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LOCAL EXACT BOUNDARY CONTROLLABILITY FOR A CLASS OF QUASILINEAR HYPERBOLIC SYSTEMS***

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(*Dedicated to the memory of Jacques-Louis Lions*)

Abstract

For a class of mixed initial-boundary value problem for general quasilinear hyperbolic systems, this paper establishes the local exact boundary controllability with boundary controls only acting on one end. As an application, the authors show the local exact boundary controllability for a kind of nonlinear vibrating string problem.

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§1. Introduction

Let us consider the following first order quasilinear hyperbolic system

$$\frac{\partial u}{\partial t}+A(u)\frac{\partial u}{\partial x}=F(u), \tag{1.1}$$

where $u=(u_1,\cdots,u_n)^T$ is an unknown vector function of (t,x), $A(u)=(a_{ij}(u))$ is an $n\times n$ matrix with suitably smooth elements $a_{ij}(u)(i,j=1,\cdots,n)$ and $F:R^n\to R^n$ is a vector function of u with suitably smooth components $f_i(u)(i=1,\cdots,n)$ such that

$$F(0)=0. \tag{1.2}$$

By the definition of hyperbolicity, on the domain under consideration, the matrix $A(u)$ has n real eigenvalues $\lambda_i(u)(i=1,\cdots,n)$ and a complete set of left eigenvectors $l_i(u)=(l_{i1}(u),\cdots,l_{in}(u))(i=1,\cdots,n)$:

$$l_i(u)A(u)=\lambda_i(u)l_i(u). \tag{1.3}$$

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We have

$$\det |l_{ij}(u)| \neq 0. \tag{1.4}$$

Moreover, we assume that on the domain under consideration, the eigenvalues satisfy the following conditions:

$$\lambda_r(u) < 0 < \lambda_s(u) \quad (r = 1, \cdots, m; s = m+1, \cdots, n). \tag{1.5}$$

Let

$$v_i = l_i(u)u \quad (i = 1, \cdots, n). \tag{1.6}$$

We set the boundary conditions as follows:

$$x = 0: \quad v_s = G_s(t, v_1, \cdots, v_m) + H_s(t) \quad (s = m+1, \cdots, n), \tag{1.7}$$

$$x = 1: \quad v_r = G_r(t, v_{m+1}, \cdots, v_n) + H_r(t) \quad (r = 1, \cdots, m), \tag{1.8}$$

where, without loss of generality, we assume that

$$G_i(t, 0, \cdots, 0) \equiv 0 \quad (i = 1, \cdots, n). \tag{1.9}$$

There is a number of publications on the exact controllability for linear hyperbolic systems (see [11] and the references therein). Especially, the exact boundary controllability for first order linear hyperbolic systems has been established by the characteristic method. J.-L. Lions introduced his Hilbert Uniqueness Method (HUM)(see [9,10]) which gives a more general and systematic framework for the study of the exact boundary controllability and the stabilisation for wave equations. Combining the HUM and Schauder's fixed point theorem, the first work on the exact controllability for semilinear wave equations was given by Zuazua[12,13]. Later, using a global inversion theorem, Lasiecka and Triggiani[3] gave an abstract result on the global exact controllability for semilinear wave equations. However, the exact controllability for the quasilinear hyperbolic systems remains quite open. To our knowledge, the first work in this direction was done by Cirinà[2] (see also [1]). Under linear boundary controls, he proved the local null controllability esstentially for quasilinear hyperbolic systems of diagonal form. In [5] and [6] the exact boundary controllability for reducible quasilinear hyperbolic systems was established by a characteristic method. Recently, these results were generalized to the case of general quasilinear hyperbolic systems. The following result was proved in [8].

