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ITERATIVE ALGORITHMS FOR DATA ASSIMILATION PROBLEMS**

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(*Dedicated to the memory of Jacques-Louis Lions*)

Abstract

Iterative algorithms for solving the data assimilation problems are considered, based on the main and adjoint equations. Spectral properties of the control operators of the problem are studied, the iterative algorithms are justified.

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§0. Introduction

The investigation of global changes has increased the interest to the observation data assimilation and data processing problems, which are applied to the modeling, retrospective analysis, and forecasting various physical and geophysical processes. From the mathematical standpoint, these problems may be formulated as the optimal control problems. Starting with the studies of Bellman and Pontryagin, these problems attract the attention of many researchers. New essential ideas were contributed to the optimization theory and methods by French mathematical school. In this connection, we must mention the works by J.-L.Lions and his disciples, which became fundamental, dedicated to investigation of problems on insensitive optimal control, nonlinear sentinels for distributed systems. The general approach (Hilbert Uniqueness Method) developed by J.-L.Lions makes it possible to prove the existence of insensitive controls in linear and nonlinear systems.

In this study, we consider numerical algorithms for the data assimilation problems based on the iterative algorithms using the main and adjoint equations. Properties of the control operators studied are used to justify the iterative algorithms.

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§1. Statement of the Problem

Consider mathematical model of a physical process that is described by the evolution problem

$$\begin{cases} \frac{d\varphi}{dt} + A(t)\varphi = f, & t \in (0,T), \\ \varphi|_{t=0} = u, \end{cases} \tag{1.1}$$

where $\varphi = \varphi(t)$ is an unknown function, $A(t)$ is an operator (generally, nonlinear) acting for each t in the Hilbert space X with the definition domain $D(A) \subset X$, $u \in X$, and $f = f(t)$ is a prescribed function.

Introduce the functional

$$J(\varphi) = \frac{1}{2}\int_0^T (C(\varphi - \hat{\varphi}), \varphi - \hat{\varphi})_X dt + \frac{\alpha}{2}(\varphi|_{t=0} - \hat{\varphi}^\circ, \varphi|_{t=0} - \hat{\varphi}^\circ)_X, \tag{1.2}$$

where $\alpha = \text{const.} \geq 0$, C is a linear operator, and $(\cdot,\cdot)_X$ is an inner product in X. The function $\hat{\varphi} = \hat{\varphi}(t)$, as a rule, is determined by a priori observation data, $\hat{\varphi}^\circ \in X$. We assume hereinafter that all spaces and functions under consideration are real.

Consider problem (1.1) with an unknown function $u \in X$ in the initial condition. The data assimilation problem can be formulated as follows: find φ and u such that they satisfy (1.1) and, on the set of solutions to equation (1.1), functional (1.2) takes the minimum value. Write this problem as

$$\begin{cases} \dfrac{d\varphi}{dt} + A(t)\varphi = f, & t \in (0,T), \\ \varphi(0) = u, \\ J(\varphi) = \inf\limits_{\tilde{u} \in H} J(\tilde{\varphi}), \end{cases} \tag{1.3}$$

where $\tilde{\varphi}$ is a solution of (1.1) when $\tilde{\varphi}(0) = \tilde{u}$.

Problems in the form (1.3) were analyzed by Pontryagin[2], Lions[3] (see also [5–14], etc.). To solve (1.3) a number of approaches may be used (see e.g. [14]). We will consider iterative algorithms for solving (1.3), assuming for simplicity that $A(t)$ is a linear operator.

The necessary optimality condition[3] reduces problem (1.3) to the system for finding the functions φ, φ^*, u:

$$\frac{d\varphi}{dt} + A(t)\varphi = f, \quad t \in (0,T); \quad \varphi(0) = u, \tag{1.4}$$

$$-\frac{d\varphi^*}{dt} + A(t)^*\varphi^* = C(\hat{\varphi} - \varphi), \quad t \in (0,T); \quad \varphi^*(T) = 0, \tag{1.5}$$

$$\alpha(u - \hat{\varphi}^\circ) - \varphi^*(0) = 0, \tag{1.6}$$

where $A(t)^*$ is the operator adjoint to $A(t)$.

