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ON NONLINEAR DIFFERENTIAL GALOIS THEORY

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(*À la mémorie de mon ami Jacques-Louis Lions*)

Abstract

Let X denote a complex analytic manifold, and let $\mathrm{Aut}(X)$ denote the space of invertible maps of a germ (X, a) to a germ (X, b); this space is obviously a groupoid; roughly speaking, a "Lie groupoid" is a subgroupoid of $\mathrm{Aut}(X)$ defined by a system of partial differential equations. To a foliation with singularities on X one attaches such a groupoid, e.g. the smallest one whose Lie algebra contains the vector fields tangent to the foliation. It is called "the Galois groupoid of the foliation". Some examples are considered, for instance foliations of codimension one, and foliations defined by linear differential equations; in this last case one recuperates the usual differential Galois group.

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This is an account of a work in course of progress. The aim is the following: define and study, for non linear differential equations, an object which generalizes the differential Galois group of linear equations. In [9], I give such a definition, and I prove the required result in the linear case. Here, I recall it shortly, and I insist on further examples and on open problems.

I should mention that another definition of a differential Galois group was proposed several years ago, by Umemura[15]; at the moment, I do not know the exact relations between both theories.

§1. General Definitions

I give the definitions in the complex analytic case; a similar theory, somewhat simpler, could be developed in the algebraic case (but, up to now, nothing is written in this context).

Let X denote a (smooth) complex analytic manifold, of dimension n; let $\mathrm{Aut}X$ be the space of germs of invertible maps $(X, a) \to (X, b)$ [e.g. the source is a, and the target b; $a, b \in X$]; let also J_k, resp. J_k^* be the space of jets of order k, resp. invertible jets of order

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k, of maps from X to X. I provide these spaces by the sheaf (on X^2) of functions which are analytic on the variables of X^2, and polynomial in the derivatives: precisely, in local coordinates, let $x_1, \cdots, x_n$ be the coordinates at the source, and $y_1, \cdots, y_n$ the coordinates at the target; with the standard notations, we have

$$\mathcal{O}_{J_k} = \mathcal{O}_{X^2}[y_j^\alpha],\ 1 \le j \le n,\ \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n,\ |\alpha| = \alpha_1 + \cdots + \alpha_n \in \{1, 2, \cdots, k\},$$

$$\mathcal{O}_{J_k^*} = \mathcal{O}_{X^2}[y_j^\alpha, \delta^{-1}],\ \delta = \det(y_j^i) \text{ the Jacobian matrix.}$$

The main objects in consideration here are what I call "D-groupoids" or "Lie groupoids" on X. Roughly speaking, there are the subgroupoids of Aut$X$ which are defined by a system of partial differential equations. More precisely, putting $\mathcal{O}_{J^*} = \cup \mathcal{O}_{J_k^*}$, we consider a sheaf of ideals $\mathfrak{J} \subset \mathcal{O}_{J^*}$ which has the following properties:

(i) $\mathfrak{J}_k = \mathcal{O}_{J_k^*} \cap \mathfrak{J}$ is coherent and reduced (i.e., equal to its radical), for every $k \ge 0$; we say for short that $\mathcal{J}$ is pseudocoherent, and reduced.

(ii) $\mathfrak{J}$ is differential, e.g. stable by derivations.

[The derivations are defined in local coordinates by the standard formula

$$D_i f = \frac{\partial f}{\partial x_i} + \sum_{j,\alpha} \frac{\partial f}{\partial y_j^\alpha} y_j^{\alpha+\varepsilon_i}\ ,\ \varepsilon_i = (0, \cdots, 1, \cdots, 0);$$

it is easy to verify that the stability by derivations is independent of the coordinates chosen.]

The two preceding properties define, generally speaking, a system of partial differential equations in Aut$X$. One has to add a third condition to have a Lie groupoid, which I explain now.

The set J_k^* is obviously a groupoid for the composition of jets (e.g. the composition $\psi\varphi$ exists if target φ = source ψ, and all elements are invertible); this property can be translated into a property of $\mathcal{O}_{J_k^*}$; now, take $\mathfrak{J}_k$ a coherent sheaf of ideals of $\mathcal{O}_{\mathfrak{J}_k^*}$, and call Y_k the ringed space $(X^2, \mathcal{O}_{J_k^*}/\mathfrak{J}_k)$. This has a sense to say that "Y_k is a subgroupoid of J_k^*"; we have to take this sentence in the sense of ideals, or in the "scheme sense", and not only in the set-theoretical sense (this can be expressed by properties of $\mathfrak{J}_k$ which I leave the reader to guess, or to look, f.i. in [7]).