Proposition 1.1. *Assume that* $l_{ij}(u), \lambda_i(u), f_i(u)$ *and* $G_i(t, \cdot)(i, j = 1, \cdots, n)$ *are all* C^1 *functions with respect to their arguments. Assume furthermore that* (1.2), (1.4)–(1.5) *and* (1.9) *hold. Let*

$$T_0 > \max_{i=1,\cdots,.n} \frac{1}{|\lambda_i(0)|}. \tag{1.10}$$

Then, for any given initial data $\phi \in C^1[0, 1]$ *and final data* $\psi \in C^1[0, 1]$ *with small* C^1 *norm, there exist boundary controls* $H_i(t) \in C^1[0, T_0](i = 1, \cdots, n)$ *with small* C^1 *norm, such that the mixed initial-boundary value problem for system* (1.1) *with the initial condition*

$$t = 0: \quad u = \phi(x) \tag{1.11}$$

and the boundary conditions (1.7)–(1.8) *admits a unique* C^1 *solution* $u = u(t,x)$ *with small* C^1 *norm on the domain*

$$R(T_0) = \{(t,x)| \quad 0 \le t \le T_0, \quad 0 \le x \le 1\},$$

which satisfies the final condition

$$t = T_0: \quad u = \psi(x). \tag{1.12}$$

As mentioned in [8], the exact controllability time T_0 given in Proposition 1.1 is optimal, however, the boundary controls are not unique.

We will prove in §2 that for a class of mixed initial-boundary value problem, the number of boundary controls can be diminished, provided that the exact controllability time is doubled. In §3 this result will be applied to show the local exact boundary controllability for a class of nonlinear vibrating string problem.

§2. Main Results

We now suppose that

$$n = 2m. \tag{2.1}$$

We suppose furthermore that the boundary condition (1.7) (resp. (1.8)) can be equivalently rewritten as

$$x = 0: \quad v_r = \overline{G}_r(t, v_{m+1}, \cdots, v_n) + \overline{H}_r(t) \quad (r = 1, \cdots, m) \tag{2.2}$$

$$\big[\text{resp.} \quad x = 1: \quad v_s = \overline{G}_s(t, v_1, \cdots, v_m) + \overline{H}_s(t) \quad (s = m+1, \cdots, n)\big],$$

where

$$\overline{G}_r(t, 0, \cdots, 0) \equiv 0 \quad (r = 1, \cdots, m) \tag{2.3}$$

$$\big[\text{resp.} \quad \overline{G}_s(t, 0, \cdots, 0) \equiv 0 \quad (s = m+1, \cdots, n)\big]$$

and

$$\text{small } C^1 \text{ norm of } H_s \Longleftrightarrow \text{small } C^1 \text{ norm of } \overline{H}_r \tag{2.4}$$

$$\big[\text{resp.} \quad \text{small } C^1 \text{ norm of } H_r \Longleftrightarrow \text{small } C^1 \text{ norm of } \overline{H}_s\big],$$

where $r = 1, \cdots, m; s = m+1, \cdots, n$.

Theorem 2.1. *Under the assumptions of Proposition* 1.1, *we suppose furthermore that conditions* (2.1)–(2.4) *hold and* $\overline{G}_r(t,\cdot)$ $(r = 1, \cdots, m)$ (*resp.* $\overline{G}_s(t,\cdot)$ $(s = m+1, \cdots, n)$) *are* C^1 *functions with respect to their arguments. Let*

$$T > 2 \max_{i=1,\cdots,n} \frac{1}{|\lambda_i(0)|}. \tag{2.5}$$

Suppose finally that $H_s(t)(s = m+1, \cdots, n)$ (*resp.* $H_r(t)(r = 1, \cdots, m)$) *are given* $C^1[0,T]$ *functions with small* C^1 *norm. Then, for any given initial data* $\phi \in C^1[0,1]$ *and final data* $\psi \in C^1[0,1]$ *with small* C^1 *norm, such that the conditions of* C^1 *compatibility are satisfied at points* $(0,0)$ *and* $(T,0)$ (*resp.* $(0,1)$ *and* $(T,1)$) *respectively, there exist boundary controls* $H_r(t) \in C^1[0,T](r = 1, \cdots, m)$ (*resp.* $H_s(t) \in C^1[0,T](s = m+1, \cdots, n)$) *with small* C^1