§2. Control Operator and Its Properties

Let $Y = L_2(0,T;X)$ be a space of abstract functions $u(t)$ with values in X, with the inner product and the norm

$$(u,v) = \int_0^T (u,v)_X dt, \quad \|u\| = \Big(\int_0^T \|u\|_X^2 dt\Big)^{1/2}, \quad u,v \in Y.$$

In the forthcoming, we suppose that the original model satisfies the following conditions:

(i) the solution to the problem

$$\begin{cases} \frac{d\psi}{dt} + A(t)\psi = f, \quad t \in (0,T), \\ \psi|_{t=0} = v, \end{cases} \tag{2.1}$$

meets the inequality

$$\|\psi\|_Y \le c_1(\|f\|_Y + \|v\|_X), \quad c_1 = \text{const.} > 0; \tag{2.2}$$

(ii) the solution of the adjoint problem

$$\begin{cases} -\frac{d\psi^*}{dt} + A^*(t)\psi^* = p, \quad t \in (0,T), \\ \psi^*\big|_{t=T} = 0, \end{cases} \tag{2.3}$$

satisfies

$$\|\psi^*\|_Y + \|\psi^*\big|_{t=0}\|_X \le c_1^*\|p\|_Y, \;\; c_1^* = \text{const.} > 0. \tag{2.4}$$

Remark 2.1. The solutions of the problems (2.1) and (2.3) are supposed to exist such that $\psi, \psi^* \in Y$, treated in a classical or a weak sense. The conditions (i), (ii) are satisfied if, for example, the operator $A(t)$ is positive definite:

$$(A(t)w, w)_Y \ge \gamma\|w\|_Y^2, \; \gamma = \text{const.} > 0, \; \forall w \in Y.$$

Indeed, from (2.1) we get

$$\Big(\frac{d\psi}{dt}, \psi\Big)_X + (A(t)\psi, \psi)_X = (f, \psi)_X,$$

whence

$$\frac{1}{2}\int_0^T \frac{d}{dt}(\psi,\psi)_X dt + \int_0^T (A(t)\psi,\psi)_X dt = \int_0^T (f,\psi)_X dt,$$

and by virtue of positive definiteness of $A(t)$,

$$\frac{1}{2}\|\psi\big|_{t=T}\|_X^2 + \gamma\|\psi\|_Y^2 \le (f,\psi)_Y + \frac{1}{2}\|\psi\big|_{t=0}\|_X^2 \le \|f\|_Y\|\psi\|_Y + \frac{1}{2}\|v\|_X^2,$$

or

$$\gamma\|\psi\|_Y^2 \le \frac{1}{2\gamma}\|f\|_Y^2 + \frac{\gamma}{2}\|\psi\|_Y^2 + \frac{1}{2}\|v\|_X^2.$$

The last inequality gives (2.2) with $c_1 = \max(\gamma^{-1}, \gamma^{-1/2})$. Similarly, the inequality (2.4) is obtained. In the finite-dimensional case, when $X = \mathbf{R}^n, n \in \mathbf{N}$, the inequalities (2.2), (2.4) are valid without positive definiteness requirement if the $n \times n$-matrix $A(t)$ is regular enough (for instance, having the elements continuous in t).

Let us introduce the operator $L: X \to X$ defined through the successive solutions of the following problems:

$$\begin{cases} \frac{d\psi}{dt} + A(t)\psi = 0, \quad t \in (0,T), \\ \psi|_{t=0} = v, \end{cases} \tag{2.5}$$

$$\begin{cases} -\frac{d\psi^*}{dt} + A^*(t)\psi^* = -C\psi, \quad t \in (0,T), \\ \psi^*\big|_{t=T} = 0, \end{cases} \tag{2.6}$$

$$Lv = \alpha v - \psi^*(0). \tag{2.7}$$

We define also $F \in X$ as the successive solutions of the following problems:

$$\begin{cases} \frac{d\phi}{dt} + A(t)\phi = f, \quad t \in (0,T), \\ \phi\big|_{t=0} = 0, \end{cases} \tag{2.8}$$

$$\begin{cases} -\frac{d\phi^*}{dt} + A^*(t)\phi^* = C(\hat{\varphi} - \phi), \quad t \in (0,T), \\ \phi^*\big|_{t=T} = 0, \end{cases} \tag{2.9}$$

$$F = \alpha\hat{\varphi}^\circ + \phi^*(0), \tag{2.10}$$

where $f, \hat{\varphi} \in Y$, $\hat{\varphi}^\circ \in X$ are introduced in (1.4)–(1.6). We suppose that $C: Y \to Y$ is a linear bounded self-adjoint positive semi-definite operator.

Then, the system (1.4)–(1.6) is reduced to the equation for the control u:

$$Lu = F, \tag{2.11}$$

and the operator $L: X \to X$ is called the control operator[13].

