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QUADRILATERAL MESH

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(*Dedicated to the memory of Jacques-Louis Lions*)

Abstract

Several quadrilateral shape regular mesh conditions commonly used in the finite element method are proven to be equivalent. Their influence on the finite element interpolation error and the consistency error committed by nonconforming finite elements are investigated. The effect of the Bi-Section Condition and its extended version $(1+\alpha)$- Section Condition on the degenerate mesh conditions is also checked. The necessity of the Bi-Section Condition in finite elements is underpinned by means of counterexamples.

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§1. Introduction

Quadrilateral mesh is widely used in the finite element method due to its simplicity and flexibility. However, numerical accuracy cannot be achieved over an arbitrary mesh, so one has to impose certain mesh conditions. There exist several mesh conditions in the literature, which can be classified into two groups. One is the shape regular mesh condition and the other is the degenerate condition. Roughly speaking, a shape regular condition requires that the element cannot be too narrow on the one hand ($(\text{C-R})_1$ hereinafter) and the interior angle of each vertex is neither too small nor too close to π on the other hand ($(\text{C-R})_2$ hereinafter). The first condition of this type belongs to Ciarlet-Raviart (C-R)[27,28]. Another two are attributed to Girault-Raviart (G-R)[33] and Arunakirinathar-Reddy (A-R)[11], all these three conditions aim for the optimal interpolation error for the isoparametric element, whereas a similar condition of Z. Zhang (Z)[68] appeared in the study of the Wilson nonconforming element. It may be interesting to ask whether such conditions are equivalent. One of the main results of this paper is a strict proof for the equivalence of C-R, G-R, A-R and Z (see Theorem 3.1 below).

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Meanwhile, there are several degenerate mesh conditions, which violate either $(\text{C-R})_1$ or $(\text{C-R})_2$, and sometimes even both of them. Such degenerate meshes are particularly effective for the finite element approximation of some physical problems with singularities (see [5]). These conditions are scattered in the literature, most of them are mutatis mutandis. We only consider two degenerate conditions, namely the Jamet condition (J)[34] and Acosta-Durans[1] Regular Decomposition Property (RDP). We will clarify their connection to the shape regular mesh condition. In particular, we show by means of a counterexample that RDP is necessary for obtaining the optimal interpolation error in the H^1-norm for the 4-node isoparametric element, thereby we solve the open problem proposed in [1].

On the other hand, the Bi-Section Condition or its extended version $(1+\alpha)$-Section Condition are two mesh conditions which quantify the deviation of an arbitrary quadrilateral from a parallelogram. These two mesh conditions are used to estimate the consistency error of the Wilson nonconforming element[54] as well as the interpolation and consistency error of the nonconforming quadrilateral rotated Q_1 element (NRQ_1) (see [40,42,46]), and the interpolation error of the lowest-order Raviart-Thomas element ($\text{RT}_{[0]}$) (see [61,43]). Süli[58] proved that the Jamet degenerate mesh condition plus the Bi-Section Condition actually imply the shape regular conditions when the mesh diameter approaches zero. We show that the requirement of the Bi-Section Condition in Süli's result can be replaced by the even weaker $(1+\alpha)$-Section Condition.

The necessity of the Bi-Section Condition for the optimal consistency error of the Wilson element is illustrated by Z. Shi[54] with a counterexample. Likewise, P. Ming[40] showed that this is also true for the NRQ_1^p (see §6 for the definition). We will propose a series of counterexamples to demonstrate that the Bi-Section Condition is also necessary for the optimal interpolation error of NRQ_1 and $\text{RT}_{[0]}$. All these facts substantiate the necessity of the Bi-Section Condition in the finite element analysis.

We mainly focus on the 4-node isoparametric element, two low-order quadrilateral nonconforming elements, i.e., the Wilson and NRQ_1 elements, and the quadrilateral $\text{RT}_{[0]}$ element. Moreover, only 2-D mesh is taken into account (see [67], for some other isoparametric elements and [31,53] for 3-D). We only consider the finite element approximation for the coercive elliptic problem, the non-coercive problem is more involved and will be addressed elsewhere (see also [1,10] for related references.)

The remaining part of this paper is organized as follows. In §2, we state all shape regular mesh conditions mentioned above. Their equivalence is proven in §3. In §4, several degenerate mesh conditions are reviewed and their connections to the shape regular mesh condition are elucidated with the aid of the Bi-Section Condition and $(1+\alpha)$-Section Condition. The influence of different mesh conditions on the interpolation error and the finite element error for the 4-node isoparametric element, the nonconforming Wilson element and NRQ_1 element, and the $\text{RT}_{[0]}$ element are included in §5, §6 and §7, respectively. Some conclusions are drawn and three open problems are proposed in the last section.

Throughout this paper, the generic constant C is independent of the element geometry unless otherwise stated.

§2. Shape Regular Mesh Condition

Before introducing mesh conditions, we fix some notations. For any convex polygon Ω with Lipschitz boundary, $H^k(\Omega)$ is defined as the standard Sobolev space[2] equipped with the norm $\|\cdot\|_k$, and semi-norm $|\cdot|_{k,\Omega}$, $H_0^k(\Omega)$ is the corresponding homogeneous space. Ω will be dropped if no confusion can occur. $⨍_\Omega u\,dx$ is defined as the integral average of u on

Ω. For any vectors $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$, $\boldsymbol{x} \otimes \boldsymbol{y}$ is a 2×2 matrix with elements $(\boldsymbol{x} \otimes \boldsymbol{y})_{ij} := x_i y_j$. For any matrix A, $\|A\|$ denotes its Euclidean norm.

2.1. Geometric Facts of Quadrilateral Mesh

Let $\mathcal{T}_h$ be a partition of $\overline{\Omega}$ by convex quadrilaterals with the mesh size $h := \max_{K \in \mathcal{T}_h} h_K$. We define h_K and $\underline{h}_K$ as the longest and shortest edges of K, respectively. ρ_K is defined as the diameter of the largest circle inscribed in K. As in Fig.1, we denote the four vertices of K by P_i with the coordinates $\boldsymbol{x}_i$. Let their edges be P_iP_{i+1} and $|P_iP_{i+1}|$ their corresponding lengths. The subtriangle of K with vertices P_{i-1}, P_i and P_{i+1} is denoted by $\mathcal{T}_i$ (i modulo 4), i.e., $\mathcal{T}_i := \langle P_{i-1}, P_i, P_{i+1} \rangle$. Similar to K, h_i and ρ_i are defined as the longest edges of $\mathcal{T}_i$ and the diameter for the largest circle inscribed in $\mathcal{T}_i$, respectively. Denote the interior angle of the vertex P_i by θ_i, and $\mu_K := \max_{1 \le i \le 4} |\cos\theta_i|$. Moreover, d_K is denoted as the distance between midpoints of two diagonals of K.

