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ON THE HYPERBOLIC OBSTACLE PROBLEM OF FIRST ORDER**

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(*Dedicated to the memory of Jacques-Louis Lions*)

Abstract

This paper presents new results for strong solutions and their coincidence sets of the obstacle problem for linear hyperbolic operators of first order. An inequality similar to the Lewy–Stampacchia ones for elliptic and parabolic problems is shown. Under nondegeneracy conditions the stability of the coincidence set is shown with respect to the variation of the data and with respect to approximation by semilinear hyperbolic problems. These results are applied to the asymptotic stability of the evolution problem with respect to the stationary coercive problem with obstacle.

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§1. Introduction

The classical obstacle problem can be formulated as the problem of finding the equilibrium position of an elastic membrane constrained to lie above an obstacle. Although, in the words of J. L. Lions, this "simple, beautiful and deep" problem is naturally associated with partial differential equations of elliptic type, it arises in many other frameworks and in different kinds of free boundary problems (see [3] or [10], and their references) and it is related to variational inequalities (see [6,7]).

Variational inequalities of first order hyperbolic type were introduced in 1973 by Bensoussan and Lions[2] for the study of deterministic cases in problems of optimal stopping time, in which their solutions can be interpreted as optimal cost functions. More recently, motivated by physical problems in petroleum engineering, some unilateral problems for scalar conservation laws have been considered by L. Lévi in [4] (see also [5]), where the existence and uniqueness of weak entropy solutions are proven for quasilinear hyperbolic operators.

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In this work we are concerned with the problem of finding a function u defined in an open smooth domain $Q \subset \mathbb{R}^N$, such that, for a given function ψ (the obstacle)

$$u \geq \psi \quad \text{a.e. in } \ Q \tag{1.1}$$

and, in the a priori unknown region where the solution u does not coincide with the obstacle ψ,

$$Hu = f \quad \text{a.e. in } \ \{u > \psi\} \tag{1.2}$$

with a Dirichlet boundary condition in a known part of the boundary of the domain

$$u = h \quad \text{a.e. on } \ \Sigma_- \subset \partial Q \tag{1.3}$$

for given functions f and h. Here H is a linear first order operator, whose principal parts determines the subset Σ_-, where the boundary condition can be imposed.

The first systematic study of the obstacle problem (1.1),(1.2),(1.3) was done by Mignot and Puel in 1976[8] in the framework of strong and weak solutions of variational inequalities of first order. Their approach, which will be followed here, is based on the general linear theory of boundary value problems for first order partial differential equations of Bardos[1]. For linear operators the boundary condition (1.3) and the definition of Σ_- are well-known and their functional spaces are recalled in Section 2, where we introduce and show the continuous dependence of strong solutions for the hyperbolic obstacle problem (1.1),(1.2),(1.3).

Always in the framework of strong solutions, in Section 3, we extend the Lewy-Stampacchia inequalities to linear first order operators by considering the approximation by solutions of semilinear hyperbolic problems. In Section 4, we show the stability of the set where the solution coincides with the obstacle, under a natural nondegeneracy condition on the obstacle and the nonhomogeneous term.

In Section 5 we extend to this case an estimate on the variation of the coincidence sets associated with the dependence on the data, including the variation of the (nondegenerating) obstacles. These results extend to first order obstacle problems the previous theory for second order linear operators (see [10], for instance) and can be applied to show the stability of the solution and coincidence set to the evolution first order obstacle problem with respect to the respective stationary one, as time goes to infinity. This is done in the final section.

§2. The Obstacle Problem of First Order

Let $Q \subset \mathbb{R}^N$ be an open domain and H be the linear first order operator defined on $\overline{Q}$ by

$$Hu \,=\, \mathbf{b} \cdot \nabla u + b_0\, u \,=\, Bu + b_0,\, u \tag{2.1}$$

where $b_0 = b_0(x) \in L^\infty(Q)$ and the vector field $B = \mathbf{b} \cdot \nabla = \sum\limits_{i=1}^{N} b_i(x) \frac{\partial}{\partial x_i}$ has coefficients $b_i \in C^1(\overline{Q}) \cap W^{1,\infty}(Q)$ and the boundary $\Sigma = \partial Q$ is C^1 piecewise in the sense of [1], i.e., admits a decomposition where

$$\Sigma_+ = \{x \in \partial Q \colon\, \mathbf{b} \cdot \mathbf{n} > 0\} \quad \text{and} \quad \Sigma_- = \{x \in \partial Q \colon\, \mathbf{b} \cdot \mathbf{n} < 0\}$$

have a finite number of C^1 piecewise subboundaries of dimension $N-2$. Here $\mathbf{n} = \mathbf{n}(x)$ is the outer normal vector at $x \in \Sigma = \partial Q$, defined almost everywhere. We define $\ell(x) = |\mathbf{b}(x) \cdot \mathbf{n}(x)|$ on Σ and we introduce the Hilbert spaces associated with the vector field B:

$$L^2_B(Q) = \{v \in L^2(Q) \colon\, Bv \in L^2(Q)\} \quad \text{and} \quad \widetilde{L}^2_B(Q) = \{v \in L^2_B(Q) \colon\, v|_\Sigma \in L^2_\ell(\Sigma)\}, \tag{2.2}$$

where $L^2_\ell(\Sigma) = \{v\colon \Sigma \to \mathbb{R} \mid \int_\Sigma v^2\,\ell\,d\Sigma < \infty\}$. We recall from [1] that the graph norm in $\widetilde{L}^2_B(Q)$ is equivalent to

$$\|v\|^2_\sim \;=\; \|Bv\|^2_{L^2(Q)} + \|v\|^2_{L^2_\ell(\Sigma_-)}, \tag{2.3}$$

that $C^1(\overline{Q})$ is dense in $\widetilde{L}^2_B(Q)$ and the following integration by parts formula holds in this space

$$\int_Q (\mathbf{b}\cdot\nabla u)\,v\,dx \;+\; \int_Q u\,\nabla\cdot(v\,\mathbf{b})\,dx \;=\; \int_{\Sigma_+\cup\Sigma_-} (\mathbf{b}\cdot\mathbf{n})\,u\,v\,d\Sigma. \tag{2.4}$$