Under the hypotheses (i), (ii) the following statement is valid.

Lemma 2.1. *The operator L acts in X with domain of definition $D(L) = X$, it is bounded, self-adjoint, and positive semi-definite. If $\alpha > 0$, the operator L is positive definite.*

Proof. Let $v \in X$, and ψ be the solution to (2.5). By (2.2), $\|\psi\|_Y \le c_1\|v\|_X$. For the solution ψ^* of (2.6) the inequality (2.4) holds:

$$\|\psi^*\|_Y + \|\psi^*\big|_{t=0}\|_X \le c_1^*\|C\psi\|_Y.$$

Hence, from (2.7),

$$\begin{aligned} \|Lv\|_X = \|\alpha v - \psi^*\big|_{t=0}\|_X &\le \alpha\|v\|_X + \|\psi^*\big|_{t=0}\|_X \\ &\le \alpha\|v\|_X + c_1^*\|C\psi\|_Y \le \alpha\|v\|_X + c_1^* c_1\|C\|\|v\|_X, \end{aligned}$$

and, therefore, L is bounded. Further, we have for $v, w \in X$,

$$\begin{aligned} (Lv, w)_X &= (\alpha v - \psi^*\big|_{t=0}, w)_X = \alpha(v,w)_X - (\psi^*\big|_{t=0}, w)_X \\ &= \alpha(v,w)_X + (C\psi, \psi_1)_Y = \alpha(v,w)_X + (\psi, C\psi_1)_Y = (v, Lw)_X, \end{aligned}$$

where ψ_1 is the solution to (2.5) with $v = w$. Hence, L is self-adjoint, and

$$(Lv, v)_X = \alpha(v,v)_X + (C\psi, \psi)_Y \ge 0,$$

that is, L is positive semi-definite. Moreover, L is positive definite if $\alpha > 0$.

Corollary 2.1. *The following estimate is valid:*

$$(Lv, v)_X \ge \mu_{\min}(v,v)_X, \ \ \forall v \in X, \tag{2.12}$$

where $\mu_{\min}$ is the lower spectrum bound of the operator L, and $\mu_{\min} \ge \alpha$.

The following solvability result holds.

Lemma 2.2. *Under the hypotheses* (i), (ii)*, for $\alpha > 0$, the control equation* (2.11) *has a unique solution $u \in X$, and*

$$\|u\|_X \le \frac{\alpha}{\mu_{\min}}\|\hat{\varphi}^\circ\|_X + \frac{c_1^*}{\mu_{\min}}\|C\hat{\varphi}\|_Y + \frac{c_1 c_1^*}{\mu_{\min}}\|C\|\|f\|_Y. \tag{2.13}$$

Proof. If $\alpha > 0$, from Corollary 2.1, there exist a unique solution u of the control equation (2.11), and

$$\|u\|_X \le \frac{1}{\mu_{\min}}\|F\|_X. \tag{2.14}$$

The solution ϕ^* of (2.9) satisfies the inequality (2.4), and

$$\|F\|_X = \alpha\|\hat{\varphi}^\circ\|_X + \|\phi^*(0)\|_X \leq \alpha\|\hat{\varphi}^\circ\|_X + c_1^*\|C(\hat{\varphi} - \phi)\|_Y,$$

where ϕ is the solution to (2.8). Due to (2.2), $\|\phi\|_Y \leq c_1\|f\|_Y$, then

$$\|F\|_X \leq \alpha\|\hat{\varphi}^\circ\|_X + c_1^*\|C\hat{\varphi}\|_Y + c_1^* c_1\|C\|\|f\|_Y. \tag{2.15}$$

From (2.14)–(2.15) we obtain (2.13). This ends the proof.

Remark 2.2. For $\alpha = 0$, the last lemma holds true also if $\mu_{\min} > 0$. It is true, for instance, in the case that $X = \mathbf{R}^n, n \in \mathbf{N}$, $C = E$ (the identity operator). The weight coefficient α is usually called a regularization parameter[4].