We define by $\mathcal{P}_k$ the space of polynomials of degree no more than k, and by $\mathcal{Q}_k$ the space of degree no more than k in each variable.

Let $\hat{K} = [-1, 1]^2$ be the reference square having the vertex $\hat{P}_i$ with the coordinates $\hat{\boldsymbol{x}}_i (1 \le i \le 4)$, then there exists a unique mapping $\mathcal{F}_K(\xi, \eta) \in \mathcal{Q}_1(\hat{K})$ such that $\mathcal{F}_K(\hat{\boldsymbol{x}}_i) = \boldsymbol{x}_i$, $1 \le i \le 4$. The Jocobian of $\mathcal{F}_K(\xi, \eta)$ is denoted by $D\mathcal{F}_K(\xi, \eta)$ which can be split as $D\mathcal{F}_K(\xi, \eta) = D\mathcal{F}_K(0, 0) + \boldsymbol{k} \otimes \boldsymbol{l}$, where $\boldsymbol{k} := (\boldsymbol{x}_1 - \boldsymbol{x}_2 + \boldsymbol{x}_3 - \boldsymbol{x}_4)^T/4$ and $\boldsymbol{l} := (\eta, \xi)^T$. The determinant of $D\mathcal{F}_K(\xi, \eta)$ is $J_K(\xi, \eta) = \det D\mathcal{F}_K(\xi, \eta) = J_0 + J_1\xi + J_2\eta$. It can be shown that

$$\max_{(\xi,\eta) \in \hat{K}} J_K(\xi, \eta) = \max_{1 \le i \le 4} |\mathcal{T}_i|/2, \qquad \min_{(\xi,\eta) \in \hat{K}} J_K(\xi, \eta) = \min_{1 \le i \le 4} |\mathcal{T}_i|/2$$

and

$$J_0 = J_K(0, 0) = |K|/4, \quad J_K(\xi, \eta) > 0.$$

Fig.1

2.2. Shape Regular Mesh Conditions

We mainly concern the following shape regular mesh conditions.

(1) Ciarlet-Raviart (C-R) (see [27,28]).

$\mathcal{T}_h$ is regular[26, p. 247], if there exist two constants $\sigma_0 \ge 1$ and $0 < \mu < 1$ such that

$$h_K/\underline{h}_K \le \sigma_0, \qquad 0 < \mu_K \le \mu < 1. \tag{2.1}$$

(2) Girault-Raviart (G-R) (see [17,33]).

Define $\overline{\rho}_K := 2 \min_{1 \le i \le 4} \rho_i$. $\mathcal{T}_h$ is regular if there exists $\sigma > 0$ such that

$$\max_{K \in \mathcal{T}_h} h_K/\overline{\rho}_K \le \sigma. \tag{2.2}$$

(3) Arunakirinathar-Reddy (A-R) (see [11]).

$\mathcal{T}_h$ is regular if there exist four constants C, C_0, C_1 and $\gamma > -1$ such that

$$\|D\mathcal{F}_K(0,0)\| \le Ch_K, \quad \|D\mathcal{F}_K^{-1}(0,0)\| \le Ch_K^{-1}, \quad C_0 h_K^2 \le J_K(0,0) \le C_1 h_K^2. \tag{2.3}$$

$$(D\mathcal{F}_K^{-1}(0,0)\boldsymbol{k}, \boldsymbol{l}) \ge \gamma. \tag{2.4}$$

(4) Z. Zhang (Z) (see [68]).

Define $\tilde{\rho}_K\colon = \min_{1\le i\le 4} \rho_i$. $\mathcal{T}_h$ is regular if there exists $\sigma > 0$ such that

$$\max_{K\in\mathcal{T}_h} h_K/\tilde{\rho}_K \le \sigma. \tag{2.5}$$

Remark 2.1. A simple manipulation yields that C_1 in (2.3) can equal 1/8, which is even sharp.

§3. Equivalence of Shape Regular Conditions

In this section, we will prove that all the aforementioned shape regular conditions are equivalent. Before the presentation, we state two lemmas for later use.

Lemma 3.1.[70] *For a family of triangular finite elements, if the ratio between the longest edge of the triangle and the radius of the biggest circle inscribed into the triangle is uniformly bounded, then there exists a constant θ_0 such that all interior angle θ_K of triangles satisfies*

$$\theta_K \ge \theta_0 > 0. \tag{3.1}$$

Lemma 3.2. *For any triangle K, the diameter of the biggest circle inscribed into K is less than the shortest edge of K.*

Theorem 3.1. *All the forgoing mentioned shape regular mesh conditions* (C-R, G-R, A-R *and* Z) *are equivalent.*

Proof. We only need to prove that all shape regular mesh conditions are equivalent to C-R. The equivalence between G-R and Z is obvious. So what we need to prove is the equivalence of C-R, G-R and A-R.

Firstly we prove that G-R implies C-R.

Given the regular condition G-R, in view of Lemma 3.2, we have

$$h_K/\underline{h}_K \le 2h_K/\overline{\rho}_K \le 2\sigma.$$

which gives (C-R)$_1$ with $\sigma_0 = 2\sigma$.

Notice that $h_i/rho_i \le 2h_K/\overline{\rho}_K \le 2\sigma$, invoking Lemma 3.1, we see that each interior angle of K is bounded below by θ_0, which in turn implies an upper bound for each angle with $\pi - 2\theta_0$, so $\theta_0 \le \theta_i \le \pi - 2\theta_0$, thus we come to (C-R)$_2$ with $\mu = \max(\cos\theta_0, |\cos 2\theta_0|)$. Therefore, C-R follows from G-R.

We are in a position to prove that C-R implies G-R. For any triangle $\mathcal{T}_i$, we have

$$\rho_i = \frac{2|\mathcal{T}_i|}{|P_{i-1}P_i| + |P_iP_{i+1}| + |P_{i+1}P_{i-1}|} \ge \frac{\underline{h}_K^2 \sin\theta_i}{3h_i} \ge \frac{(1-\mu^2)^{1/2}}{3{\sigma_0}^2} h_i,$$

where $|\mathcal{T}_i|$ is the area of the triangle $\mathcal{T}_i$ and $\sin\theta_i \ge (1-\mu^2)^{1/2}$ since $|\cos\theta_i| \le \mu$. The above inequality immediately leads to

$$h_i/\rho_i \le 3\sigma_0^2/(1-\mu^2)^{1/2}. \tag{3.3}$$

Let $\overline{\rho}_K = 2\rho_{i_0}$. Using (3.3), we get

$$h_K/\overline{\rho}_K \le (h_K/h_{i_0})(h_{i_0}/2\rho_{i_0}) \le \sigma_0 \frac{3\sigma_0^2/2}{(1-\mu^2)^{1/2}} = :\sigma,$$

that is just the G-R condition.