So, now, the third condition to impose to $\mathfrak{J}$ is the following

(iii-1) For every k, Y_k contains the identity (e.g. the ideal of the identity contains $\mathfrak{J}_k$), and Y_k is stable by the inverse.

(iii-2) For every $U \subset X$, open relatively compact (I abreviate $U \subset\subset X$), there exists Z, closed analytic subset of codimension ≥ 1 of U such that, on $(U - Z)^2$, Y_k is a subgroupoid of J_k^* for k large.

[Note that, in the algebraic case, the "U" would be unnecessary, and also probably the restriction "k large".]

One can see that this definition has reasonably good properties, and covers the cases of interest, for the following reasons:

(a) Any increasing sequence $\mathfrak{J}^\ell \subset \mathfrak{J}^{\ell+1} \subset \cdots$ of ideals defining a D-groupoid is locally stationary.

(b) Let $\mathfrak{J}_k \subset \mathcal{O}_{J_k^*}$ be a coherent sheaf of ideals, not necessarily reduced defining a subgroupoid of J_k^* outside of some $Z \subset X$ of codimension ≥ 1; let $\mathfrak{J}'$ be the pseudocoherent and differential ideal generated by $\mathfrak{J}_k$; then the radical $\mathfrak{J}'^{\mathrm{rad}}$ of $\mathfrak{J}'$ defines a Lie groupoid; and, on every $U \subset\subset X$, one has $\mathfrak{J}'^{\mathrm{rad}} = \mathfrak{J}'$ outside a subset of codimension ≥ 1.

This situation occurs practically when $\mathfrak{J}_k$ is the ideal expressing the condition that the transformation stabilizes a "differential structure of order k, meromorphic on X with poles on Z" (whichever be the reasonable meaning given to this expression); standard examples are given by the groupoids preserving a 2-form (for instance, symplectic structures), one form up to invertible factors (f.i. contact structures), etc. In the literature, these objects are usually called "Lie pseudogroups", and considered in the nonsingular case; but here, I need to accept singularities.

(c) From (a) and (b), one deduces that any intersection of D-groupoids is locally finite, and is a D-groupoid (if the groupoids are defined by the sheaf of ideals $\mathfrak{J}^\alpha \subset \mathcal{O}_{J^*}$, their intersection is the smallest pseudocoherent reduced differential sheaf of ideals which contains the $\mathfrak{J}^\alpha$).

For these properties, see [9].

§2. *D*-Envelope and Galois Groupoid

Call "D-Lie-algebra" any system of linear partial differential equations on Θ, the sheaf of vector fields of X, such that its solutions are stable by Lie bracket. A D-groupoid has a D-Lie algebra, which is simply the linearized differential system along the identity.

But the converse is not true. This is similar to the fact that a Lie subalgebra of $\mathrm{Gl}(n, \mathbb{C})$ is not necessarily the Lie algebra of an algebraic subgroup of $\mathrm{Gl}(n, \mathbb{C})$. Given a D-Lie algebra $\mathcal{L}$, we can therefore define its D-envelope, as the smallest D-groupoid whose Lie algebra contains $\mathcal{L}$; this is meaningful, according to the property 1-c.

An especially interesting case is the case of the foliations with singularities. This is defined, f.i. by a coherent subsheaf N of Ω^1 (the sheaf of 1-forms on X) of locally constant rank outside of $Z \subset X$ of codimension ≥ 1 and verifying outside of Z the Frobenius condition $dN \subset \Omega^1 \wedge N$. We can suppose, by increasing a little bit N, that any local section which is in N outside of Z is actually in N. Then N defines a D-Lie algebra $\mathcal{F}$, the vectors fields orthogonal to N (in fact, $\mathcal{F}$ is defined by equations of order 0).

Definition *The Galois groupoid $G(\mathcal{F})$ of $\mathcal{F}$ is its D-envelope.*

Before giving examples, a few explanations are necessary. First, I will use the following facts:

(i) The solutions of a D-groupoid Y make a subgroupoid, in the usual sense, of $\mathrm{Aut}X$.

(ii) These solutions determine Y. Therefore, I will often identify both objects implicitely.

The first result is essentially obvious. The second one is a general fact of differential algebra (see [10], or [13] for the algebraic case).