norm, such that the mixed initial boundary value problem (1.1), (1.11) *and* (1.7)–(1.8) *admits a unique* C^1 *solution* $u = u(t, x)$ *with small* C^1 *norm on the domain*

$$R(T) = \{(t, x)| \quad 0 \le t \le T, \quad 0 \le x \le 1\},$$

which satisfies the final condition

$$t = T: \quad u = \psi(x). \tag{2.6}$$

In order to prove Theorem 2.1, it suffices to establish the following

Lemma 2.1. *Under the assumptions of Theorem* 2.1, *for any given initial data* $\phi \in C^1[0, 1]$ *and final data* $\psi \in C^1[0, 1]$ *with small* C^1 *norm, such that the conditions of* C^1 *compatibility are satisfied at points* $(0, 0)$ *and* $(T, 0)$ (*resp.* $(0, 1)$ *and* $(T, 1)$) *respectively, the quasilinear hyperbolic system* (1.1) *with the boundary condition* (1.7) (*resp.* (1.8)) *admits a* C^1 *solution* $u = u(t, x)$ *with small* C^1 *norm on the domain* $R(T)$, *which satisfies* (1.11) *and* (2.6).

In fact, let $u = u(t, x)$ be a C^1 solution on the domain $R(T)$ given by Lemma 2.1. Taking the boundary controls as

$$H_r(t) = (v_r - G_r(t, v_{m+1}, \cdots, v_n))|_{x=1} \quad (r = 1, \cdots, m) \tag{2.7}$$

$$\big[\text{resp.} \quad H_s(t) = (v_s - G_s(t, v_1, \cdots, v_m))|_{x=0} \quad (s = m + 1, \cdots, n)\big],$$

the C^1 norm of which is small, we obtain the exact boundary controllability desired by Theorem 2.1.

We now prove Lemma 2.1. For fixing the idea, in what follows we consider only the case that the boundary controls are given at the end $x = 1$.

Noting (2.5), there exists an $\epsilon_0 > 0$ such that

$$T > 2 \max_{|u| \le \epsilon_0, i=1,\cdots,.n} \frac{1}{|\lambda_i(u)|}. \tag{2.8}$$

Let

$$T_1 = \max_{|u| \le \epsilon_0, i=1,\cdots,.n} \frac{1}{|\lambda_i(u)|}. \tag{2.9}$$

We first consider the forward mixed initial-boundary value problem for system (1.1) with the initial data

$$t = 0: \quad u = \phi(x), \quad 0 \le x \le 1 \tag{2.10}$$

and the boundary conditions (1.7) and

$$x = 1: \quad v_r = \bar{f}_r(t) \quad (r = 1, \cdots, m), \tag{2.11}$$

where $\bar{f}_r(t)$ are any given functions of t with small $C^1[0, T_1]$ norm. We assume that the conditions of C^1 compatibility are satisfied at point $(0, 1)$. By [7], there exists a unique semi-global C^1 solution $u = u^{(1)}(t, x)$ with small C^1 norm on the domain

$$\{(t, x)| \quad 0 \le t \le T_1, \quad 0 \le x \le 1\}. \tag{2.12}$$

Thus we can uniquely determine the corresponding value of $u = u^{(1)}(t, x)$ on $x = 0$ as

$$x = 0: \quad u = a(t), \quad 0 \le t \le T_1 \tag{2.13}$$

and the $C^1[0, T_1]$ norm of $a(t)$ is suitably small.