We remain to prove that C-R and A-R are equivalent. Firstly we show that C-R implies A-R. The first part of A-R is easily deduced from C-R, we omit details for simplicity. Further, a simple manipulation shows $(D\mathcal{F}_K(0,0)^{-1}\boldsymbol{k},\boldsymbol{l}) = (J_K - J_0)/J_0$, then

$$J_K/J_0 \geq \frac{\min_{(\xi,\eta)\in\hat{K}} J_K(\xi,\eta)}{|K|/4} = 2\min_{1\leq i\leq 4}|\mathcal{T}_i|/|K|. \tag{3.4}$$

It is seen that

$$|\mathcal{T}_i| \geq 1/2\underline{h}_K^2(1-\mu^2)^{1/2} \quad \forall 1\leq i\leq 4, \qquad |K|\leq h_K^2.$$

Inserting the above two inequalities into (3.4) leads to

$$J_K/J_0 \geq (1-\mu^2)^{1/2}/\sigma_0^2.$$

Let $\gamma\colon = (1-\mu^2)^{1/2}/\sigma_0^2 - 1 > -1$, then we obtain the A-R condition.

To deduce C-R from A-R, without loss of generality, let $\mathcal{T}_1$ include the shortest edge $\underline{h}_K$. The second part of A-R implies

$$2|\mathcal{T}_1|/|K| \geq \min_{(\xi,\eta)\in\hat{K}} J_K(\xi,\eta)/J_0 \geq 1+\gamma. \tag{3.5}$$

Using (3.2) and noticing that $|\mathcal{T}_1| \leq 1/2h_K\underline{h}_K$, we obtain (C-R)$_1$ with $\sigma_0 = (4C_0(1+\gamma))^{-1}$. Since the area of any triangle $\mathcal{T}_i$ can be expressed as $|\mathcal{T}_i| \leq 1/2h_K^2\sin\theta_i$, so repeating the above procedure using $|\mathcal{T}_i| \leq 1/2h_K^2\sin\theta_i$ instead of $|\mathcal{T}_i| \leq 1/2h_K\underline{h}_K$, we get $\sin\theta_i \geq 4C_0(1+\gamma)$, which implies (C-R)$_2$ with $\mu = (1-16C_0^2(1+\gamma)^2)^{1/2}$. We complete the proof.

§4. Degenerate Mesh Condition

In what follows, we discuss some degenerate mesh conditions. As a preparation, we introduce the $(1+\alpha)$-Section Condition.

Definition 4.1. $(1+\alpha)$-*Section Condition* $(0\leq\alpha\leq 1)$

$$d_K = \mathcal{O}(h_K^{1+\alpha}),$$

uniformly for all elements K as $h\to 0$.

If $d_K = 0$, K degenerates into a parallelogram. In case $\alpha > 0$, we recover the Condition A in [54]. In case α equals to 1, we obtain Condition B, or the Bi-Section Condition[54].

Angle condition is another kind of degenerate mesh condition, which is introduced in [50] and used to measure the deviation of a quadrilateral from a parallelogram. Define σ_K as

$$\sigma_K\colon = \max(\,|\pi-\alpha_1|, |\pi-\alpha_2|\,).$$

Here α_1 is the angle between the outward normal of two opposite sides of K and α_2 is the angle between the outward normal of other two sides. We call a mesh satisfying the Angle Condition if $\sigma_K = \mathcal{O}(h_K)$, i.e., if σ_K/h_K is uniformly bounded for all elements. It is seen that $0\leq\sigma_K<\pi$, and $\sigma_K = 0$ iff K is a parallelogram.

Assuming that C-R holds, H. S. Chow et al (see [24, Theorem 3.2]) proved that the Angle Condition and the Bi-Section Condition are equivalent.

Remark 4.1. Observe that the h^2-parallelogram mesh condition in [32] is actually equivalent to the Bi-Section Condition.

Motivated by the Bi-Section Condition, we define a kind of mesh which will be shown to be quite useful for the convergence analysis of finite elements.

Definition 4.2. *We call $\mathcal{T}_h$ an asymptotically regular parallelogram mesh if it satisfies C-R as well as the Bi-Section Condition.*

Notice that any polygon can be meshed by asymptotically regular parallelograms with a mesh size tending to zero. Indeed, if we begin with any mesh of convex quadrilaterals, and refine it by dividing each quadrilateral into four by connecting two midpoints of opposite edges. As in Fig.2, the resulting mesh is an asymptotically regular parallelogram mesh.

Fig.2

There is also another kind of method for generating such an asymptotically regular parallelogram mesh. For example, the $\mathcal{C}^2$-grid in [8] which results from a mapping of uniform grids is actually an asymptotically regular parallelogram mesh. This approach has already appeared in [71] (see also [25, Remark 2.3]).

As to the generation of a shape regular mesh satisfying the $(1+\alpha)$-Section Condition, we refer to [23] for the work of Whiteman's school.

Degenerate Mesh Conditions

(1) Jamet condition (J) (see [34,36]).

$\mathcal{T}_h$ is regular if there exists a constant $\sigma > 0$ such that

$$h_K/\rho_K \le \sigma.$$

(2) Acosta-Duran Regular Decomposition Property (RDP) (see [1]).

$\mathcal{T}_h$ is regular with constant $N \in \mathbb{R}$ and $0 < \psi < \pi$, or shortly RDP(N, ψ), if we can divide K into two triangles along one of its diagonals, which will always be called d_1, the other is d_2 in such a way that $|d_2|/|d_1| \le N$ and both triangles satisfy the maximum angle condition, i.e., each interior angle of these two triangles is bounded from above by ψ.

(3) Süli condition (S) (see [58]).

$\mathcal{T}_h$ is regular if it satisfies the J condition and the Bi-Section Condition simultaneously.

Remark 4.2. Notice that Süli's condition was firstly appeared in the convergence proof of a kind of cell vertex finite volume method for hyperbolic problems and has recently got renewed interest in the mixed finite volume method[25].

As addressed in [36], C-R implies the J condition with $\sigma = (1-\mu^2)^{1/2}/\sigma_0$. On the contrary, the J condition allows for the degeneration of a quadrilateral K into a triangle since the ratio $h/\underline{h}$ can be arbitrarily large and the largest angle of K can equal π, i.e., the J condition may violate either $(\text{C-R})_1$ or $(\text{C-R})_2$. This is shown in Fig.3 below.

Fig.3

Both elements in Fig.3 satisfy the J condition since $h_K/\rho_K \le \sqrt{2}+1$. However, as to the left element, the ratio $h_K/\underline{h}_K = (a^2+(a-b)^2)^{1/2}/b \le \sqrt{2}a/b$ blows up as b tends to zero, which obviously violates $(\text{C-R})_1$. As to the right element, the interior angle $\angle ADC = \pi - x$, which approaches π as x tends to zero, thereby it violates $(\text{C-R})_2$. However, not the whole $(\text{C-R})_2$ is violated since the J condition excludes the interior angle from becoming too small.