Now consider, outside Z, the groupoid of automorphisms of X which preserve $\mathcal{F}$ (or N); it is easy to prove that the Zariski closure of the corresponding D-groupoid is a D-groupoid on X; denote it by $\mathrm{Aut}\mathcal{F}$. It is obvious that its D-Lie algebra contains $\mathcal{F}$; therefore, $\mathrm{Aut}\mathcal{F}$ contains $\mathrm{Gal}\mathcal{F}$.

Call "admissible" (w.r.t. $\mathcal{F}$) any D-groupoid Y contained in $\mathrm{Aut}\mathcal{F}$ and containing $\mathrm{Gal}\mathcal{F}$. If we consider Y near a pair $(a, b) \in (X - Z)^2$, no condition for Y occurs along the leaves, and Y is given by equations on the variables transversal to the foliation (f.i., in the case of $\mathrm{Aut}\mathcal{F}$, there are no such equations).

More precisely, suppose $\mathcal{F}$ of codimension d outside Z, and call "transversal" a locally closed submanifold T of X of dimension d which, outside of a set of codimension ≥ 1, does not meet Z and is transverse to $\mathcal{F}$. Call $\mathcal{T}$ the disjoint union of all the transversals; it is easy to see that, on $\mathcal{T}$, an admissible groupoid Y defines a groupoid: the "transversal groupoid" defined by Y. The philosophy is the following one: from a geometrical point of view, the interesting object is the transversal groupoid (which will correspond, f.i. to some transversal structure). But, to verify the admissibility, we have to go back to X, and to verify the possibility of extension along Z (actually, meromorphic extension in a suitable sense will be sufficient, since, then, one has just to take Zariski closures of the corresponding varieties $Y_k \hookrightarrow J_k^*$).

§3. Examples

The first example is treated in [9]; the second one will be developed elsewhere.

(i) The Linear Case

Let C be a complex nonsingular connected curve (= manifold of dimension one), and $X \overset{\pi}{\to} C$ a vector bundle over C. Let S be a discrete subset of C, and $Z = \pi^{-1}S$; we suppose given a connection ∇ on X, meromorphic on S (in local coordinates, this is simply a linear differential system $Y' = aY$, with a meromorphic S) with poles on S; note that we could equally consider flat meromorphic connections in the sense of [4] on vector bundles over a nonsingular analytic manifold of any dimension; the result would be similar.

On the total space X of the bundle, ∇ defines a foliation $\mathcal{F}$ of dimension one, with singularities on Z. The Galois groupoid of $\mathcal{F}$ can be described in the following way:

(a) Outside Z, the linear structures of $X \to C$ gives a "transversal linear structure", and a corresponding subgroupoid of Aut$\mathcal{F}$; one proves that this groupoid extends to X (in a unique way) into an admissible groupoid Lin$\mathcal{F}$.

(b) Admissible subgroupoids of Lin$\mathcal{F}$ can be described in the following way: choose a base point $a \in C - S$, and put $X_a = \pi^{-1}a$. Let G be an algebraic subgroup of Gl(X_a), containing the monodromy of ∇ at a.

Now, let Iso$X$ be the groupoid consisting of the family of linear isomorphisms $X_b \overset{\sim}{\to} X_c$, for $b, c \in C$, and let $\widetilde{G}$ the subgroupoid of Iso $X|X-Z$ generated by G and the isomorphisms of monodromy (= the parallel transport along any path in $X - Z$). Then, $\widetilde{G}$ defines a subgroupoid of Lin$F|X - Z$. This subgroupoid extends to an admissible groupoid if and only if $\widetilde{G}$ extends to an analytic subvariety of Iso$X$.

(c) Call "admissible" such a G; then there is a smallest admissible G; one proves that it is the differential Galois group of ∇, in the "tannakian" sense (for the definition, see [5]).

Roughly speaking, this means that the differential Galois group of ∇ "depends only" on the corresponding foliation and "does not depend" on the further structures of X and ∇.

(ii) Codimension One

The D-Lie algebras in dimension one, outside the singularities, have been determined by Lie; locally, their spaces of solutions are of dimension $0, 1, 2, 3, \infty$; in suitable coordinates, these solutions can be written in the following way:

Dimension 0: 0

Dimension 1: $\lambda \frac{d}{dx}$, $\lambda \in \mathbb{C}$ (structure of translation)

Dimension 2: $\lambda \frac{d}{dx} + \mu x \frac{d}{dx}$ (linear affine structure)

Dimension 3: $\lambda\frac{d}{dx}+\mu x\frac{d}{dx}+\nu x^2\frac{d}{dx}$ (projective structure)

Dimension ∞: all vector fields; no equation.