§3. Spectrum Bounds of the Control Operator

In the general case, from Corollary 2.1, for the lower spectrum bound of the operator L we have $\mu_{\min} \geq \alpha$. Moreover, from the proof of Lemma 2.1, we get

$$(Lv, v)_X = \alpha(v, v)_X + (C\psi, \psi)_Y, \quad v \in X,$$

where ψ is the solution of (2.5). Hence, due to (2.2),

$$(Lv, v)_X \leq \alpha(v, v)_X + \|C\|\|\psi\|_Y^2 \leq \alpha\|v\|_X^2 + c_1\|C\|\|v\|_X^2,$$

and for the upper spectrum bound $\mu_{\max}$ of the operator L we get

$$\mu_{\max} \leq \alpha + c_1\|C\|. \tag{3.1}$$

In the case that $C = E$ (the identity operator), sharper estimates may be derived. The following result is valid.

Theorem 3.1. *The spectrum $\sigma(L)$ of the operator L defined by* (2.5)–(2.7) *for $C = E$ satisfies the estimates*

$$m \leq \sigma(L) \leq M, \tag{3.2}$$

where

$$m = \alpha + \int_0^T e^{-\int_0^t \lambda_{\max}(\tau)d\tau}dt, \quad M = \alpha + \int_0^T e^{-\int_0^t \lambda_{\min}(\tau)d\tau}dt,$$

and $\lambda_{\min}, \lambda_{\max}$ are the lower and the upper spectrum bounds of the operator $A + A^*$, respectively.

Proof. For the operator L defined by (2.5)–(2.7) for $C = E$ the following representation is valid:

$$(Lu, u) = \alpha(u, u) + \int_0^T (\varphi(t), \varphi(t))dt, \ u \in X, \tag{3.3}$$

where $\varphi(t)$ is the solution to (2.5) for $v = u$. From (2.5),

$$\frac{d}{dt}\|\varphi\|^2 + ((A + A^*)\varphi, \varphi) = 0,$$

then

$$-\lambda_{\max}(t)\|\varphi\|^2 \leq \frac{d}{dt}\|\varphi\|^2 \leq -\lambda_{\min}(t)\|\varphi\|^2,$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the lower and the upper spectrum bounds of the operator $A + A^*$, respectively. Therefore, the function $F(t) = \ln\|\varphi\|^2$ meets the inequality

$$-\lambda_{\max}(t) \leq \frac{dF}{dt} \leq \lambda_{\min}(t).$$

By integrating this inequality with respect to t from 0 to t, we get

$$-\int_0^t \lambda_{\max}(\tau)d\tau \le F(t) - F(0) \le -\int_0^t \lambda_{\min}(\tau)d\tau,$$

or

$$-\int_0^t \lambda_{\max}(\tau)d\tau \le \ln \frac{\|\varphi(t)\|^2}{\|u\|^2} \le -\int_0^t \lambda_{\min}(\tau)d\tau.$$

Hence

$$e^{-\int_0^t \lambda_{\max}(\tau)d\tau} \le \frac{\|\varphi(t)\|^2}{\|u\|^2} \le e^{-\int_0^t \lambda_{\min}(\tau)d\tau}.$$

Integrating the last inequality with respect to t from 0 to T and taking into account (3.3), we obtain

$$\int_0^T e^{-\int_0^t \lambda_{\max}(\tau)d\tau} dt \le \frac{(\bar{L}u, u)}{(u, u)} \le \int_0^T e^{-\int_0^t \lambda_{\min}(\tau)d\tau} dt,$$

where $\bar{L}$ is the operator L for $\alpha = 0$. Thus, the spectrum bounds of the operator L are defined by (3.2). This ends the proof.

If $A(t) = A : X \to X$ is a linear closed operator independent of time and being unbounded self-adjoint positive definite operator in X with the compact inverse, then the eigenvalues μ_k of the operator $\bar{L}$ are defined by the formula[13]

$$\mu_k = \frac{1 - e^{-2\lambda_k T}}{2\lambda_k},$$

where λ_k are the eigenvalues of the operator A. Then in (3.2) $\lambda_{\min} = 2\lambda_1,\ \lambda_{\max} = \infty$, and m, M are given in the explicit form

$$m = \alpha, \quad M = \alpha + \frac{1 - e^{-2\lambda_1 T}}{2\lambda_1}, \tag{3.4}$$

where λ_1 is the least eigenvalue of the operator A. By this is meant that the estimates (3.2) are exact.