This fact is hidden in [58, Lemma 1] which is stated as follows.

Lemma 4.1. *If $\mathcal{T}_h$ satisfies Süli's condition, then for sufficiently small h, $\mathcal{T}_h$ is shape regular in the sense of C-R.*

Proceeding along the same line of the above lemma, we obtain

Corollary 4.1. *If $\mathcal{T}_h$ satisfies the J condition as well as the $(1+\alpha)$-Section Condition, then for sufficiently small h, $\mathcal{T}_h$ is shape regular in the sense of C-R.*

Notice that C-R does not imply the S condition. Indeed, considering the trapezoid mesh as that in Fig. 6, it is seen that the S condition is stronger than C-R. In fact, it is a strong shape regular mesh condition instead of a degenerate one. Invoking [1, Remark 2.7], RDP(N,ψ) is weaker than both the J condition and C-R, therefore it is weaker than the S condition. Moreover, RDP(N,ψ) together with the Bi-Section Condition does not imply C-R. It is due to the fact that a rectangular element which satisfies both RDP$(1,\pi/2)$ and the Bi-Section Condition simultaneously may still have its anisotropic aspect ratio arbitrarily large.

Remark 4.3. Schmidt[51] replaced the Bi-Section Condition in Lemma 4.1 by the arbitrary smallness of the ratio d_K/h_K.

Remark 4.4. Zlámal[71] proposed the k-strongly regular mesh condition for investigating the superconvergence of isoparametric elements. Following the analysis of Theorem 3.1, we find that Zlámal's 1-strongly regular condition is actually equivalent to the S condition. Notice that the 1-strongly regular condition has already appeared in [29] but in a slightly different form.

Remark 4.5. Braess[18,p.99, Remark 2] proposed a new shape regular mesh condition which consists of (C-R)$_1$, the J condition and the maximum interior angle condition. Obviously, it is equivalent to C-R. However, due to its superfluous complexity this new condition is not advisable.

§5. 4-Node Isoparametric Element

Denoting by $\mathcal{Q}$ the standard Lagrangian interpolant for the 4-node isoparametric element, we look for a geometric condition under which the estimate

$$\|u-\mathcal{Q}u\|_{0,K}+h_K|u-\mathcal{Q}u|_{1,K}\le Ch_K^2|u|_{2,K}$$

holds uniformly for $K\in\mathcal{T}_h$.

In view of [1,Theorem 4.7], the optimal interpolation error estimate with respect to the L^2 norm holds with a constant independent of the geometry of K, so we only consider

$$|u-\mathcal{Q}u|_{1,K}\le Ch_K|u|_{2,K}. \tag{5.1}$$

Ciarlet and Raviart[27] proved (5.1) under C-R. However, C-R prohibits the quadrilateral from either reducing to a triangle or becoming too flat. Jamet[35,36] derived (5.1) under the J condition which allows a quadrilateral degenerating to a triangle, but not too flat. Ženisek and Vanmaele[66] also proved (5.1). They required that the two longest sides of the element be opposite and almost parallel, but the constant C in (5.1) depends on an angle, which somehow is the minimum angle of the element K. Apel[4] derived the following estimate

$$|u-\mathcal{Q}u|_{1,K}\le C\Big(h_1\Big\|\frac{\partial}{\partial x_1}\nabla u\Big\|_{0,K}+h_2\Big\|\frac{\partial}{\partial x_2}\nabla u\Big\|_{0,K}\Big). \tag{5.2}$$

Here h_1 and h_2 are the element sizes in the direction of x_1 and x_2, respectively. Acosta and Durán[1] derived (5.1) under RDP(N,ψ), which seems the weakest mesh condition up to now under which (5.1.) is valid (see [1, Remark 2.4–Remark 2.7]). One may ask whether

$\mathrm{RDP}(N,\psi)$ is also necessary. Acosta and Durán put it as an open problem in [1]. The following example shows that this condition is indeed necessary.

Counterexample

Fig.4

Consider an element K like Fig.4 which does not satisfy $\mathrm{RDP}(N,\psi)$. If we decompose it by the diagonal AC, then the triangles $\langle A,B,C\rangle$ and $\langle A,C,D\rangle$ indeed satisfy the maximal angle condition since all interior angles in these two triangles are bounded from above by $\pi/2$. However, $|BD|/|AC| = 1/a$ which cannot be bounded by any constant as a tends to zero. If we decompose it by the diagonal BD, a simple computation leads to $\sin\angle DAB = 2a/(1{+}a^2)$, so the angle $\angle DAB$ approaches π as a tends to zero, thus it also violates $\mathrm{RDP}(N,\psi)$.

Let $u(x,y) = x^2$, then $|u|^2_{2,K} = 8a$. A direct manipulation shows that $\left\|\frac{\partial(u-\mathcal{Q}u)}{\partial y}\right\|^2_{0,K} = (3a)^{-1}$, so we have

$$|u-\mathcal{Q}u|^2_{1,K}/|u|^2_{2,K} \geq \left\|\frac{\partial(u-\mathcal{Q}u)}{\partial y}\right\|^2_{0,K}\Big/|u|^2_{2,K} = (1/24)1/a^2.$$

Since the diameter of K is 2, (5.1) does not hold with a constant independent of a.

To sum up, we have the following interpolation result for the 4-node isoparametric element.

Theorem 5.1. *For any* $u \in H^2(\Omega)$, *if* $\mathrm{RDP}(N,\psi)$ *holds, then there exists a constant* $C = C(N,\psi)$ *such that*

$$|u-\mathcal{Q}u|_{1,K} \leq Ch_K|u|_{2,K}. \tag{5.3}$$

Moreover, $\mathrm{RDP}(N,\psi)$ *is also necessary for the validity of* (5.3).

By Céa Lemma, the error bound of the 4-node isoparametric element solution is bounded by its interpolation error, i.e.,

$$\|u-u_h\|_1 \leq C\inf_{v\in V_h}\|u-v\|_1. \tag{5.4}$$

Here

$$V_h := \{v \in H^1_0 \mid v_{|K} \in \mathcal{Q}_1(K), \quad \forall K \in \mathcal{T}_h\}.$$

However, the experience with triangle elements indicates that the interpolation error actually says nothing about the approximation error of the finite element. When triangles with uncontrolled maximal angles are taken into account, examples in [12] show that the approximation error grows to infinity as the interpolation error with respect to the H^1-norm grows to infinity. However, in [6], another example shows that the finite element solution converges while the interpolation error with respect to the H^1-norm also grows to infinity. So it is equally interesting to ask such question for the quadrilateral element approximation.