This framework allowed Bardos[1] to show the existence and uniqueness of the solution w in $\widetilde{L}^2_B(Q)$ of the linear first order problem

$$Hw = f \quad \text{in } Q, \qquad w = h \quad \text{on } \Sigma_- \tag{2.5}$$

under the coercivity assumption

$$b_0(x) - \frac{1}{2}(\nabla\cdot\mathbf{b})(x) \;\geq\; \beta \;>\; 0, \qquad \forall\, x \in Q, \tag{2.6}$$

for any given data

$$f \in L^2(Q) \qquad \text{and} \qquad h \in L^2_\ell(\Sigma_-). \tag{2.7}$$

Consider now an obstacle $\psi = \psi(x)$ such that

$$\psi \in \widetilde{L}^2_B(Q), \qquad \psi \leq h \quad \text{on } \Sigma_-, \tag{2.8}$$

and introduce the non-empty convex subsets

$$K_\psi = \{v \in L^2(Q)\colon\, v \geq \psi \;\;\text{a.e. in } Q\} \qquad \text{and} \qquad \widetilde{K}_\psi = K_\psi \cap \widetilde{L}^2_B(Q). \tag{2.9}$$

Following Mignot and Puel[8], we consider the strong formulation of the obstacle problem for the first order operator (2.1)

$$u \in \widetilde{K}_\psi, \quad u|_{\Sigma_-} = h\colon \qquad \int_Q (Hu - f)\,(v-u)\,dx \;\geq\; 0, \quad \forall\, v \in K_\psi. \tag{2.10}$$

Under the assumptions (2.7) and (2.8) the existence and uniqueness of a strong solution to the hyperbolic variational inequality (2.10) are shown in [8]. Actually the conditions on ψ in (2.8) can be taken in a weaker sense, since only the regularity $\widetilde{\psi} = \sup(\psi, w) \in \widetilde{L}^2_B(Q)$ and $\widetilde{\psi} \leq h$ on Σ_- are necessary. Indeed, it was shown that if w and u are the solutions of (2.5) and (2.10), respectively, then $u \geq w$ a.e. in Q. So also $u \geq \widetilde{\psi}$ and to solve the variational inequality (2.10) in $\widetilde{K}_\psi$ and in $\widetilde{K}_{\widetilde{\psi}}$ are equiv a lent problems, and (2.8) is, therefore, a natural assumption to obtain strong solutions.

We also recall the property of $\widetilde{L}^2_B(Q)$ as a Dirichlet space, i.e., v^+, v^- and $|v| \in \widetilde{L}^2_B(Q)$ if $v \in \widetilde{L}^2_B(Q)$, which can be proved as in the Sobolev space $H^1(\Omega)$. Similarly, one has, for instance, $Bv^+ = Bv$ in $\{v > 0\}$ and $Bv^+ = 0$ in $\{v \leq 0\}$, provided $v \in \widetilde{L}^2_B(Q)$, in the almost everywhere sense. We can also show for $v \in \widetilde{L}^2_B(Q)$ that

$$Bv = 0 \qquad \text{a.e. in } \;\{x \in Q\colon\, v(x) = 0\}. \tag{2.11}$$

Here we shall use the standard notations

$$u \vee v = \sup(u, v), \qquad v^+ = v \vee 0 \qquad \text{and} \qquad v^- = (-v)^+.$$

We do not restrict the generality in taking

$$\psi = 0, \tag{2.12}$$

since we can reformulate the obstacle problem (2.10) into an equivalent one for the translated functions

$$\widetilde{u} = u - \psi, \quad \widetilde{h} = h - \psi|_{\Sigma_-} \quad \text{and} \quad \widetilde{f} = f - H\psi. \tag{2.13}$$

Indeed, (2.10) is easily seen to be equivalent to

$$\widetilde{u} \in \widetilde{K}_0, \quad \widetilde{u}|_{\Sigma_-} = \widetilde{h}: \qquad \int_Q (H\widetilde{u} - \widetilde{f})\,(v - \widetilde{u})\,dx \,\geq\, 0, \quad \forall\, v \in K_0. \tag{2.14}$$

We have the following continuous dependence estimate for strong solutions.

Proposition 2.1. *Let u_i denote the solution of* (2.10) *corresponding to the data f_i, h_i and ψ_i under the assumptions* (2.6), (2.7), (2.8) *for $i = 1, 2$, respectively. Then*

$$\|u_1 - u_2\|_{L^2(Q)} \,\leq\, C\,(\|f_1 - f_2\|_{L^2(Q)} + \|h_1 - h_2\|_{L^2_\ell(\Sigma_-)} + \|\psi_1 - \psi_2\|_\sim), \tag{2.15}$$

where $\|\cdot\|_\sim$ denotes the norm (2.3) *and $C > 0$ is a constant independent of the data.*

Proof. Using (2.13) we may assume $\psi_1 = \psi_2 = 0$, for $i = 1, 2$. We may take $v = \widetilde{u}_2$ in the inequality for $\widetilde{u}_1$ and $v = \widetilde{u}_1$ in the one for $\widetilde{u}_2$. Setting $w = \widetilde{u}_1 - \widetilde{u}_2 = u_1 - u_2 - (\psi_1 - \psi_2)$ and denoting $\overline{f} = \widetilde{f}_1 - \widetilde{f}_2 = f_1 - f_2 - H(\psi_1 - \psi_2)$ and $\overline{h} = h_1 - h_2 - (\psi_1 - \psi_2)|_{\Sigma_-}$ we obtain

$$\int_Q w\,H\,w\,dx \,\leq\, \int_Q \overline{f}\,w\,dx \,\leq\, \frac{\beta}{2}\int_Q w^2\,dx + \frac{1}{2\beta}\int_Q \overline{f}^2\,dx. \tag{2.16}$$