§4. Iterative Algorithms

To solve (1.4)–(1.6) we consider a class of iterative algorithms:

$$\frac{d\varphi^k}{dt} + A(t)\varphi^k = f,\ t \in (0, T); \quad \varphi^k(0) = u^k, \tag{4.1}$$

$$-\frac{d\varphi^{*k}}{dt} + A^*(t)\varphi^{*k} = C(\hat{\varphi} - \varphi^k),\ t \in (0, T); \quad \varphi^{*k}(T) = 0, \tag{4.2}$$

$$u^{k+1} = u^k - \alpha_{k+1} B_k(\alpha(u^k - \hat{\varphi}^\circ)\varphi^{*k}\big|_{t=0}) + \beta_{k+1} C_k(u^k - u^{k-1}), \tag{4.3}$$

where $B_k, C_k : H \to H$ are some operators, and $\alpha_{k+1},\ \beta_{k+1}$ the iterative parameters. Let m and M be the spectral bounds of the control operator L defined by (3.2). We introduce the following notations:

$$\tau_{opt} = 2(M + m)^{-1}, \quad \theta = (M + m)(M - m)^{-1}, \tag{4.4}$$

$$\tau_k = 2(M + m - (M - m)\cos\omega_k\pi)^{-1}, \quad k = 1, 2, \ldots, s, \tag{4.5}$$

$$\alpha_{k+1} = \begin{cases} 2(M+m)^{-1}, & k=0, \\ 4(M-m)^{-1}\frac{T_k(\theta)}{T_{k+1}(\theta)}, & k>0, \end{cases}$$

$$\beta_{k+1} = \begin{cases} 0, & k=0, \\ \frac{T_{k-1}(\theta)}{T_{k+1}(\theta)}, & k>0, \end{cases} \tag{4.6}$$

$$e_k = \begin{cases} 0, & k=0, \\ p_k\|\xi^k\|_H^2/\|\xi^{k-1}\|_H^2, & k>0, \end{cases} \tag{4.7}$$

$$p_{k+1} = \alpha + (\eta^k, \eta^k)/\|\xi^k\|_H^2 - e_k, \ \ k = 0, 1, \cdots, \tag{4.8}$$

where $\omega_k = (2i-1)/2s$, T_k is the k-th degree Chebyshev polynomial of the first kind,

$$\xi^k = \alpha u^k - \varphi^{*k}(0),$$

and η^k is the solution of the problem

$$\frac{d\eta^k}{dt} + A\eta^k = 0, \ t \in (0,T); \quad \eta^k(0) = \xi^k.$$

Theorem 4.1. (i) *If* $\alpha_{k+1} = \tau, \ B_k = E, \ \beta_{k+1} = 0$, *then the condition* $0 < \tau < 2/(M+m)$ *is a sufficient condition for the convergence of the iterative process* (4.1)–(4.3). *For* $\tau = \tau_{opt}$ *defined by* (4.4) *the following convergence rate estimates are valid*:

$$\|\varphi - \varphi^k\|_W \le c_1 q_k, \ \ \|\varphi^* - \varphi^{*k}\|_W \le c_2 q_k, \ \ \|u - u^k\|_H \le c_3 q_k, \tag{4.9}$$

where $q_k = 1/\theta^k, \theta$ *is given by* (4.4), *and the constants* c_1, c_2, c_3, c_4 *do not depend on the number of iterations and on the functions* $\varphi, \varphi^k, \varphi^*, \varphi^{*k}, u, u^k, k > 0$.

(ii) *If* $B_k = E, \ \beta_{k+1} = 0$, *and* $\alpha_{k+1} = \tau_k$, *where the parameters* τ_k *are defined by* (4.5) *and repeated cyclically with the period* s, *then the error in the iterative process* (4.1)–(4.3) *is suppressed after each cycle of the length* s. *After* $k = ls$ *iterations the error estimates* (4.9) *are valid with* $q_k = (T_s(\theta))^{-l}$.

(iii) *If* $B_k = C_k = E$ *and* $\alpha_{k+1}, \beta_{k+1}$ *are defined by* (4.6), *then the error in the algorithm* (4.1)–(4.3) *is suppressed for each* $k \ge 1$, *and the estimates* (4.9) *hold for* $q_k = (T_k(\theta))^{-1}$.

(iv) *If* $B_k = C_k = E$ *and* $\alpha_{k+1} = 1/p_{k+1}, \ \beta_{k+1} = e_k/p_{k+1}$, *where* e_k, p_{k+1} *are defined by* (4.7), (4.8), *then the iterative process* (4.1)–(4.3) *is convergent, and the convergence rate estimates* (4.9) *are valid with* $q_k = (T_k(\theta))^{-1}$.