§6. Nonconforming Quadrilateral Element

As to the nonconforming quadrilateral element approximation, the situation is less satisfactory. The main nonconforming quadrilateral elements are the Wilson element and nonconforming rotated $\mathcal{Q}_1$ element. The former is well-known in the engineering community and has a long history (see [65]). The latter is the simplest nonconforming quadrilateral element, and was proposed for solving the incompressible flow problem in [50] and widely used in the software FEATFLOW[62]. It has also been applied to the crystalline microstructure problem[37], the Chappman-Ferraro problem[38] and the Reissner-Mindlin plate bending problem[41,45].

By Strang Lemma[57], it is common to split the finite element error into two parts, i.e., the interpolation error and the consistency error, thus the impact of the mesh conditions on both errors has to be checked.

For any v belonging to a nonconforming finite element space, we define the discrete H^1-norm as

$$\|v\|_{1,h} := \Big(\sum_{K\in\mathcal{T}_h} \|v\|^2_{1,K}\Big)^{1/2}.$$

The interpolation error of numerous nonconforming elements can be checked case by case. As to the consistency error, it is common to estimate the following piecewise integral functional

$$\sum_{K\in\mathcal{T}_h}\int_{\partial K} \boldsymbol{\psi}\cdot\boldsymbol{n} v\, ds.$$

Usually the nonconforming element has some sort of continuities, so to bound the above piecewise integral functional is boiled down to the estimate of the following integral functional, i.e.,

$$\|u-\mathcal{J}(u)\|_{0,\mathcal{E}} \quad \forall\mathcal{E}\subset\partial K.$$

Here $\mathcal{J}(u)$ usually takes two forms as

$$\mathcal{J}(u)=\mathcal{J}^a_{\mathcal{E}}(u) := \fint_{\mathcal{E}} u\, ds \quad \text{or} \quad \mathcal{J}(u)=\mathcal{J}_K(u) := \fint_K u\, dx, \qquad u\in H^1(K)$$

and $\mathcal{J}(u)=\mathcal{J}^p_{\mathcal{E}}(u) := u(\mathcal{M})$ in case of $u\in H^1(K)\cap\mathcal{C}^0(\overline{K})$, where $\mathcal{M}$ is the middle point of the edge $\mathcal{E}$. The following lemma bounds the above functional.

Lemma 6.1. *For $u\in H^1(K)$ or $u\in H^1(K)\cap\mathcal{C}^0(\overline{K})$, and for any $\mathcal{E}\subset\partial K$, we have*

$$\|u-\mathcal{J}(u)\|_{0,\mathcal{E}} \le C(|\mathcal{E}|/|K|)^{1/2} h_K \|\nabla u\|_{0,K}. \tag{6.1}$$

Proof. In case of $u\in H^1(K)$, since $\|u-\fint_{\mathcal{E}} u\, ds\|_{0,\mathcal{E}} = \inf_{C\in\mathbb{R}}\|u-C\|_{0,\mathcal{E}}$, we have

$$\|u-\mathcal{J}^a_{\mathcal{E}}(u)\|_{0,\mathcal{E}} \le \|u-\mathcal{J}_K(u)\|_{0,\mathcal{E}}.$$

By a sharp trace inequality in [63, Lemma 3.2], the right hand side of the above inequality is bounded by

$$(2|\mathcal{E}|/|K|)^{1/2}(\|u-\mathcal{J}_K(u)\|_{0,K} + h_K\|\nabla u\|_{0,K}).$$

Since K is convex, by the Poincaré Inequality[48] we have the first term as

$$\|u-\mathcal{J}_K(u)\|_{0,K} \le h_K/\pi \|\nabla u\|_{0,K}. \tag{6.2}$$

A combination of the above three inequalities yields (6.1).

When $u\in H^1(K)\cap\mathcal{C}^0(\overline{K})$, proceeding along the same line of the above procedure and employing the scaling trick instead of the Poincaré Inequality on the last step complete the proof.

To be more specific, we further bound the expression $C(|\mathcal{E}|/|K|)^{1/2}h_K$ appeared on the right hand side of (6.1) as $M\sigma^{1/2}h_K^{1/2}$ provided that the J condition holds, where M is independent of h.

The following simple example shows that the dependence on σ is essential.

Counterexample

Consider an element K centered at the origin with the lengths $2h_x$ and $2h_y$ in the x and y directions, respectively. Without loss of generality, we assume that $h_x < h_y$, and define $\sigma := (h_x^2 + h_y^2)^{1/2}/\rho_K$. Let $u(x,y) = x^2 + y^2$ and $\mathcal{E}$ be one vertical edge. A simple computation leads to

$$\|u - \mathcal{J}_{\mathcal{E}}^a(u)\|_{0,\mathcal{E}}^2 = 8h_y^5/45, \quad \|\nabla u\|_{0,K}^2 = 16h_xh_y(h_x^2 + h_y^2)/3.$$

A combination of these two identities leads to

$$\|u - \mathcal{J}_{\mathcal{E}}^a(u)\|_{0,\mathcal{E}}/\|\nabla u\|_{0,K} \ge \sigma^{1/2}(h_x^2 + h_y^2)^{1/4}/2\sqrt{30}.$$

On the other hand, (6.1) gives the upper bound as

$$\|u - \mathcal{J}_{\mathcal{E}}^a(u)\|_{0,\mathcal{E}}/\|\nabla u\|_{0,K} \le C\sigma^{1/2}(h_x^2 + h_y^2)^{1/4}.$$

The above two inequalities illustrate the sharpness of the element geometry dependence of the constant in the right hand side of (6.1). Moreover, this example also works for the other two cases when $\mathcal{J}(u)$ is $\mathcal{J}_K(u)$ or $\mathcal{J}_{\mathcal{E}}^p(u)$.

6.1. Wilson Element

The Wilson nonconforming finite element space[65] is

$$\{\, v \in L^2(\Omega) \mid v \circ \mathcal{F}_K \in \mathcal{P}(\hat{K}) \quad \forall K \in \mathcal{T}_h \,\},$$

where $\mathcal{Q}_1(\hat{K}) \subset \mathcal{P}(\hat{K}) \subset \mathcal{P}_2(\hat{K})$. We write $\mathcal{P}(\hat{K}) = \mathcal{Q}_1(\hat{K}) + \mathcal{B}(\hat{K})$, where $\mathcal{B}(\hat{K})$ contains the nonconforming part:

$$\mathcal{B}(\hat{K}) = \mathrm{Span}(\xi^2 - 1, \eta^2 - 1).$$

There are also another two types of Wilson-like elements (see [68] for a review).

Let Π_h denote the interpolant for the Wilson element, we sum up the interpolation results in the following theorem.