On the other hand, from the coercivity condition (2.6), we find

$$\begin{aligned}\int_Q w\,H\,w\,dx &= \int_Q \Big(b_0 - \frac{1}{2}\,\nabla\cdot\mathbf{b}\Big)\,w^2\,dx + \frac{1}{2}\int_{\Sigma_+\cup\Sigma_-} (\mathbf{b}\cdot\mathbf{n})\,w^2\,d\Sigma \\ &\geq \beta\int_Q w^2\,dx - \frac{1}{2}\int_{\Sigma_-} w^2\,\ell\,d\Sigma,\end{aligned} \tag{2.17}$$

which combined with (2.16) yields

$$\beta\int_Q w^2\,dx \,\leq\, \frac{1}{\beta}\int_Q \overline{f}^2\,dx \,+\, \int_{\Sigma_-} \overline{h}^2\,\ell\,d\Sigma.$$

This implies the conclusion (2.15) by the definitions and the equivalence of the norm $\|\cdot\|_\sim$ in $\widetilde{L}^2_B(Q)$.

§3. An Inequality for Strong Solutions

The strong solution of the obstacle problem, solving the first order variational inequality, is also a solution to the nonlinear complementary problem

$$u \geq \psi, \quad Hu - f \geq 0 \;\text{ and }\; (Hu - f)\,(u - \psi) = 0 \quad \text{a.e. in } \; Q. \tag{3.1}$$

Indeed, it suffices to take $v = \psi$ and $v = 2u - \psi$ in (2.10) to conclude the third condition from the first two. The second one, which follows from (2.10) with $v = u + w$ for arbitrary $w \in L^2(Q)$, $w \geq 0$, provides a lower bound for Hu. The aim of this section is to show an upper bound, extending to H the well-known Lewy–Stampacchia inequalities obtained first for second order obstacle problem of elliptic type (see [10], for references).

Theorem 3.1. *Under the assumptions* (2.6), (2.7), (2.8) *the strong solution u of the first order obstacle problem* (2.10) *satisfies the inequalities*

$$f \leq Hu \leq f \vee H\psi \quad \textit{a.e. in } \; Q. \tag{3.2}$$

The proof of this result follows easily by recalling that $f \vee H\psi = f + (H\psi - f)^+$ and the fact that u can be approximated in $L^2(Q)$ by the solution $u_\varepsilon \in \widetilde{L}^2_B(Q)$ of the semilinear first order equation

$$Hu_\varepsilon + \xi\,\vartheta_\varepsilon(u_\varepsilon - \psi) = f + \xi \quad \text{in } Q, \quad u_\varepsilon = h \quad \text{on } \Sigma_-. \tag{3.3}$$

Here we consider for each $\varepsilon > 0$, the nondecreasing Lipschitz function $\vartheta_\varepsilon\colon \mathbb{R} \to [0,1]$ defined by

$$\vartheta_\varepsilon(t) = 0, \quad t \le 0, \quad \vartheta_\varepsilon(t) = t/\varepsilon, \quad 0 < t \le \varepsilon \quad \text{and} \quad \vartheta_\varepsilon(t) = 1, \quad t > \varepsilon, \tag{3.4}$$

and the nonnegative function $\xi \in L^2(Q)$ given by

$$\xi = (H\psi - f)^+. \tag{3.5}$$

We can prove the very precise approximation result.

Theorem 3.2. *If u and u_ε denote the solutions of* (2.10) *and* (3.3)*, respectively, under the previous assumptions we have*

$$u_\varepsilon \in \widetilde{K}_\psi, \tag{3.6}$$

$$u_\varepsilon \ge u_{\widehat{\varepsilon}} \quad \textit{in } Q \qquad \textit{if} \quad \varepsilon > \widehat{\varepsilon} > 0, \tag{3.7}$$

$$\|u_\varepsilon - u\|^2_{L^2(Q)} \le \frac{\varepsilon}{\beta}\,\|(H\psi - f)^+\|_{L^1(Q)} \qquad \textit{as} \quad \varepsilon \to 0. \tag{3.8}$$

Proof. Since ϑ_ε is monotone and H is coercive, the existence and uniqueness of $u_\varepsilon \in \widetilde{L}^2_B(Q)$ follows by the results of Bardos[1].

To prove (3.6), we must show that $u_\varepsilon \ge \psi$ in Q. Take $z = (\psi - u_\varepsilon)^+ \in \widetilde{L}^2_B(Q)$ and note that by (2.8) we have $z|_{\Sigma_-} = 0$. Since $\vartheta_\varepsilon(u_\varepsilon - \psi) = 0$ whenever $\psi > u_\varepsilon$ we obtain $Hu_\varepsilon = f + \xi \ge H\psi$ if $\psi > u_\varepsilon$ and

$$\int_Q z\,Hz\,dx \;=\; \int_Q (H\psi - Hu_\varepsilon)\,(\psi - u_\varepsilon)^+\,dx \;\le\; 0,$$

since $z\,Hz = (\psi - u_\varepsilon)^+\,H(\psi - u_\varepsilon)$ a.e. in Q. Hence, using (2.17)

$$0 \;\ge\; \int_Q z\,Hz\,dx \;\ge\; \beta \int_Q z^2\,dx$$

we conclude $z = 0$ a.e. in Q and (3.6) follows.