A study of these algebras, and the corresponding D-groupoids, near a singularity, will be published by G. Casale. I just mention the following facts: in dimension 0, one can have many groupoids (f.i. actions on X of finite groups); in dimension 1, the correspondence groupoids λ algebras is neither injective nor surjective.

In dimension 2 or 3, one has just one groupoid, e.g. the groupoid of automorphisms of the corresponding Lie algebra (in case of a D-Lie algebra of dimension one, the groupoid of its automorphisms has dimension 2).

Finally, the case "∞" is trivial: the corresponding groupoid is Aut$X$.

Now, the foliation, with singularities of codimension one on a manifold X, can be classified according to the dimension of the transversal Galois groupoid; if X is connected, it is easy to verify that this dimension is independent of the chosen transversal. Here, I will only look at the local situation, e.g. at a foliation in the neighborhood of $0\in\mathbb{C}^n$; it can be defined by a holomorphic 1-form ω, with $\omega\wedge d\omega=0$ (we can even suppose that ω has only singularities in codimension 2).

Even in this local case, I have no complete results when Gal$\mathcal{F}$ has transversal dimension 0 or 1.

(a) Dimension 0. This will be the case if ω has a first integral f, e.g. if there exists a meromorphic f such that $\omega\wedge df=0$; then, Gal$\mathcal{F}$ is contained in the groupoid which fixes f. I have no necessary and sufficient condition; the question is related to the quotient by an equivalence relation.

(b) Dimension 1. This will be the case if ω has an integrating factor, e.g. if there exist g meromorphic such that $d(g\omega)=0$; in that case, Gal$\mathcal{F}$ is contained in the groupoid which fixes $g\omega$; to prove this, it is sufficient to show that its D-Lie algebra "contains $\mathcal{F}$", e.g. contains the vector fields tangent to the foliation; but, if ξ is such a vector field, one has with standard notations $L_\xi(g\omega)=i_\xi d(g\omega)+d\langle\xi,g\omega\rangle=0$.

A classical theorem of Lie says the following: if ξ is a meromorphic vector field not in $\mathcal{F}$, but preserving the foliation, e.g. $L_\xi\omega\wedge\omega=0$, then $\langle\xi,\omega\rangle^{-1}$ is an integrating factor; in fact, write $\omega=\langle\xi,\omega\rangle\pi$; one has also $L_\xi\pi\wedge\pi=0$; as $d\langle\xi,\pi\rangle=0$, this can be written as $(i_\xi d\pi)\wedge\pi=0$; but, one has $d\pi\wedge\pi=0$, therefore $0=i_\xi(d\pi\wedge\pi)=i_\xi d\pi\wedge\pi+d\pi\cdot\langle\xi,\pi\rangle$, and $d\pi=0$.

Conversely, if g is an integrating factor, choose a ξ such that $\langle\xi,\omega\rangle=g^{-1}$ (ξ is determined mod$\mathcal{F}$); the same calculation shows that ξ preserves the foliation.

Transversally, this can be interpreted in the following way: ξ defines a vector field $\bar\xi$ on the transversals, which is fixed by the holonomy; then $\bar\xi^{-1}$ is a 1-form $\bar\pi$ on the transversals, and $\pi=\langle\xi,\omega\rangle^{-1}\omega$ is just the inverse image of $\bar\pi$ on X.

I will return to this example later (see Remark (iii)).

One can ask for a necessary and sufficient condition for $\mathcal{F}$ to have an admissible groupoid of transversal dimension one; it seems to me likely that such a condition is: there exists a finite ramified covering $\widetilde{X}\xrightarrow{\pi}X$ such that $\pi^*\omega$ admits a meromorphic integrating factor.

Of course, if Gal$\mathcal{F}$ has transversal dimension one, such an admissible groupoid exists. But I do not know if it is always the case when Gal$\mathcal{F}$ has transversal dimension 0.