Theorem 6.1. *For any $u \in H^2(\Omega)$, if $\mathcal{T}_h$ is C-R shape regular, then there exists a constant $C = C(\sigma_0, \mu)$ such that*

$$\|u - \Pi_h u\|_0 + h\|u - \Pi_h u\|_{1,h} \le Ch^2\|u\|_2. \tag{6.3}$$

Moreover, if $u \in H^3(\Omega)$, $\mathcal{T}_h$ is C-R shape regular and the $(1+\alpha)$-Section Condition holds, then there exists a constant $C = C(\sigma_0, \mu)$ such that

$$\|u - \Pi_h u\|_0 + h\|u - \Pi_h u\|_{1,h} \le Ch^{2+\alpha}\|u\|_3. \tag{6.4}$$

Proof. (6.3) has already been included in [39, Theorem 1]. Notice that $\mathcal{P}_2(\hat{K}) \subset \mathcal{P}(\hat{K})$, the standard interpolation argument yields (6.4).

One may ask if the shape regular condition can be relaxed in the above theorem. There is no such result for a general quadrilateral mesh, however, the following result indicates that it is not hopeless at least for a rectangular mesh.

Lemma 6.2. *If Ω is covered by a uniform rectangular mesh with $h_x(h_1)$ and $h_y(h_2)$ in the x and y directions, respectively, then*

$$\|u - \Pi_h u\|_0 \le Ch^2\Big(\Big\|\frac{\partial^2 u}{\partial x^2}\Big\|_0 + \Big\|\frac{\partial^2 u}{\partial y^2}\Big\|_0\Big), \tag{6.5}$$

$$\Big\|\frac{\partial(u - \Pi_h u)}{\partial x_i}\Big\|_0 \le C\sum_{j=1}^{2} h_j\Big\|\frac{\partial^2 u}{\partial x_i \partial x_j}\Big\|_0, \quad i = 1, 2. \tag{6.6}$$

Moreover, if $u \in H^3$, then

$$\|u - \Pi_h u\|_0 \le Ch^3 |u|_3, \tag{6.7}$$

$$\Big\|\frac{\partial(u - \Pi_h u)}{\partial x_i}\Big\|_0 \le C \sum_{j,k=1}^{2} h_j h_k \Big\|\frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}\Big\|_0, \quad i = 1, 2. \tag{6.8}$$

To estimate the consistency error, we follow that in [54]. The consistency functional can be decomposed into

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} \int_{\partial K} \boldsymbol{\psi}\cdot\boldsymbol{n} v\, ds = &\sum_{K\in\mathcal{T}_h,\mathcal{E}\subset\partial K} \int_{\mathcal{E}} \Big(\boldsymbol{\psi}\cdot\boldsymbol{n} - \fint_K \boldsymbol{\psi}\cdot\boldsymbol{n}\Big)\Big(v - \fint_K v\Big)\, ds \\ &+ \sum_{K\in\mathcal{T}_h,\mathcal{E}\subset\partial K} \int_{\mathcal{E}} \Big(\boldsymbol{\psi}\cdot\boldsymbol{n} - \fint_K \boldsymbol{\psi}\cdot\boldsymbol{n}\Big) \fint_K v\, ds \\ &+ \sum_{K\in\mathcal{T}_h,\mathcal{E}\subset\partial K} \fint_K \boldsymbol{\psi}\cdot\boldsymbol{n} \int_{\mathcal{E}} v\, ds.\end{aligned} \tag{6.9}$$

Lemma 6.1 bounds the first two terms on the right hand side of (6.9), thus the J condition is needed. Moreover, in [54, Theorem 2], a counterexample is presented to show that the Bi-Section Condition is necessary for estimating the third term on the right hand side of (6.9). Summing up and using Lemma 4.1, we see that the asymptotically regular parallelogram mesh condition is both sufficient and necessary for obtaining the optimal consistency error, at least for the above decomposition. Moreover, the optimal interpolation error requires that $\mathcal{T}_h$ is shape regular which may be weakened as indicated by Lemma 6.1. However, it is still unknown whether or not the asymptotically regular parallelogram mesh is really necessary for the optimal error estimate of the Wilson-type elements.

6.2. Nonconforming Rotated $\mathcal{Q}_1$ Element

Two types of the quadrilateral rotated $\mathcal{Q}_1$ finite element spaces can be defined as follows. Let

$$\overline{Q}_1 := \{\, q \circ \mathcal{F}_K^{-1} \mid q \in \mathrm{Span}\langle 1, x, y, x^2 - y^2\rangle \,\}.$$

Denote $\mathcal{J}_{\mathcal{E}}^{a/p}$ for $\mathcal{J}_{\mathcal{E}}^{a}$ as well as $\mathcal{J}_{\mathcal{E}}^{p}$. The finite element spaces are defined as

$$V_h^{a/p} := \{\, v \in L^2(\Omega) \mid v_{|K} \in \overline{Q}_1(K),\ v \text{ is continuous regarding } \mathcal{J}_{\mathcal{E}}^{a/p}(\cdot) \,\},$$

and the corresponding homogeneous spaces as

$$V_{0,h}^{a/p} := \{\, v \in V_h^{a/p} \mid \mathcal{J}_{\mathcal{E}}^{a/p}(v) = 0, \text{ if } \mathcal{E} \subset \partial\Omega \,\}.$$

A global interpolation operator π_h is realized by the forgoing local interpolation operator $\mathcal{J}_{\mathcal{E}}^{a/p}$, i.e., $\pi_{h|K} = \mathcal{J}_{\mathcal{E}}^{a/p} \quad \forall \mathcal{E} \subset \partial K$. We have the following interpolation result for π_h.

Theorem 6.2.[50, Lemma 1] *For any $u \in H^2 \cap H_0^1$, if $\mathcal{T}_h$ is C-R shape regular, then*

$$\|u - \pi_h u\|_0 + h\|u - \pi_h u\|_{1,h} \le Ch(h + \sigma_h)\|u\|_2. \tag{6.10}$$

Here C depends on σ_0 and μ.

As a direct consequence of the above result, we have

Corollary 6.1. *For any $u \in H^2 \cap H_0^1$, if the $(1+\alpha)$-Section Condition holds, then*

$$\|u - \pi_h u\|_0 \le C_1 h^{1+\alpha}\|u\|_2. \tag{6.11}$$

Here the constant C_1 is independent of the geometry of $K \in \mathcal{T}_h$. If $\mathcal{T}_h$ is an asymptotically regular parallelogram mesh, then

$$\|u - \pi_h u\|_{1,h} \le C_2 h\|u\|_2. \tag{6.12}$$

Here C_2 depends on σ_0 and μ.

