2\mathbf{u}^n$ in (6.15), then take into account $\nabla \cdot \mathbf{u}^n = 0$, relations (6.6) and (6.7) and $\int_\Omega \mathbf{u}^n \cdot \mathbf{u}^{n-1} dx \leq \|\mathbf{u}^n\|_{0,\Omega} \|\mathbf{u}^{n-1}\|_{0,\Omega}$; it follows then from (6.15) that

$$\begin{cases} \dfrac{\rho}{\Delta t} \|\mathbf{u}^n\|_{0,\Omega} (\|\mathbf{u}^n\|_{0,\Omega} - \|\mathbf{u}^{n-1}\|_{0,\Omega}) + \lambda_0 \mu \|\mathbf{u}^n\|_{0,\Omega}^2 + g\sqrt{2}\gamma^{-1} \|\mathbf{u}^n\|_{0,\Omega} \leq 0, \quad \forall n \geq 1, \\ \|\mathbf{u}^0\|_{0,\Omega} = \|\mathbf{u}_0\|_{0,\Omega}. \end{cases} \tag{6.17}$$

Suppose that $\mathbf{u}^n \neq \mathbf{0}$, $\forall n \geq 0$. We have then $\|\mathbf{u}^n\|_{0,\Omega} > 0$, $\forall n \geq 1$, which combined with (6.17) yields

$$\begin{cases} \dfrac{\rho}{\Delta t} (\|\mathbf{u}^n\|_{0,\Omega} - \|\mathbf{u}^{n-1}\|_{0,\Omega}) + \lambda_0 \mu \|\mathbf{u}^n\|_{0,\Omega} + g\sqrt{2}\gamma^{-1} \leq 0, \quad \forall n \geq 1, \\ \|\mathbf{u}^0\|_{0,\Omega} = \|\mathbf{u}_0\|_{0,\Omega}. \end{cases} \tag{6.18}$$

It follows from (6.18) that

$$\begin{cases} \|\mathbf{u}^n\|_{0,\Omega} + g\sqrt{2}(\lambda_0 \mu \gamma)^{-1} \leq \Big(1 + \dfrac{\lambda_0 \mu}{\rho} \Delta t\Big)^{-1} [\|\mathbf{u}^{n-1}\|_{0,\Omega} + g\sqrt{2}(\lambda_0 \mu \gamma)^{-1}], \quad \forall n \geq 1, \\ \|\mathbf{u}^0\|_{0,\Omega} = \|\mathbf{u}_0\|_{0,\Omega}, \end{cases}$$

which implies in turn that

$$\|\mathbf{u}^n\|_{0,\Omega} + g\sqrt{2}(\lambda_0 \mu \gamma)^{-1} \leq \Big(1 + \frac{\lambda_0 \mu}{\rho} \Delta t\Big)^{-n} [\|\mathbf{u}_0\|_{0,\Omega} + g\sqrt{2}(\lambda_0 \mu \gamma)^{-1}], \quad \forall n \geq 0. \tag{6.19}$$

Since $\lim_{n \to +\infty} \big(1 + \frac{\lambda_0 \mu}{\rho} \Delta t\big)^{-n} = 0$, relation (6.19) makes no sense if $n > n_c$, with

$$n_c = \frac{\ln\Big(1 + \dfrac{\lambda_0 \mu \gamma}{g\sqrt{2}} \|\mathbf{u}_0\|_{0,\Omega}\Big)}{\ln\Big(1 + \dfrac{\lambda_0 \mu}{\rho} \Delta t\Big)}; \tag{6.20}$$

we have thus

$$\mathbf{u}^n = \mathbf{0}, \quad \forall n > n_c; \tag{6.21}$$

relation (6.21) is a discrete analogue of $\mathbf{u}(t) = 0$, $\forall t \geq t_c$. It is worth while noticing that, as expected,

$$\lim_{\Delta t \to 0_+} n_c \Delta t = \frac{\rho}{\lambda_0 \mu} \ln\Big(1 + \frac{\lambda_0 \mu \gamma}{g\sqrt{2}} \|\mathbf{u}_0\|_{0,\Omega}\Big) = t_c.$$

We have shown thus that the solution $\{\mathbf{u}^n\}_{n \geq 1}$ of problem (6.14)–(6.16) behaves "discretely" like the solution of problem (2.6)–(2.9). To prove (and have) the same result after space discretization it will definitely help to have

$$\int_\Omega (\mathbf{u}_h^n \cdot \nabla)\mathbf{u}_h^n \cdot \mathbf{u}_h^n dx = 0, \ \forall n \geq 1. \tag{6.22}$$

This will not be the case, in general, if one employs the Hood-Taylor or Bercovier-Pironneau finite element methods to approximate problem (2.6)–(2.9). An easy way to overcome this difficulty, and recover the convergence to zero in finite discrete time, would be to replace $\int_\Omega(\mathbf{u}_h^n\cdot\nabla)\mathbf{u}_h^n\cdot(\mathbf{v}_h-\mathbf{u}_h^n)dx$ by

$$\int_\Omega\Big[(\mathbf{u}_h^n\cdot\nabla)\mathbf{u}_h^n+\frac{1}{2}(\nabla\cdot\mathbf{u}_h^n)\mathbf{u}_h^n\Big]\cdot(\mathbf{v}_h-\mathbf{u}_h^n)dx,$$

an idea (due to R. Temam) used by many authors.

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