A similar argument applies to $z = (u_{\widehat{\varepsilon}} - u_\varepsilon)^+$, by using

$$z\,Hz \;=\; z\,H(u_{\widehat{\varepsilon}} - u_\varepsilon) \;=\; z\,\xi[\vartheta_\varepsilon(u_\varepsilon - \psi) - \vartheta_{\widehat{\varepsilon}}(u_{\widehat{\varepsilon}} - \psi)] \le 0 \quad \text{in } Q,$$

since if $u_{\widehat{\varepsilon}} > u_\varepsilon$, then $\vartheta_{\widehat{\varepsilon}}(u_{\widehat{\varepsilon}} - \psi) \ge \vartheta_{\widehat{\varepsilon}}(u_\varepsilon - \psi) \ge \vartheta_\varepsilon(u_\varepsilon - \psi)$.

Finally, remarking that for any $v \in K_\psi$ we have

$$[1 - \vartheta_\varepsilon(u_\varepsilon - \psi)]\,(v - u_\varepsilon) \;\ge\; [1 - \vartheta_\varepsilon(u_\varepsilon - \psi)]\,(\psi - u_\varepsilon) \;\ge\; -\varepsilon,$$

we first obtain

$$\int_Q (Hu_\varepsilon - f)\,(v - u_\varepsilon)\,dx \;=\; \int_Q \xi[1 - \vartheta_\varepsilon(u_\varepsilon - \psi)]\,(v - u_\varepsilon)\,dx \;\ge\; -\,\varepsilon \int_Q \xi\,dx. \tag{3.9}$$

Setting $v = u$ in (3.9) and $v = u_\varepsilon$ in (2.10), we conclude (3.8) with the help of (2.17) for $w = u_\varepsilon - u$

$$\beta \int_Q w^2\,dx \;\le\; \int_Q w\,Hw\,dx \;=\; \int_Q (u_\varepsilon - u)\,H(u_\varepsilon - u)\,dx \;\le\; \varepsilon \int_Q \xi\,dx.$$

Remark 3.1. The proof of Theorem 3.2 actually also shows the existence of the solution u to (2.10), since $0 \le \vartheta_\varepsilon \le 1$ implies that the approximating solution u_ε of (3.3) are bounded in $W = \{v \in \widetilde{L}^2_B(Q)\colon\ v|_{\Sigma_-} = h\}$, uniformly in $\varepsilon > 0$. Hence the lower semi-continuity of

$$w \mapsto \int_Q w\,Hw\,dx \quad \text{in } \ W$$

allows to pass to the limit $\varepsilon \to 0$ in (3.9) by showing that if $u_\varepsilon \rightharpoonup u$ in $\widetilde{L}^2_B(Q)$ then u solves (2.10). By uniqueness, which is a consequence of the coercivity in W, the whole sequence converges.

Remark 3.2. In [8] the existence of a strong solution was obtained with a different approximation (see [6]) by considering the penalized problem for $\varepsilon > 0$,

$$Hw_\varepsilon - \frac{1}{\varepsilon}\,(\psi - w_\varepsilon)^+ \,=\, f \quad \text{ in } \ Q,$$

with the same boundary condition $w_\varepsilon = h$ on Σ. While this is a more natural way to penalize the constraint $u \ge \psi$, this method does not allow to conclude the second inequality in (3.2).

§4. Stability of the Coincidence Set

A main feature in the obstacle problem is the presence, in general, of the coincidence set

$$I = \{u = \psi\} = \{x \in Q\colon\ u(x) = \psi(x)\}. \tag{4.1}$$

In the complementary set Λ of this measurable subset, from (3.1), we have

$$Hu = f \quad \text{ a.e. in } \ \Lambda = \{u > \psi\} = Q\backslash I. \tag{4.2}$$

It is clear that, in general, I and Λ are measurable subsets defined up to a null set. This is however sufficient for our purposes in this work, since we are interested in their characteristic functions. Set

$$\chi \,=\, \chi_{(u=\psi)} = \begin{cases} 1 & \text{if } \ x \in \{u = \psi\}, \\ 0 & \text{if } \ x \in \{u > \psi\}. \end{cases} \tag{4.3}$$

As a consequence of (4.2) and property (2.11), we may conclude that the solution u of (2.10) solves the equation

$$Hu - (H\psi - f)\,\chi \,=\, f \quad \text{ a.e. in } \ Q. \tag{4.4}$$

This important remark allows us to include the first order obstacle problem in the general framework of stability of the coincidence set with respect to perturbation of data (see [10, p. 204], for the elliptic theory).

Theorem 4.1. *Suppose u_n and $\chi_n = \chi_{\{u_n=\psi_n\}}$ denote the solution of* (2.10) *and the characteristic function of its coincidence set associated with a sequence f_n, h_n and ψ_n satisfying* (2.7), (2.8) *and*

$$f_n \to f \ \text{ in } L^2(Q), \quad h_n \to h \ \text{ in } L^2_\ell(\Sigma_-) \quad \text{ and } \quad \psi_n \to \psi \ \text{ in } \widetilde{L}^2_B(Q).$$

If u and χ refer to the corresponding limit problem in which we assume

$$H\psi \ne f \quad \text{ a.e. in } \ \ Q, \tag{4.5}$$

then the coincidence sets converge in measure, or equivalently

$$\chi_n \to \chi \quad \text{ in } \ L^p(Q), \ \ 1 \le p < \infty. \tag{4.6}$$

Proof. We remark $0 \le \chi_n \le 1$, so that there is a function $\chi_* \in L^\infty(Q)$, $0 \le \chi_* \le 1$, and a subsequence

$$\chi_n \rightharpoonup \chi_* \quad \text{in } \ L^\infty(Q)\text{-weak*}.$$

By Proposition 2.1, we know that

$$u_n \to u \quad \text{in } \ L^2(Q),$$

and, from remark (4.4) for u_n,

$$Hu_n - (H\psi_n - f_n)\,\chi_n = f_n \quad \text{a.e. in } \ Q.$$

So we may pass to the limit and obtain

$$Hu - (H\psi - f)\,\chi_* = f \quad \text{a.e. in } \ Q. \tag{4.7}$$

Comparing (4.7) with (4.4) and using the assumption (4.5) we immediately conclude

$$\chi_* = \chi = \chi_{\{u=\psi\}},$$

i.e., the whole sequence converges $\chi_n \to \chi$ first weakly in $L^p(Q)$ and, since they are characteristic functions, also strongly for any $p < \infty$.