By Theorem 5.1, RDP(N,ψ) is sufficient and necessary for obtaining the optimal interpolation error for the 4-node isoparametric element. One may ask whether this is the same for the NRQ$_1$ element. The following lemma and a counterexample give a negative answer.

Lemma 6.3. *If Ω is covered by a uniform rectangular mesh with $h_x(h_1)$ and $h_y(h_2)$ in the x and y directions, respectively. Then for $u \in H^2 \cap H_0^1$, we have*

$$\|u-\pi_h u\|_0 \le C\sum_{i,j=1}^{2} h_i h_j \Big\|\frac{\partial^2 u}{\partial x_i \partial x_j}\Big\|_0, \tag{6.13}$$

$$|u-\pi_h u|_{1,h} \le Ch(1+\sigma)|u|_2. \tag{6.14}$$

Here σ is the anisotropic ratio which is defined by $\sigma := \max(h_x/h_y, h_y/h_x)$.

Proof. (6.14) is included in [42, Theorem 2.1], from which we get (6.13).

We adopt a counterexample in [42] to show that the anisotropic ratio σ appeared in the right hand side of (6.14) is essential.

Counterexample

Consider an element K centered at the origin with the lengths $2h_x$ and $2h_y$ in the x and y directions, respectively. Without loss of generality, we assume that $h_y < h_x$, and define the anisotropy ratio as $\sigma := h_x/h_y$. Let $u(x,y) = 2x^2$, then $|u|_{2,K}^2 = 64h_xh_y$. As to this u, we can verify that $|u-\pi_h u|_{1,K}^2 = 16h_x^3h_y(1+\sigma^2)/3$, therefore

$$|u-\pi_h u|_{1,K}/|u|_{2,K} = ((1+\sigma^2)/3)^{1/2}h_x/2.$$

So (6.14) cannot hold with a constant independent of the anisotropic ratio σ.

Notice that the asymptotically regular parallelogram mesh condition is sufficient for the optimal interpolation error for the NRQ$_1$ element, the following counterexample shows that the Bi-Section Condition is also necessary.

Counterexample

Consider the element K as that in Fig. 5. Let $u(x,y) := 1/(1+x)$. It is seen that

$$|u|_{2,K}^2 \le 4|K| = 8(2-a)h_K^2. \tag{6.15}$$

A simple manipulation shows that

$$\|u-\pi_h u\|_{0,K}^2 = \int_{\hat K} h_K^2(1-a(1+\eta)/2)(1+\xi)(ah_K\xi\eta/2+\mathcal{O}(h_K^2))^2/f(\xi,\eta)\,d\hat x,$$

where $f(\xi,\eta)$ is defined as

$$f(\xi,\eta) := (1+h_K)^2(1+(1-a)h_K)^2(1+(2-a)h_K)^2(1+(1-a(1+\eta)/2)(1+\xi)h_K)^2.$$

Notice that $0 \le a < 1$, so $|f(\xi,\eta)| \le 36$, thus a combination of the above two identities yields

$$\|u-\pi_h u\|_{0,K}^2 \ge a^2h_K^4/648 + \mathcal{O}(h_K^6), \tag{6.16}$$

which together with (6.15) leads to

$$\|u-\pi_h u\|_{0,K}/|u|_{2,K} \ge \sqrt{2}ah_K/18 = \sqrt{2}d_K/18. \tag{6.17}$$

Proceeding along the same line, we obtain

$$\frac{|u-\pi_h u|_{1,K}}{|u|_{2,K}} \ge \frac{\Big\|\frac{\partial(u-\pi_h u)}{\partial x}\Big\|_{0,K}}{|u|_{2,K}} \ge (\sqrt{3}/36)d_K/h_K. \tag{6.18}$$

The above two inequalities clearly show the necessity of the Bi-Section Condition for the optimal interpolation error bound of the NRQ$_1$ element with respect to both L^2-norm and H^1-norm.

Fig.5

Degradation of the interpolation error will occur on real quadrilateral meshes, particularly, on the following trapezoid meshes. Nevertheless, as shown by Corollary 6.1, if the quadrilateral mesh is an asymptotic parallelogram (see Fig.2), such degradation will not occur.

Fig.6

It remains to consider the consistency error of NRQ_1. As to NRQ_1^a, the consistency functional can be decomposed into

$$\sum_{K\in\mathcal{T}_h,\mathcal{E}\subset\partial K}\int_{\mathcal{E}}\Big(\boldsymbol{\psi}\cdot\boldsymbol{n}-\fint_{\mathcal{E}}\boldsymbol{\psi}\cdot\boldsymbol{n}\Big)\Big(v-\fint_{\mathcal{E}}v\Big)\,ds.$$

By Lemma 6.1, the J condition is needed for the optimal consistency error estimate. As to NRQ_1^p, besides the J condition the Bi-Section Condition is also needed. The latter is even necessary in some sense for NRQ_1^p as shown in [42, Theorem 3.2]. However, for the rectangular NRQ_1, a different argument yields the optimal consistency error bound which is independent of the J condition (see [44] for more details). We do not know whether a similar argument works for the quadrilateral NRQ_1.

The above discussion indicates that there is a convergence degradation of the NRQ_1 element over a degenerate mesh. Indeed, such degradation was observed by the numerical results in [50, 62], both the interpolation and consistency error commit such degradation. There are many works on modifications of this element to accommodate the degenerate mesh (see [7, 21, 22, 44]).

To sum up, neither the Wilson element nor the NRQ_1 element can be used for a degenerate mesh. Is there another kind of lower-order nonconforming quadrilateral element which can be used over a fully degenerate mesh while retaining its excellent stability property simultaneously? Such an element would be a grail for the finite element circus.

§7. RT Element of Lowest-Order

Up to now, there are only few results available to explain the mesh dependence of the interpolation error for mixed elements, like $\mathrm{R\text{-}T}_{[k]}$, $\mathrm{BDFM}_{[k]}$ and $\mathrm{BDM}_{[k]}$ et al[20].[1]

[1]see [64, 56] for the definitions of $\mathrm{RT}_{[k]}$, $\mathrm{BDDM}_{[k]}$ and $\mathrm{BDFM}_{[k]}$ over an arbitrary quadrilateral mesh.

Wang and Mathew[64] gave the optimal interpolation error for all these mixed finite elements over an arbitrary quadrilateral, unfortunately, the mesh dependence is not clearly stated therein.

Raviart and Thomas[49] derived the optimal interpolation error for the R-T element over a shape regular parallelogram. Proceeding along the same line of [1, Lemma 4.1, Lemma 4.2], one can easily get the following interpolation and stability estimates for the $\mathrm{RT}_{[0]}$ element (see [1, Remark 4.1] and [8]) for the case when $\mathcal{T}_h$ is a rectangular mesh.

