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ON THE CLASSIFICATION OF INITIAL DATA FOR NONLINEAR WAVE EQUATIONS**

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(*Dedicated to the memory of Jacques-Louis Lions*)

Abstract

The purpose of the present paper is to call for attention to the following question: Which of the initial data (nonsmall) admit global smooth solutions to the Cauchy problem for nonlinear wave equations. A few cases and examples are sketched, showing that the general answer of this question may be quite complicated.

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§1. Introduction

For hyperbolic equations the most fundamental problem is the Cauchy problem. In the linear case, the well-posedness of the Cauchy problem has been established successfully long time ago[2,6]. In the nonlinear case the existence and uniqueness of the local solution is also well-known[2,12]. One of main problems to be settled is the existence of global smooth solutions. There are a series of important works in this direction. In particular, for a big class of nonlinear wave equations, the existence of global solutions to the Cauchy problem with sufficient small initial data is quite clear[7−14]. However, the existence of global solutions to the simplest wave equation

$$u_{tt} - u_{xx} = f(u) \tag{1.1}$$

is still to be elucidated if the initial data

$$u(0,x) = \phi_0(x), \quad u_t(0,x) = \phi_1(x) \tag{1.2}$$

are nonsmall. For simplifying the statement we suppose that each function appeared is of C^∞. Define $\Sigma[f]$ as the set of initial data for which the Cauchy problem (1.1) and (1.2) admits a global smooth solution on the entire t-x plane. The main problem is to identify

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$\Sigma[f]$. The purpose of the present paper is to call for attention to this problem. Some examples will be sketched so as to explain some approaches to the construction of $\Sigma[f]$. It is expected that these examples will give some insight to the further study.

§2. Hyperbolic Liouville Equation

The hyperbolic Liouville equation is

$$u_{tt} - u_{xx} = -e^u. \tag{2.1}$$

The equation has a general solution of the form[1]

$$u = \ln \frac{-2\psi'(\xi)\phi'(\eta)}{[\psi(\xi)+\phi(\eta)]^2}, \tag{2.2}$$

where $\xi = \frac{t+x}{2}$, $\eta = \frac{t-x}{2}$. It is seen that

(i) The expression (2.2) is meaningful if and only if

$$\psi'(\xi)\phi'(\eta) < 0, \quad \psi(\xi)+\phi(\eta) \neq 0. \tag{2.3}$$

(ii) u is unchanged if ψ and ϕ are replaced by

$$\psi_1 = a\psi + b, \quad \phi_1 = a\phi - b, \quad (a, b = \text{const.}) \tag{2.4}$$

(iii) u is a global solution if ψ, ϕ are defined on $(-\infty,\infty)$ and (2.3) holds for every $\xi \in (-\infty,\infty)$, $\eta \in (-\infty,\infty)$.

(iv) The expression (2.2) exhausts all C^∞ solutions not only in local sense but also and in global sense if (iii) holds.

(i), (ii) and (iii) are abvious, (iv) follows from the fact that the unique solution to the Gousart problem with the boundary condition

$$u|_{\xi=0} = \alpha(\eta), \quad u|_{\eta=0} = \beta(\xi) \quad (\alpha(0) = \beta(0)) \tag{2.5}$$

can be expressed in the form (2.2). Thus the initial data

$$u(0,x) = \phi_0(x), \quad u_t(0,x) = \phi_1(x) \tag{2.6}$$

admit a global solution on the t-x plane iff there exist ϕ, ψ defined on $(-\infty,\infty)$ and satisfy (iii) such that

$$\phi_0(x) = \ln \frac{-2\psi'(\frac{x}{2})\phi'(-\frac{x}{2})}{\left[\psi(\frac{x}{2})+\phi(-\frac{x}{2})\right]^2}, \tag{2.7}$$

$$\phi_1(x) = \frac{1}{2}\frac{\psi''(\frac{x}{2})}{\psi'(\frac{x}{2})} + \frac{1}{2}\frac{\phi''(-\frac{x}{2})}{\phi'(-\frac{x}{2})} - \frac{\psi'(\frac{x}{2})+\phi'(-\frac{x}{2})}{\psi(\frac{x}{2})+\phi(-\frac{x}{2})}. \tag{2.8}$$

Consequently, the set of all C^∞ initial data with global solutions, i.e. $\Sigma[-e^u]$, has been obtained explicitly. However, it is quite difficult to check whether a given pair $(\phi_0(x), \phi_1(x))$ can be initial data of a global smooth solution. The problem is reduced to seeing whether the solution (ϕ, ψ) of (2.7) and (2.8) is a global one and satisfies (iii) or not.

Remark: In [1], there is a table of special nonlinear PDEs with explicit solutions. Some of them can be treated in a similar way.