Remark 4.1. As a consequence of Theorem 4.1, we can immediately conclude also that $u_n \to u$ in $\widetilde{L}^2_B(Q)$-strong under the assumption (4.5), which however is not necessary, as we shall see in Theorem 5.1.

On the other hand, we know that u is approximated by the solution u_ε of (3.3), i.e., we have

$$Hu_\varepsilon - (H\psi - f)^+\,q_\varepsilon = f \quad \text{in } \ Q, \tag{4.8}$$

where we set

$$0 \le q_\varepsilon \equiv 1 - \vartheta_\varepsilon(u_\varepsilon - \psi) \le \chi_\varepsilon \le 1 \quad \text{a.e. in } \ Q. \tag{4.9}$$

Here we have introduced χ_ε as the characteristic function of the "approximating coincidence set"

$$I_\varepsilon = \{x \in Q\colon\ \psi(x) \le u_\varepsilon(x) < \psi(x) + \varepsilon\}.$$

To prove (4.9) it is sufficient to recall the definition of ϑ_ε: since $u_\varepsilon \ge \psi$ always, if $u_\varepsilon(x) \ge \psi(x) + \varepsilon$ (i.e. $\chi_\varepsilon(x) = 0$), then $q_\varepsilon(x) = 0$.

As $\varepsilon \to 0$, we may consider subsequences such that

$$q_\varepsilon \rightharpoonup q \quad \text{and} \quad \chi_\varepsilon \rightharpoonup \chi_* \ \text{ in } L^\infty(Q)\text{-weak*} \tag{4.10}$$

for some functions q and χ_* such that

$$0 \le q \le \chi_* \le 1 \quad \text{a.e. in } \ Q. \tag{4.11}$$

From (4.8) we find

$$Hu - (H\psi - f)^+\,q = f \quad \text{in } \ Q. \tag{4.12}$$

In the coincidence set $I = \{u = \psi\}$ we have $Hu = H\psi$ a.e. and, if we assume $H\psi \ne f$, from (4.12) we must have $q = 1$ in I, since we have always $Hu \ge f$ by (3.1). Therefore the nondegeneracy condition (4.5) implies

$$q \ge \chi = \chi_{\{u=\psi\}} \quad \text{a.e. in } \ Q. \tag{4.13}$$

But the definition of q_ε and (4.10) with the convergence of $u_\varepsilon \to u$ in $L^2(Q)$ yield as $\varepsilon \to 0$,

$$0 = (u_\varepsilon - \psi - \varepsilon)^+ q_\varepsilon \rightharpoonup (u-\psi)^+ q = 0,$$

and this implies $q = 0$ if $u > \psi$, i.e.,

$$q \le \chi = \chi_{\{u=\psi\}} \quad \text{a.e. in } Q. \tag{4.14}$$

Then (4.11), (4.13) and (4.14) imply

$$q = \chi_* = \chi = \chi_{\{u=\psi\}}. \tag{4.15}$$

By (4.9) we remark $q_\varepsilon^2 \le q_\varepsilon$ and from

$$\int_Q q = \lim_{\varepsilon\to 0}\int_Q q_\varepsilon \ge \liminf_{\varepsilon\to 0}\int_Q q_\varepsilon^2 \ge \int_Q q^2 = \int_Q \chi$$

we may conclude the strong convergences as $\varepsilon \to 0$

$$\chi_\varepsilon \to \chi \text{ and } q_\varepsilon \to \chi \quad \text{in } L^p(Q)\text{-strong}, \ \forall\, p < \infty.$$

Then (4.8) implies also $Hu_\varepsilon \to Hu$ in $L^2(Q)$-strong, and we have proven the following result on the strong approximation of the first order obstacle problem by solutions of semilinear hyperbolic problems (3.3).

Theorem 4.2. *Let u_ε and u denote the solutions of* (2.10) *and* (3.3) *respectively, under the nondegeneracy assumption* (4.5).

Then, as $\varepsilon \to 0$, we have

$$u_\varepsilon \to u \quad \text{in } \widetilde{L}^2_B(Q)\text{-strong}$$

and

$$\lim_{\varepsilon\to 0}[1 - \vartheta_\varepsilon(u_\varepsilon - \psi)] = \lim_{\varepsilon\to 0}\chi_{\{\psi \le u_\varepsilon < \psi+\varepsilon\}} = \chi_{\{u=\psi\}}$$

for the strong topologies of $L^p(Q)$, $\forall\, p$, $1 \le p < \infty$.