Lemma 7.1. *If Ω is covered by a uniform rectangular mesh with the diameter h_1 and h_2 in the x and y directions, respectively, then*

$$\|(\boldsymbol{u}-RT\boldsymbol{u})_i\|_0 \le C\sum_{j=1}^{2} h_j \Big\|\frac{\partial \boldsymbol{u}_i}{\partial x_j}\Big\|_0, \quad i=1,2 \tag{7.1}$$

with a constant C independent of the ratio between h_1 and h_2. Moreover, we have the following stability estimate

$$\Big\|\frac{\partial (RT\boldsymbol{u})_i}{\partial x_i}\Big\|_0 \le \Big\|\frac{\partial \boldsymbol{u}_i}{\partial x_i}\Big\|_0, \quad i=1,2. \tag{7.2}$$

Remark 7.1. As to a simple proof for (7.2), see [8, Lemma 5.7].

Similar to [43, Theorem 3.2], we have the following result for the quadrilateral $\mathrm{RT}_{[0]}$.

Theorem 7.1. *If $\mathcal{T}_h$ is an asymptotically regular parallelogram mesh, then for $\boldsymbol{u} \in \boldsymbol{H}^1(\mathrm{div})$, there holds*

$$\|\boldsymbol{u}-RT\boldsymbol{u}\|_0 \le Ch\|\boldsymbol{u}\|_1, \tag{7.3}$$

$$\|\mathrm{div}(\boldsymbol{u}-RT\boldsymbol{u})\|_0 \le Ch\|\boldsymbol{u}\|_{H^1(\mathrm{div})}. \tag{7.4}$$

Here the constant C depends on σ_0 and μ.

Remark 7.2. Similar results can be found in [25, Lemma 3.2]. However, the Bi-Section Condition is missing therein, which is actually necessary for obtaining the optimal interpolation error. This will be illustrated by the example below.

Similar to [43, Theorem 3.1], we have the refined interpolation results for $\mathrm{RT}_{[0]}$ as follows.

Theorem 7.2. *If $\mathcal{T}_h$ satisfies C-R as well as the $(1+\alpha)$-Section Condition, then for any $\boldsymbol{u} \in \boldsymbol{H}^\beta(\mathrm{div})$ with $\beta \in (0,1]$, there holds*

$$\|\boldsymbol{u}-RT\boldsymbol{u}\|_0 \le C(h^\beta|\boldsymbol{u}|_{H^\beta} + h^\alpha\|\boldsymbol{u}\|_0 + h\|\mathrm{div}\boldsymbol{u}\|_0), \tag{7.5}$$

$$\|\mathrm{div}(\boldsymbol{u}-RT\boldsymbol{u})\|_0 \le C(h^\beta|\mathrm{div}\boldsymbol{u}|_{\boldsymbol{H}^\beta} + h^\alpha\|\mathrm{div}\boldsymbol{u}\|_0). \tag{7.6}$$

If $\beta > 1/2$, the last term in (7.5) can be dropped. Morover, if $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ and the Bi-Section Condition holds, we have

$$\|\boldsymbol{u}-RT\boldsymbol{u}\|_0 \le Ch\|\boldsymbol{u}\|_1, \tag{7.7}$$

$$\|\mathrm{div}(\boldsymbol{u}-RT\boldsymbol{u})\|_0 \le Ch\|\mathrm{div}\boldsymbol{u}\|_1. \tag{7.8}$$

Here the constant C depends on σ_0 and μ.[2]

Comparing Theorem 7.1 and Theorem 7.2 with Lemma 7.1, one may ask whether the mesh conditions in Theorem 7.1 and Theorem 7.2 can be relaxed. The following example shows the necessity of the $(1+\alpha)$-Section Condition or the Bi-Section Condition for the optimality of the interpolation error. It seems that the argument based on the Piola transform as that in [43] and [25] is less hopeful for further relaxing the mesh condit ions.

Counterexample

[2]The definitions of $\boldsymbol{H}^\beta$ and $\boldsymbol{H}^\beta(\mathrm{div})$ with $\beta \in (0,1]$ can be found in [3].

Consider the element as in Fig. 5. Let $\boldsymbol{u}=(1,0)$, a simple manipulation yields

$$\|\boldsymbol{u}-RT\boldsymbol{u}\|_{0,K}^2=2(ah_K/2)^2\int_{-1}^{1}\frac{\eta^2}{1-a/2-a/2\eta}\,d\eta\geq a^2h_K^2/3. \tag{7.9}$$

It is seen that

$$\|\boldsymbol{u}\|_{1,K}^2=2(2-a)h_K^2.$$

A combination of the above two inequalities leads to

$$\|\boldsymbol{u}-RT\boldsymbol{u}\|_{0,K}/\|\boldsymbol{u}\|_{1,K}\geq(1/2\sqrt{3})a=(1/2\sqrt{3})d_K/h_K. \tag{7.10}$$

The above example clearly shows the necessity of the $(1+\alpha)$-Section Condition for obtaining the optimal interpolation error bound (7.5).

Let $\boldsymbol{u}=(x,0)$; proceeding along the same line of the above procedure, we obtain

$$\|\mathrm{div}(\boldsymbol{u}-RT\boldsymbol{u})\|_{0,K}/\|\mathrm{div}\boldsymbol{u}\|_{1,K}\geq(1/\sqrt{6})d_K/h_K, \tag{7.11}$$

which shows the necessity of the $(1+\alpha)$-Section Condition for the optimal interpolation error bounds in (7.6). Naturally, (7.10) and (7.11) also demonstrate the necessity of the Bi-Section Condition for the optimal interpolation error bounds in (7.7) and (7.8), respectively.

§8. Conclusions and Open Problems

In this paper, some commonly used shape regular mesh conditions are proven to be equivalent and their connections to some degenerate mesh conditions are also clarified.

We have checked the influence of mesh conditions on the interpolation error for the 4-node isoparametric element, quadrilateral nonconforming element and $\mathrm{RT}_{[0]}$ element. The asymptotically regular parallelogram mesh is found to be indispensable for the successful application of either Wilson, NRQ_1 or $\mathrm{RT}_{[0]}$ element, otherwise, the degradation of the convergence order will occur which is not widely appreciated, and was casually observed in numerical experiments (see [69, §8.7] and [9]).

Before closing this paper, we propose three open problems.

(1) Is RDP(N,ψ) also necessary for the optimal finite element approximation error of the 4-node isoparametric element for the 2-order elliptic problem?

(2) What is the necessary and sufficient mesh condition for the convergence as well as the optimal error bounds of the Wilson and NRQ_1 element for the 2-order elliptic problem?

(3) What is the necessary and sufficient mesh condition for obtaining the optimal interpolation error bounds of $\mathrm{RT}_{[k]}$, $\mathrm{BDDM}_{[k]}$ and $\mathrm{BDFM}_{[k]}$?

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