§3. Some Cases for $\Sigma[f]=C^\infty\times C^\infty$

There is a big class of nonlinear wave equations, for which any given initial data ϕ_0, ϕ_1 will admit a global smooth solution. Among these nonlinear wave equations we mention

(i) Equations with $f(u)$ of weak nonlinearity

Let $h(t,x)$ be the solution of wave equation

$$\frac{\partial^2 h}{\partial t^2} - \frac{\partial^2 h}{\partial x^2} = 0$$

satisfying the initial data (ϕ_0, ϕ_1). The equation (1.1) with the same initial data is equivalent to the integral equation

$$u(x,t) = h(t,x) + \frac{1}{2}\int_{\Delta_{(t,x)}} f(u(\tau,\xi))d\tau d\xi, \tag{3.1}$$

where $\Delta_{(t,x)}$ is the characteristic triangle bounded by the x-axis and two characteristic lines passing through (t,x). The function $f(u)$ is called to be of weak nonlinearity if

$$|f(u)| < |u|^{1+r}, \quad \text{for} \quad |u| > A \ (A \ \text{is a positive constant}, \ \ r < 1). \tag{3.2}$$

It is well-known that the solution $u(x,t)$ exists globally. Example: sine-Gordon equation

$$u_{tt} - u_{xx} = \sin u.$$

(ii) Equations with strong nonlinear dissipation

Suppose that $f(u) = -F'(u)$ and $F(u) > ku^2$ (k is a positive constant). Let u be a solution on the characteristic triangle $\Delta_{(a,0)}$ with the vertices $(a,0)$, $(0,-a)$ and $(0,a)$. Then we have the energy inequality

$$\int_{l_\tau} (u_t^2 + u_x^2 + F(u))dx \le \int_{l_0} (\phi_1^2 + \phi_{0x}^2 + F(\phi_0))dx \quad (0 \le \tau \le a), \tag{3.3}$$

where l_τ is the interval $t = \tau$, $-a+\tau \le x \le a-\tau$. It follows that u is bounded in $\Delta_{(a,0)}$ and hence the solution $u(t,x)$ exists on $\Delta_{(a,0)}$ for any $a > 0$. Example: $f(u) = -\sinh u$, i.e. the minus sinh-Gordon equation

$$u_{tt} - u_{xx} = -\sinh u. \tag{3.4}$$

§4. Dual Relation

If we know $\Sigma[f]$, then $\Sigma[-f]$ can be constructed in the following way.

Let $(\phi_0, \phi_1) \in \Sigma[f]$. By solving the Cauchy problem for (1.1) with the initial data (ϕ_0, ϕ_1) we obtain the solution $u(t,x)$ which is defined on the whole x-t plane. Let

$$\psi_0 = u(t,0), \quad \psi_1 = u_x(t,0). \tag{4.1}$$

By interchanging t and x, it is obvious that the Cauchy problem for

$$v_{tt} - v_{xx} = -f(v) \tag{4.2}$$

with the initial data $v(0,x) = \psi_0(x)$, $v_t(0,x) = \psi_1(x)$ on the line $t = 0$ admits the global solution

$$v(t,x) = u(x,t). \tag{4.3}$$

The transformation $(\phi_0, \phi_1) \to (\psi_0, \psi_1)$ is denoted by L_f. Thus $\Sigma[-f]$ can be obtained from $\Sigma[f]$ via the transformation

$$L_f\{\phi_0, \phi_1\} = \{\psi_0, \psi_1\}. \tag{4.4}$$

The domain and range of L_f are $\Sigma[f]$ and $\Sigma[-f]$ respectively.

It is obvious that if $\Sigma[f]$ is nonempty, then $\Sigma[-f]$ is nonempty too and $L_{-f} = (L_f)^{-1}$. For example, let $f = \sinh u$. Then $\Sigma[\sinh u] = L_{(-\sinh u)} : \Sigma[\sinh(-u)] = L_{(-\sinh u)} : C^\infty \times C^\infty$.

Even in this case, the structure of $\Sigma[\sinh u]$ is not very clear and it is difficult to answer whether a given $\{\phi_0, \phi_1\}$ belongs to $\Sigma[\sinh u]$ or not. However, $\Sigma[\sinh u]$ is quite big and there exists a 1-1 mapping from $C^\infty \times C^\infty$ to $\Sigma[\sinh u]$. In $\Sigma[\sinh u]$, $|\phi_0|$ may be infinitely large, since in $\Sigma[-\sinh u]$, ϕ_0 can be arbitrary and $\psi_0(0) = \phi_0(0)$ if $(\psi_0, \psi_1) \in L_{(-\sinh u)}(\phi_0, \phi_1)$.

§5. Case of Hyperbolic Systems

The problem is more difficult even in the case of $1 + 1$ dimension. However, we can mention the following facts. Besides those, in [9,10,12] there are a series of results on global classical solutions.

(i) For the Euler equations for isentropic gas, the only global smooth solutions are the trivial ones (i.e. u=const, ρ=const.)[3.10].

(ii) For the harmonic maps from R^{1+1} to any complete Riemann manifold (so called wave maps) any smooth initial data admits a global solution[4].

(iii) For the harmonic maps from R^{1+1} to the Lorentz surface $S^{1,1}$, the classification of initial data according to the global existence has been discussed in [5]. The situation is quite complicated.

In short, the problem of determining initial data such that the Cauchy problem admits global smooth solution is of significance and worthy to be studied further.

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