Remark 4.2. We observe that here the nondegeneracy assumption $H\psi \ne f$ a.e. in Q is required, as in Theorem 4.1 on the stability of the coincidence sets. Analogously Theorem 4.2 yields a stability in Lebesgue measure of the approximation of the coincidence set $\{u = \psi\}$, i.e., we have

$$\{\psi \le u_\varepsilon < \psi + \varepsilon\} \to \{u = \psi\} \quad \text{in measure.}$$

§5. An Estimate on the Coincidence Set

Let $\beta > 0$ be the constant of (2.6) and $\alpha > 0$, such that

$$\frac{1}{2}\,|(\nabla\cdot\mathbf{b})(x)| \le \alpha, \quad \forall\, x \in Q.$$

Denote by S the monotone graph corresponding to the sign function, i.e.,

$$S(t) = 1 \text{ if } t > 0, \quad S(t) = -1 \text{ if } t < 0 \quad \text{and} \quad S(0) = [-1, 1].$$

Lemma 5.1. *For any $w \in \widetilde{L}^2_B(Q)$ and any measurable function s such that $s(x) \in S(w(x))$, a.e. $x \in Q$, we have*

$$\int_Q s\, Hw\, dx \ge (\beta - \alpha)\int_Q |w|\, dx - \int_{\Sigma_-} |w|\,\ell\, d\Sigma. \tag{5.1}$$

Proof. By the property (2.11), we remark that we have

$$s\,Hw \;=\; \operatorname{sign}(w)\,Hw \quad \text{ a.e. in } \; Q, \tag{5.2}$$

where $\operatorname{sign}(t) = 1$ if $t > 0$, $\operatorname{sign}(t) = -1$ if $t < 0$ and $\operatorname{sign}(0) = 0$. Hence it is sufficient to prove (5.1) with s replaced by $\operatorname{sign}(w)$, which can be approximated in $L^2(Q)$ by the sequence of functions $s_\delta(w) \in \widetilde{L}^2_B(Q)$, where $s_\delta(t)$ are smooth functions approximating the sign, such that $|s_\delta(t)| \le 1$, $s'_\delta \ge 0$, $s_\delta(0) = 0$ and $s_\delta(t) \to \operatorname{sign}(t)$ as $\delta \to 0$ for $t \in \mathbb{R}$.

Integrating by parts and setting $m_\delta(t) = \int_0^t s_\delta(\tau)\,d\tau$ we have

$$\begin{aligned}\int_Q s_\delta(w)\,Hw\,dx \;&=\; \int_Q [b_0\,s_\delta(w)\,w + \mathbf{b}\cdot\nabla m_\delta(w)]\,dx\\ &=\; \int_Q [b_0\,s_\delta(w)\,w - (\nabla\cdot\mathbf{b})\,m_\delta(w)]\,dx \;+ \int_{\Sigma_+\cup\Sigma_-}(\mathbf{b}\cdot\mathbf{n})\,m_\delta(w)\,d\Sigma.\end{aligned} \tag{5.3}$$

Noting that $m_\delta(w) \to |w|$ in $L^2(Q)$ and in $L^2_\ell(\Sigma)$ as $\delta \to 0$, from (5.3) we obtain

$$\begin{aligned}\int_Q \operatorname{sign}(w)\,H(w)\,dx &= \int_Q \Big[b_0 - (\nabla\cdot\mathbf{b})\Big]\,|w|\,dx \;+ \int_{\Sigma_+\cup\Sigma_-}(\mathbf{b}\cdot\mathbf{n})\,|w|\,d\Sigma\\ &\ge \int_Q \Big[b_0 - \frac{1}{2}(\nabla\cdot\mathbf{b})\Big]\,|w|\,dx \;-\; \frac{1}{2}\int_Q(\nabla\cdot\mathbf{b})\,|w|\,dx \;+ \int_{\Sigma_-}(\mathbf{b}\cdot\mathbf{n})\,|w|\,d\Sigma\\ &\ge (\beta-\alpha)\int_Q |w|\,dx \;-\; \int_{\Sigma_-}|w|\,\ell\,d\Sigma\end{aligned}$$

and (5.1) follows from (5.2).

Remark 5.1. If $\beta \ge \alpha$ or if we assume instead

$$b_0(x) - \nabla\cdot\mathbf{b}(x) \;\ge\; 0 \quad \text{ a.e. in } \; Q, \tag{5.4}$$

the estimate (5.1) reduces to

$$\int_Q s\,Hw\,dx \;\ge -\int_{\Sigma_-}|w|\,\ell\,d\Sigma, \quad \forall\, w \in \widetilde{L}^2_B(Q), \tag{5.5}$$

provided $s(x) \in S(w(x))$ a.e. $x \in Q$.

These estimates may be used "to measure" the stability of the coincidence set in the nondegenerate case.

We recall the Lewy-Stampacchia type inequality (3.2) in the form

$$0 \;\le\; Hu - f \;\le\; (H\psi - f)^+ \quad \text{ a.e. in } \; Q$$

and, recalling that the solution u of the obstacle problem also solves the equation (4.4), we have $\zeta = \zeta(u) \ge 0$, where

$$\zeta \;=\; Hu - f \;=\; (H\psi - f)\,\chi_{\{u=\psi\}} \;=\; (H\psi - f)^+\,\chi_{\{u=\psi\}}. \tag{5.6}$$

Lemma 5.2. *Let u_i for $i = 1, 2$ denote the solution to* (2.10) *for data f_i, h_i and ψ_i under the assumptions* (2.6),(2.7), (2.8) *respectively and set $\zeta_i = \zeta(u_i)$. Then*

$$\|\zeta_1 - \zeta_2\|_{L^1(Q)} \;\le\; C_1\,(\|f_1 -_2\|_{L^2(Q)} + \|h_1 - h_2\|_{L^2_\ell(\Sigma_-)} + \|\psi_1 - \psi_2\|_\sim), \tag{5.7}$$

where $C_1 > 0$ is a constant independent of the data.

Proof. As in Proposition 2.1, by using the translation argument, we may assume $\psi_1 = \psi_2 = 0$ without loss of generality.