Similarly, we consider the backward mixed initial-boundary value problem for system (1.1) with the initial condition

$$t = T: \quad u = \psi(x), \qquad 0 \le x \le 1 \tag{2.14}$$

and the boundary conditions (2.2) and

$$x = 1: \quad v_s = \bar{g}_s(t) \qquad (s = m+1, \cdots, n), \tag{2.15}$$

where $\bar{g}_s(t) \quad (s = m+1, \cdots, n)$ are any given functions of t with small $C^1[T-T_1, T]$ norm. We assume that the conditions of C^1 compatibility are satisfied at point $(T, 1)$. Once again by [7], there exists a unique semi-global C^1 solution $u = u^{(2)}(t, x)$ with small C^1 norm on the domain

$$\{(t, x) | \quad T - T_1 \le t \le T, \quad 0 \le x \le 1\}. \tag{2.16}$$

Thus we can uniquely determine the corresponding value of $u = u^{(2)}(t, x)$ on $x = 0$ as

$$x = 0: \quad u = b(t), \quad T - T_1 \le t \le T \tag{2.17}$$

and the $C^1[T - T_1, T]$ norm of $b(t)$ is suitably small. Noting that both $a(t)$ and $b(t)$ satisfy the boundary condition (1.7), we can find a $C^1[0, T]$ function $c(t)$ with small C^1 norm, such that

$$c(t) = \begin{cases} a(t), & 0 \le t \le T_1, \\ b(t), & T - T_1 \le t \le T \end{cases} \tag{2.18}$$

and $c(t)$ satisfies the boundary condition (1.7) on the whole interval $[0, T]$.

Now we change the order of the variables t and x, then system (1.1) is rewritten in the following form

$$\frac{\partial u}{\partial x} + A^{-1}(u) \frac{\partial u}{\partial t} = \widetilde{F}(u) := A^{-1}(u) F(u). \tag{2.19}$$

We notice that

$$\widetilde{F}(0) = 0. \tag{2.20}$$

Noting (1.5), the eigenvalues of the inverse matrix $A^{-1}(u)$ satisfy

$$\frac{1}{\lambda_r(u)} < 0 < \frac{1}{\lambda_s(u)} \quad (r = 1, \cdots, m; \quad s = m+1, \cdots, n). \tag{2.21}$$

Moreover, since the matrices $A(u)$ and $A^{-1}(u)$ have the same left eigenvectors, we can still define the variables $v_i (i = 1, \cdots, n)$ by the same formula (1.6).

We now consider the mixed initial-boundary value problem for system (2.19) with the initial condition

$$x = 0: \quad u = c(t), \qquad 0 \le t \le T \tag{2.22}$$

and the boundary conditions

$$t = 0: \quad v_s = \Phi_s(t) \qquad (s = m+1, \cdots, n), \tag{2.23}$$

$$t = T: \quad v_r = \Psi_r(t) \qquad (r = 1, \cdots, m), \tag{2.24}$$

where

$$\Phi_i(x) = l_i(\phi(x))\phi(x) \quad (i = 1, \cdots, n), \tag{2.25}$$

$$\Psi_i(x) = l_i(\psi(x))\psi(x) \quad (i = 1, \cdots, n), \tag{2.26}$$

the C^1 norm of which is small. It is easy to see that the mixed initial-boundary value problem (2.19) and (2.22)–(2.24) satisfies the conditions of C^1 compatibility at points $(0,0)$ and $(T,0)$ respectively. Therefore, by [7] there exists a unique semi-global C^1 solution $u=u(t,x)$ with small C^1 norm on the domain

$$R(T)=\{(t,x)|\quad 0\le t\le T,\quad 0\le x\le 1\}.$$

In order to finish the proof of Lemma 2.1, it is only necessary to check that

$$t=0:\quad u=\phi(x),\quad 0\le x\le 1, \tag{2.27}$$

$$t=T:\quad u=\psi(x),\quad 0\le x\le 1. \tag{2.28}$$

In fact, the C^1 solutions $u=u(t,x)$ and $u=u^{(1)}(t,x)$ satisfy the system (2.19) (namely, (1.1)), the initial condition

$$x=0:\quad u=a(t),\quad 0\le t\le T_1 \tag{2.29}$$

and the boundary condition

$$t=0:\quad v_s=\Phi_s(t)\qquad (s=m+1,\cdots,n). \tag{2.30}$$

Because of the finiteness of the speed of wave propagation and the choice of T_1 given by (2.9), the mixed-initial boundary problem (2.19) and (2.29)–(2.30) has a unique C^1 solution on the domain

$$\{(t,x)|\quad 0\le t\le T_1(1-x),\quad 0\le x\le 1\} \tag{2.31}$$

(see [4]). Then it follows that

$$u(t,x)\equiv u^{(1)}(t,x) \tag{2.32}$$

on this domain. In particular, we obtain (2.27). We can get (2.28) in a similar way.