(c) Dimensions 2 and 3. The answer is related with Godbillon-Vey sequence. Starting with ω meromorphic near 0, $\omega \wedge d\omega = 0$, one can construct recursively $\omega_1, \cdots, \omega_n, \cdots$ meromorphic, such that

$$\begin{aligned} d\omega &= \omega \wedge \omega_1, \\ d\omega_1 &= \omega \wedge \omega_2, \\ &\vdots \\ d\omega_n &= \omega \wedge \omega_{n+1} + \sum_1^n \binom{n}{k} \omega_k \wedge \omega_{n-k+1}. \end{aligned}$$

The following trick is due to J. Martinet: put $\Omega = dt + \sum \frac{t^n}{n!}\omega_n$ ($\omega_0 = \omega$); then, one has $d\Omega = \Omega \wedge \frac{\partial \Omega}{\partial t}$.

Given a germ of codimension one foliation, the Godbillon-Vey sequence is not unique: one can replace ω_0 by $g\omega_0$; and, at each step of the recurrence, one can add to ω_n a multiple of ω_0. We say that "$G-V$ sequence stops at order i", $i = 1, 2, 3$, if we can choose the ω_i's in such a way that $\omega_n = 0$ for $n \geq i$. To stop at order 1, it is necessary and sufficient that ω_0 has an integrating factor. To stop at order 2 (resp. 3), we must arrive at ω_1, with $d\omega_1 = 0$ (resp. ω_2, with $d\omega_2 = \omega_1 \wedge \omega_2$). In the nonsingular case and $\mathcal{C}^\infty$ context the following fact is well-known: the existence of a transversal affine (resp. projective) structure equivalent to the possibility of stopping $G-V$ sequence at order 2 (resp. 3) (cf. [6]).

The same result is true in the present context (this fact was suggested to me by [2]; see also [14], for an interpretation of "case 2", in an algebraic context, in terms of liouvillian solutions).

Let me give a few words of explanations; the details will be published elsewhere. I will look at the "case 3"; the other one is similar, and simpler.

Let X be a neighborhood of $0 \in \mathbb{C}^n$ on which ω, ω_1, ω_2 are meromorphic, and giving a $G-V$ sequence stopping at order 3. On $X \times P_1$, the form $\Omega = dt + \omega_0 + \omega_1 t + \omega_2 \frac{t^2}{2}$ defines a foliation transverse to the fibers $\{x\} \times \mathbb{P}_1$; outside of the singularities, this defines a transverse structure of type 3, which, on $\{x\} \times \mathbb{P}_1$, is simply the standard projective structure; [this is well known; in fact the equation $\Omega = 0$ comes from the integrable system $dy_1 = \frac{\omega_1}{2}y_1 + \frac{\omega_2}{2}y_2$, $dy_2 = -\omega_0 y_1 - \frac{\omega_1}{2}y_2$ by taking $t = \frac{y_2}{y_1}$]. One sees that the corresponding groupoid extends to $X \times \mathbb{P}_1$ into an admissible one. Taking the restriction to $t = 0$ gives an admissible groupoid of transversal dimension 3.

Now, the result is: conversely any admissible groupoid of dimension 3 of the foliation defined by ω can be obtained in this way, in particular, there exists a $G-V$-sequence $\omega_0, \omega_1, \omega_2$ stopping at order 3, with $\omega_0 = f\omega$.

(d) Dimension ∞. The corresponding groupoid is Aut$\mathcal{F}$; and there is nothing else to say from the point of view considered here. This case, which is of course the general case, should be studied by other methods (recurrence, attractors, etc), familiars in the theory of dynamical systems.

(iii) Remark: Lie Symmetries and Galois Symmetries

The result of Lie, mentionned in **(ii)**, (b) is sometimes stated in a slightly confusing way, as f.i. "if one has a one parameter group of symmetries of the equation, one can solve it"; this could induce a confusion between symmetries of the data ("Lie symmetries"), and Galois symmetries, e.g. the Galois groupoid.

It seems to me that the precise relations should be stated in the following way. We give on X a foliation with singularities $\mathcal{F}$; first of all, call "symmetries of $\mathcal{F}$" the global automorphisms of $\mathcal{F}$ which preserve $\mathcal{F}$, e.g. which are solutions of Aut$\mathcal{F}$. In fact here, we are not interested in them, but in the corresponding D-Lie algebra; call meraut$\mathcal{F}$ the sheaf of its meromorphic solutions; call similarly mer$\mathcal{F}$ the germs of meromorphic vector fields tangent to $\mathcal{F}$; incidentally, note that meraut$\mathcal{F}$/mer$\mathcal{F}$ is a sheaf of Lie algebras, (which can be interpreted as vector fields on the transversals).