From (5.6) for $i = 1, 2$ we obtain

$$\zeta_1 - \zeta_2 \;=\; H(u_1 - u_2) - (f_1 - f_2) \quad \text{ a.e. in } \; Q. \tag{5.8}$$

We define almost everywhere in Q the measurable function s by

$$s(x) = \begin{cases} -1 & \text{on } \{u_1 < u_2\} \cup \{\zeta_2 < \zeta_1\}, \\ 0 & \text{on } \{u_1 = u_2\} \cap \{\zeta_1 = \zeta_2\}, \\ 1 & \text{on } \{u_1 > u_2\} \cup \{\zeta_2 > \zeta_1\}, \end{cases} \tag{5.9}$$

and we observe that $s \in S(u_1 - u_2)$ a.e. in Q. Indeed, if $\zeta_2 > \zeta_1 \geq 0$, by (5.6) and (3.1) we have $u_2 = 0$ and the subset $\{u_2 > u_1\} \cap \{\zeta_1 < \zeta_2\}$ cannot have positive measure. Similarly the same conclusion holds for $\{u_2 < u_1\} \cap \{\zeta_1 > \zeta_2\}$ and s given by (5.9) is a.e. well-defined.

Multiplying (5.8) by s and using (5.1) with $w = u_1 - u_2$, we obtain

$$\begin{aligned} \int_Q |\zeta_1 - \zeta_2| \, dx &= \int_Q (\zeta_2 - \zeta_1) \, s \, dx = - \int_Q s \, Hw \, dx + \int_Q s(f_1 - f_2) \, dx \\ &\leq (\alpha - \beta) \int_Q |u_1 - u_2| \, dx + \int_{\Sigma_-} |h_1 - h_2| \, \ell \, d\Sigma + \int_Q |f_1 - f_2| \, dx \end{aligned}$$

and using the estimate (2.15) we easily conclude (5.7).

Remark 5.2. Under the assumption (5.4) (or if $\alpha \leq \beta$) when $\psi_1 = \psi_2 = 0$, we may improve the estimate (5.7) by exactly the simpler one

$$\|\zeta_1 - \zeta_2\|_{L^1(Q)} \leq \|f_1 - f_2\|_{L^1(Q)} + \|h_1 - h_2\|_{L^1_\ell(\Sigma_-)}, \tag{5.10}$$

as a simple consequence of Remark 5.1 and the above proof.

As an immediate consequence of (5.7) we have the strong continuous dependence in $\widetilde{L}^2_B(Q)$ of the first order obstacle problem with respect to the data.

Theorem 5.1. *If we assume in the obstacle problem* (2.10)

$$f_n \to f \quad \textit{in } L^2(Q), \quad h_n \to h \quad \textit{in } L^2_B(\Sigma_-) \quad \textit{and} \quad \psi_n \to \psi \quad \textit{in } \widetilde{L}^2_B(Q),$$

the respective strong solutions satisfy the strong convergence

$$u_n \to u \quad \textit{in } \widetilde{L}^2_B(Q).$$

Perhaps a more interesting consequence of (5.7) can be obtained, exactly as in the elliptic theory of [10], for estimating locally the Lebesgue measure of the variation of the coincidence set associated with different data, under the local nondegeneracy assumption in an arbitrary measurable subset $\mathcal{O} \subset Q$:

$$f_1 - H\psi_1 \leq -\lambda < 0 \quad \text{and} \quad f_2 - H\psi_2 \leq -\lambda < 0 \qquad \text{a.e. on } \mathcal{O}. \tag{5.11}$$

For the coincidence subsets $I_1 = \{u_1 = \psi_1\}$ and $I_2 = \{u_2 = \psi_2\}$ we denote by $\div$ the symmetric difference

$$I_1 \div I_2 = I_1 \backslash I_2 \cup I_2 \backslash I_1,$$

where $A \backslash B = A \cap B^{\mathrm{C}}$ as usual.

Theorem 5.2. *Under the assumption* (5.11), *we have*

$$\textit{meas}\Big((I_1 \div I_2) \cap \mathcal{O}\Big) \leq \frac{C_1}{\lambda} (\|f_1 - f_2\|_{L^2(Q)} + \|h_1 - h_2\|_{L^2_\ell(\Sigma_-)} + \|\psi_1 - \psi_2\|_\sim). \tag{5.12}$$

Proof. It suffices to remark from (5.6) that (5.11) implies

$$\lambda \, |\chi_{\{u_1 = \psi_1\}} - \chi_{\{u_2 = \psi_2\}}| \leq |\zeta_1 - \zeta_2| \qquad \text{a.e. in } \mathcal{O}$$

and, using (5.7), (5.12) follows from

$$\text{meas}((I_1 \div I_2) \cap \mathcal{O}) = \int_{\mathcal{O}} |\chi_{\{u_1 = \psi_1\}} - \chi_{\{u_2 = \psi_2\}}| \, dx \leq \frac{1}{\lambda} \, \|\zeta_1 - \zeta_2\|_{L^1(Q)}.$$

Remark 5.3. Under the additional assumptions (5.4) and $\psi_1 = \psi_2 = 0$, (5.12) reduces to

$$\mathrm{meas}(\mathcal{O} \cap \Big(\{u_1 = 0\} \div \{u_2 = 0\})) \leq \frac{1}{\lambda}(\|f_1 - f_2\|_{L^1(Q)} + \|h_1 - h_2\|_{L^1_\ell(\Sigma_-)}).$$

§6. The Stability of the Evolution Problem

In this section we set $Q = \Omega\times]0,T[$, $T > 0$, and $\Sigma' = \partial\Omega\times]0,T[$, with the assumptions of Section 2 and where $\Omega \subset \mathbb{R}^n$ with $N = n+1$.