Thus $u=u(t,x)$ is the desired C^1 solution. The proof of Lemma 2.1 is complete.

Remark 2.1. The exact controllability time given by Theorem 2.1 is optimal.

Remark 2.2. The boundary controls in Theorem 2.1 are not unique.

§3. Application to a Class of Nonlinear Vibrating String Problem

In this section, we will use the previous results to show the local exact boundary controllability for the following nonlinear vibrating string equation

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial x}\Big(K\Big(\frac{\partial u}{\partial x}\Big)\Big)=F\Big(\frac{\partial u}{\partial x},\frac{\partial u}{\partial t}\Big), \tag{3.1}$$

where $K=K(v)$ is a given C^2 function of v, such that

$$K'(v)>0, \tag{3.2}$$

and $F=F(v,w)$ is a C^1 function of v and w, satisfying

$$F(0,0)=0. \tag{3.3}$$

We consider the exact boundary controllability only with one control applied at one end of the string. The boundary condition at the end $x=0$ is of Dirichlet type:

$$u=h(t), \tag{3.4}$$

where $h(t)$ is a given C^2 function of t; while the boundary condition at the end $x = 1$ is one of the following types:

$$u = \bar{h}(t), \tag{3.5.1}$$

$$u_x = \bar{h}(t), \tag{3.5.2}$$

$$u_x + \alpha u = \bar{h}(t), \tag{3.5.3}$$

$$u_x + \alpha u_t = \bar{h}(t), \tag{3.5.4}$$

where α is a positive constant and $\bar{h}(t)$, as boundary control, is a C^2 function (in case (3.5.1)) or a C^1 function (in cases (3.5.2)–(3.5.4)).

We want to find a time $T > 0$ and suitable boundary control $\bar{h}(t)$ with small C^1 norm $\|\bar{h}'\|_{C^1[0,T]}$ in case (3.5.1) or $\|\bar{h}\|_{C^1[0,T]}$ in cases (3.5.2)–(3.5.4), such that for any given initial data $(\phi(x), \psi(x))$ and final data $(\Phi(x), \Psi(x))$ with small C^1 norms

$$\|\phi'\|_{C^1[0,1]}, \quad \|\psi\|_{C^1[0,1]}, \quad \|\Phi'\|_{C^1[0,1]}, \quad \|\Psi\|_{C^1[0,1]},$$

satisfying the conditions of C^2 compatibility at points $(0,0)$ and $(T,0)$ respectively, the C^2 solution $u = u(t,x)$ to the mixed initial-boundary value problem for equation (3.1) with the initial condition

$$t = 0: \quad u = \phi(x), \qquad u_t = \psi(x) \tag{3.6}$$

and the boundary conditions (3.4)–(3.5) satisfies the final condition

$$t = T: \quad u = \Phi(x), \qquad u_t = \Psi(x). \tag{3.7}$$

We will prove

Theorem 3.1. *Let*

$$T > \frac{2}{\sqrt{K'(0)}}. \tag{3.8}$$

Then for any given initial data $\phi(x) \in C^2[0,1], \psi(x) \in C^1[0,1]$ *and final data* $\Phi(x) \in C^2[0,1], \Psi(x) \in C^1[0,1]$ *with small* C^1 *norms*

$$\|\phi'\|_{C^1[0,1]}, \quad \|\psi\|_{C^1[0,1]} \quad \text{and} \quad \|\Phi'\|_{C^1[0,1]}, \quad \|\Psi\|_{C^1[0,1]}$$