Proof. It is not difficult to show that the iterative process (4.1)–(4.3) is equivalent to the following iterative algorithm

$$u^{k+1} = u^k - \alpha_{k+1}B_k(Lu^k - F) + \beta_{k+1}C_k(u^k - u^{k-1}) \tag{4.10}$$

for solving the control equation (2.11) with the right-hand side F defined by (2.8)–(2.10).

The bounds m and M of the spectrum of the control operator L are given by (3.2). Thus, for $\alpha > 0$ for solving the equation $Lu = F$ we may use the well-known iterative algorithms with optimal choice of parameters. The theory of these methods is well developed[15]. Taking into account the explicit form of the bounds for m and M and applying for the equation $Lu = F$ the simple iterative method, the Chebyshev acceleration methods (s-cyclic and two-step ones), and the conjugate gradient method in the form (4.10), we arrive at the conclusions of the Theorem, using the well-known convergence results [15] for these methods.

Remark 4.1. In case $\alpha_k = 1/\alpha, \ B_k = E, \ \beta_k = 0$, the iterative algorithm (4.1)–(4.3) coincides with the Krylov-Chernousko method[16].

The numerical analysis of the above-formulated algorithms has been done in [17] for the data assimilation problem with a linear parabolic state equation. The numerical experiments are in good agreement with theoretical results on the convergence of the iterative algorithms.

References

[1] Bellman, R., Dynamic programming [M], Princeton Univ. Press, New Jersey, 1957.

[2] Pontryagin, L. S., Boltyanskii, V. G., Gamkrelidze, R. V. & Mischenko, E. F., The mathematical theory of optimal processes [M], John Wiley, New York, 1962.

[3] Lions, J. L., Contrôle Optimal des Systèmes Gouvernés par des Équations aux Dérivées Partielles [M], Dunod, Paris, 1968.

[4] Tikhonov, A. N., On the solution of ill-posed problems and the regularization method [J], *Dokl. Akad. Nauk SSSR*, **151:**3(1963), 501–504.

[5] Le Dimet, F. X. & Talagrand, O., Variational algorithms for analysis and assimilation of meteorological observations: theoretical aspects [J], *Tellus*, **38A**(1986), 97–110.

[6] Navon, I. M., A review of variational and optimization methods in meteorology [A], In: Variational methods in geosciences [C], (Ed. Sasaki Y.K.) – New York, Elsevier, 1986, 29–34.

[7] Glowinski, R. & Lions, J. L., Exact and approximate controllability for distributed parameter systems [J], *Acta Numerica*, **1**(1994), 269–378.

[8] Lions, J. L., On controllability of distributed system [J], *Proc. Natl. Acad. Sci. USA*, **94**(1997), 4828–4835.

[9] Agoshkov, V. I. & Marchuk, G. I., On solvability and numerical solution of data assimilation problems [J], *Russ. J. Numer. Anal. Math. Modelling*, **8**(1993), 1–16.

[10] Marchuk, G. I. & Zalesny, V. B., A numerical technique for geophysical data assimilation problem using Pontryagin's principle and splitting-up method [J], *Russ. J. Numer. Anal. Math. Modelling*, **8**(1993), 311–326.

[11] Marchuk, G. I. & Shutyaev, V. B., Iteration methods for solving a data assimilation problem [J], *Russ. J. Numer. Anal. Math. Modelling*, **9**(1994), 265–279.

[12] Marchuk, G. I., Agoshkov, V. I. & Shutyaev, V. P., Adjoint equations and perturbation algorothms in nonlinear problems [M], New York, CRC Press Inc., 1996.

[13] Shutyaev, V. P., Some properties of the control operator in the problem of data assimilation and iterative algorithms [J], *Russ. J. Numer. Anal. Math. Modelling*, **10:**4(1995), 357–371.

[14] Marchuk, G., Shutyaev, V. & Zalesny, V., Approaches to the solution of data assimilation problems [A], In: Optimal control and partial differential equations [M], Amsterdam: IOS Press, 2001, 489–497.

[15] Marchuk, G. I. & Lebedev, V. I., Numerical methods in the theory of neutron transport [M], Harwood Academic Publishers, New York, 1986.

[16] Krylov, I. A. & Chernousko, F. L., On a successive approximation method for solving optimal control problems [J], *Zh. Vychisl. Mat. Mat. Fiz.*, **2:**6(1962), 1132–1139.

[17] Parmuzin, E. I. & Shutyaev, V. P., Numerical analysis of iterative methods for solving evolution data assimilation problems [J], *Russ. J. Numer. Anal. Math. Modelling*, **14**(1999), 275–289.