Now, if we have a subset $\Gamma \subset \Gamma(X, \text{meraut}\mathcal{F}/\text{mer}\mathcal{F})$, we get an admissible groupoid G by taking the solutions of Aut$\mathcal{F}$ which fix Γ. Therefore, the bigger is Γ, the smaller is G. But this is only one procedure among others to have admissible groupoids as small as possible.

Of course, if one has a Lie group acting on X, and preserving $\mathcal{F}$, its infinitesimal transformations give sections of meraut$\mathcal{F}$ and a fortiori sections of meraut$\mathcal{F}$/mer$\mathcal{F}$.

§4. Further Examples and Problems

(i) Can one extend the results of §3 **(ii)** (c) to some cases of higher codimension?

(ii) Let ω be a closed form (f.i. meromorphic) of any degree on X. Then, the condition $i_\xi \omega = 0$ (ξ, vector field) defines a foliation with singularities $\mathcal{F}$, because one has $L_\xi \omega = di_\xi \omega + i_\xi d\omega = di_\xi \omega = 0$. An admissible groupoid is therefore obtained by taking the φ, solutions of Aut$\mathcal{F}$ which verify $\varphi^* \omega = \omega$.

Find "interesting" examples where some further reduction, on no further reduction can be obtained.

The most interesting case is probably the case where ω is a 2-form; then according to a classical theorem of Darboux, one has, outside of the singularities, a transversal symplectic structure.

For instance, consider the differential equation $y'' = f(x, y)$; writing this equation as the Pfaff system $dy - y\ dx$, $dz - f\ dx$, put $\omega = (dy - z\ dx) \wedge (dz - f\ dx)$; one has $d\omega = 0$, because f does not contain y'; and the foliation associated to ω is just the foliation corresponding to the equation.

In particular, it is a classical problem to prove that no further reduction occur for the "Painlevé 1" equation $y'' = y^2 + x$; in fact, Painlevé claimed this result with insufficient proof, see [12] (I owe this reference to J. P. Ramis; see also [15]).

The same problem could be considered for the other Painlevé equations, for which Okamoto has constructed transverse symplectic structures[11].

(iii) More generally, Hitchin[8] has given a transverse symplectic structure on the Schlessinger equations for isomonodromic deformations; this result has been extended by Boalch[3] to the irregular case.

Do these structures extend meromorphically to the singularities, e.g. do they give admissible groupoids? If this is the case, which seems to me likely, are there further reductions, or not?

(iv) One can consider other problems of D-envelopes that the problems arising from foliations. For instance

(a) Find the smallest D-groupoid whose given automorphisms of X are solutions. F.i., if f is a germ of automorphism of $(\mathbb{C}, 0)$, and if $f'(0)$ is not a root of unit, f can be embedded in a germ of non-trivial D-groupoid iff f is linearizable (see the forthcoming paper by G. Casale mentioned above). The case of roots of unit is more complicated; see loc. cit.

(b) Given a vector field ξ, find its D-envelope, e.g. the smallest D-groupoid whose D-Lie algebra contains ξ as a solution. The case of symplectic vector fields on a symplectic manifold is of special interest: the D envelope is contained obviously in the D-groupoid of symplectic transformations fixing ξ; but further reductions could occur.

It would be, f.i. quite interesting to look at the following case: ξ is the germ at 0, in $\mathbb{C}^4$ of the symplectic gradient of a Morse function (=case of 2 coupled oscillators).

In the same order of ideas, I mention the following result, which I owe to J. P. Ramis: Let X be a symplectic manifold of dimension $2n$, f a holomorphic function on X, and ξ its symplectic gradient. Suppose that ξ is Liouville-integrable, e.g. suppose that there exist $f = f_1, f_2, \cdots, f_n$, with $df_1, \cdots, df_n$ generically independent, such that the Poisson brackets $\{f_i, f_j\}$ vanish. Then the D-Lie algebra of the D-envelope of ξ is abelian.

Actually, near a point a where the df_i are independent, we can complete $f_1, \cdots, f_p$ with $p_1, \cdots, p_n$ to have a system of coordinates such that the symplectic form is $\sum df_i \wedge dp_i$. Then, one proves easily that a local symplectic vector field at a fixing $f_1, \cdots, f_n$ is the symplectic gradient of a function $\varphi(f_1, \cdots, f_n)$.

This fact is related to the methods used by Ziglin and Moralès-Ramis to prove the non-integrability of some hamiltonian systems. On this subject, a lot of very nice work has been made recently; the reader could consult the survey [1] by M. Audin.

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