and any given function $h(t) \in C^2[0,T]$ *with small* C^1 *norm* $\|h'\|_{C^1[0,T]}$*, satisfying the following conditions of* C^2 *compatibility at points* $(0,0)$ *and* $(T,0)$ *respectively:*

$$\begin{cases} h(0) = \phi(0), \quad h'(0) = \psi(0), \\ h''(0) = K'(\phi'(0))\phi''(0) + F(\phi'(0), \psi(0)) \end{cases} \tag{3.9}$$

and

$$\begin{cases} h(T) = \Phi(0), \quad h'(T) = \Psi(0), \\ h''(T) = K'(\Phi'(0))\Phi''(0) + F(\Phi'(0), \Psi(0)), \end{cases} \tag{3.10}$$

there exits a boundary control $\bar{h}(t) \in C^2[0,T]$ *with small* C^1 *norm* $\|\bar{h}'\|_{C^1[0,T]}$ *in case* (3.5.1) *or* $\bar{h}(t) \in C^1[0,T]$ *with small* C^1 *norm* $\|\bar{h}\|_{C^1[0,T]}$ *in cases* (3.5.2)–(3.5.4)*, such that the mixed initial-boundary value problem for equation* (3.1) *with the initial condition* (3.6)*, the boundary condition* (3.4) *at the end* $x = 0$ *and one of the boundary conditions* (3.5) *at the end* $x = 1$ *admits a unique* C^2 *solution* $u = u(t,x)$ *on the domain*

$$R(T) = \{(t,x) | \quad 0 \le t \le T, \quad 0 \le x \le 1\},$$

which satisfies the final condition (3.7).

In order to prove Theorem 3.1, setting

$$v = \frac{\partial u}{\partial x}, \qquad w = \frac{\partial u}{\partial t}, \tag{3.11}$$

equation (1.1) is reduced to the following first order quasilinear system

$$\begin{cases} \dfrac{\partial v}{\partial t} - \dfrac{\partial w}{\partial x} = 0, \\ \dfrac{\partial w}{\partial t} - \dfrac{\partial K(v)}{\partial x} = F(v, w). \end{cases} \tag{3.12}$$

The system is strictly hyperbolic with two distinct real eigenvalues $\pm\lambda$, where

$$\lambda = \sqrt{K'(v)} > 0. \tag{3.13}$$

We introduce the Riemann invariants r and s as follows:

$$2r = G(v) + w, \quad 2s = -G(v) + w, \tag{3.14}$$

where

$$G(v) = \int_0^v \sqrt{K'(v)}dv \tag{3.15}$$

with

$$G(0) = 0, \quad G'(v) = \sqrt{K'(v)} > 0. \tag{3.16}$$

Let H be the inverse function of G. It follows from (3.14) that

$$w = r + s, \quad v = H(r - s). \tag{2.17}$$

Using the Riemann invariants r and s, (3.12) can be rewritten into the diagonal form

$$\begin{cases} \dfrac{\partial r}{\partial t} - \lambda \dfrac{\partial r}{\partial x} = f(r, s), \\ \dfrac{\partial s}{\partial t} + \lambda \dfrac{\partial s}{\partial x} = f(r, s), \end{cases} \tag{3.18}$$

where

$$f(r, s) = \frac{1}{2} F(H(r - s), r + s) \tag{3.19}$$

with

$$f(0, 0) = 0. \tag{3.20}$$

Correspondingly, the inial and final conditions (3.6)–(3.7) yield

$$t = 0: \quad \begin{cases} r = r_0(x) := \frac{1}{2}(G(\phi'(x)) + \psi(x)), \\ s = s_0(x) := \frac{1}{2}(-G(\phi'(x)) + \psi(x)), \end{cases} \tag{3.21}$$

$$t = T: \quad \begin{cases} r = r_1(x) := \frac{1}{2}(G(\Phi'(x)) + \Psi(x)), \\ s = s_1(x) := \frac{1}{2}(-G(\Phi'(x)) + \Psi(x)), \end{cases} \tag{3.22}$$

while the boundary condition (3.4) implies that

$$x = 0: \quad r + s = h'(t). \tag{3.23}$$

Moreover it follows from the last two equalities in (3.9)–(3.10) that the conditions of C^1 compatibility at point $(0, 0)$:

$$\begin{cases} r_0(0) + s_0(0) = h'(0), \\ h''(0) = \lambda(r_0(0), s_0(0))(r_0'(0) - s_0'(0)) + 2f(r_0(0), s_0(0)) \end{cases} \tag{3.24}$$

and at point $(0,T)$:

$$\begin{cases} r_1(0)+s_1(0)=h'(T), \\ h''(T)=\lambda(r_1(0),s_1(0))(r_1'(0)-s_1'(0))+2f(r_1(0),s_1(0)) \end{cases} \tag{3.25}$$

are satisfied. Then it is easy to check that we can apply Lemma 2.1 to get the following

Lemma 3.1. *Assume that* $h(t)\in C^2[0,T]$ *with small* C^1 *norm* $\|h'\|_{C^1[0,T]}$, *where* T *is defined by* (3.8)*. Then for any given initial data* $r_0,s_0\in C^1[0,1]$ *and final data* $r_1,s_1\in C^1[0,1]$ *with small* C^1 *norm, satisfying the conditions of* C^1 *compatibility* (3.24)–(3.25)*, the quasilinear hyperbolic system* (3.18) *associated with the boundary condition* (3.23) *admits a* C^1 *solution* $(r,s)=(r(t,x),s(t,x))$ *with small* C^1 *norm on the domain*

$$R(T)=\{(t,x)|\quad 0\le t\le T,\quad 0\le x\le 1\},$$

which satisfies the initial and final conditions (3.21)–(3.22)*.*

Proof of Theorem 3.1. Applying Lemma 3.1, we can find a C^1 solution $(v,w)=(v(t,x),w(t,x))$ to system (3.12) on the domain $R(T)$, which satisfies the initial and final conditions

$$t=0:\quad v=\phi'(x),\qquad w=\psi(x), \tag{3.26}$$

$$t=T:\quad v=\Phi'(x),\qquad w=\Psi(x) \tag{3.27}$$

and the boundary condition

$$x=0:\quad w=h'(t). \tag{3.28}$$

Let

$$u(t,x)=h(t)+\int_0^x v(t,y)dy. \tag{3.29}$$

It is easy to see that $u=u(t,x)$ is a C^2 function on the domain $R(T)$, such that

$$u(t,0)=h(t) \tag{3.30}$$

and

$$\frac{\partial}{\partial x}u(t,x)=v(t,x). \tag{3.31}$$

On the other hand, noting the first equation in (3.12) and using (3.28), we have

$$\frac{\partial}{\partial t}u(t,x)=h'(t)+\int_0^x \frac{\partial}{\partial x}w(t,y)dy=w(t,x). \tag{3.32}$$

Therefore, it follows from the second equation in (3.12) that the function u defined by (3.29) satisfies the string equation (3.1) and the boundary condition (3.4). Moreover, noting the first equality in (3.9)–(3.10), we check easily that u satisfies the initial and final conditions (3.6)–(3.7):

$$t=0:\ u=h(0)+\int_0^x v(0,y)dy=h(0)+\int_0^x \phi'(y)dy=\phi(x), \tag{3.33}$$

$$t=T:\ u=h(T)+\int_0^x v(T,y)dy=h(T)+\int_0^x \Phi'(y)dy=\Phi(x). \tag{3.34}$$

We now define the control $h(t)$ at the end $x = 1$ by one of the following expressions:

$$\bar{h}(t) =: u, \tag{3.35.1}$$

$$\bar{h}(t) =: u_x, \tag{3.35.2}$$

$$\bar{h}(t) =: u_x + \alpha u, \tag{3.35.3}$$

$$\bar{h}(t) =: u_x + \alpha u_t. \tag{3.35.4}$$

The proof of Theorem 3.1 is thus finished.

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