Then $\Sigma = \partial Q = \Sigma'\cup\Omega_0\cup\Omega_T$ ($\Omega_k = \Omega\times\{k\}$, $k = 0, T$) and we redefine $x = (x_1, \cdots, x_n) \in \overline{\Omega}$, $x_N = t \in [0,T]$, $b_i = a_i$, $i = 0, 1, \cdots, n$, and we set $b_N = 1$. Then the first order evolutionary operator becomes with $\partial_t = \partial/\partial t$:

$$Hu \,=\, \partial_t u + \mathbf{a}\cdot\nabla u + a_0\, u \,=\, \partial_t u + Au,$$

where the coefficients a_i, $i = 1, \cdots, n$ belong to $C^1(\overline{Q})$, $a_0 \in L^\infty(Q)$ and may depend on t but do not satisfy necessarily the coercivity assumption (2.6). We still define $\ell = \ell(x,t) = \Big|\sum_{i=1}^n n_i\, a_i(x,t)\Big|$ along Σ', with the external normal $\mathbf{n}$ to Ω, and analogously the norm of $\widetilde{L}^2_B(Q)$ is given by (2.3), where now

$$\|u\|^2_{L^2_\ell(\Sigma_-)} = \|u(x,0)\|^2_{L^2(\Omega)} + \|u|_{\Sigma'_-}\|^2_{L^2_\ell(\Sigma'_-)}$$

since $\Sigma_- = \Omega_0 \cup \Sigma'_-$, with $\Sigma'_- = \{(x,t) \in \Sigma\colon\, \mathbf{a}(x,t)\cdot\mathbf{n} < 0\}$.

The strong formulation of the evolutionary first order obstacle problem can now be rewritten in the form

$$u \in \widetilde{K}_\psi, \quad u|_{\Sigma'_-\cup\Omega_0} = h\colon \int_Q (\partial_t u + Au - f)\,(v - u)\,dx\,dt \,\geq\, 0, \quad \forall\, v \in K_\psi. \tag{6.1}$$

Here K_ψ and $\widetilde{K}_\psi$ are given also by (2.9) and

$$h|_{\Sigma'_-} = g \in L^2_\ell(\Sigma'_-) \quad \text{ and } \quad h|_{\Omega_0} = u_0 \in L^2(\Omega) \tag{6.2}$$

for $g = g(x,t)$ and $u_0 = u_0(x)$ compatible with the obstacle in the sense of (2.8).

It is clear that all the results of the preceding sections still hold for the solution of (6.1) as a consequence of the following proposition.

Proposition 6.1. *Under the preceding assumptions* (2.7), (2.8), (6.2) *the unique strong solution of* (6.1) *satisfies the estimates* (2.15) *and* (3.2).

Proof. If the operator $H = \partial_t + A$ does not satisfy the condition (2.6), we consider a constant $\mu > 0$ such that, for all $t \in [0,T]$,

$$\mu + a_0 - \frac{1}{2}\,(\nabla\cdot\mathbf{a}) \,\geq\, \beta \,>\, 0, \quad \forall\, x \in \Omega. \tag{6.3}$$

Setting $u = e^{\mu t}\,\widehat{u}$ it is easy to see that $\widehat{u}$ solve the coercive problem

$$\widehat{u} \in \widetilde{K}_{\widehat{\psi}}, \quad \widehat{u}|_{\Sigma'_-\cup\Omega_0} = \widehat{h}: \int_Q (\partial_t\widehat{u} + A\widehat{u} + \mu\,\widehat{u} - \widehat{f})\,(v - \widehat{u})\,dx\,dt \,\geq\, 0, \quad \forall\, v \in K_{\widehat{\psi}}$$

with $\widehat{f} = e^{-\mu t}\, f$, $\widehat{h} = e^{-\mu t}\, h$ and $\widehat{\psi} = e^{-\mu t}\,\psi$, for which all previous results apply.

When Q is a cylinder, we may use the integration by parts formula (2.4) in a subset

$Q_{\sigma,t} = \Omega\times]\sigma, t[$, $0 \le \sigma < t < T$, in the following form for any $w \in \widetilde{L}^2_B(Q_{\sigma,t})$ and $\lambda \in \mathbb{R}$:

$$\int_\sigma^\tau \int_\Omega w(\partial_t w + Aw)\, e^{\lambda t}\, dx\, dt \;=\; \int_\sigma^\tau \int_\Omega \Big(a_0 - \frac{1}{2}\,\nabla\cdot\mathbf{a} - \frac{\lambda}{2}\Big)\, w^2\, e^{\lambda t}\, dx\, dt$$
$$+\; \frac{1}{2}\int_\sigma^\tau \int_{\partial\Omega} (\mathbf{a}\cdot\mathbf{n})\, w^2\, e^{\lambda t}\, d\tau\, d\Gamma + \frac{1}{2}\int_\Omega [w^2(\tau)\, e^{\lambda\tau} - w^2(\sigma)\, e^{\lambda\sigma}]\, dx. \tag{6.4}$$

In order to consider the stability of the evolutionary problem as $t \to \infty$ we need to consider first the corresponding stationary problem in Ω.

We shall assume from now on that the coefficients of A are time independent, i.e.,

$$a_i = a_i(x) \in C^1(\overline{\Omega}),\quad i = 1,\cdots,n, \qquad \text{and} \qquad a_0 = a_0(x) \in L^\infty(\Omega), \tag{6.5}$$

and coercive, i.e., satisfying (6.3) with $\mu = 0$.

Decomposing $\Gamma = \partial\Omega$ as in Section 2, we set $\Gamma_- = \{u \in \partial\Omega\colon\ \mathbf{a}\cdot\mathbf{n} < 0\}$, $\ell(x) = |\mathbf{a}(x)\cdot\mathbf{n}(x)|$ and we consider the Hilbert spaces

$$L^2_A(\Omega) = \{v \in L^2(\Omega)\colon\ Av \in L^2(\Omega)\} \qquad \text{and} \qquad \widetilde{L}^2_A(\Omega) = \{v \in L^2_A(\Omega)\colon\ v|_\Gamma \in L^2_\ell(\Gamma)\}$$

and we also consider in $\widetilde{L}^2_A(\Omega)$ the norm

$$\|v\|^2_\# \;=\; \|Av\|^2_{L^2(\Omega)} + \|v\|^2_{L^2_\ell(\Gamma_-)}. \tag{6.6}$$

For the stationary problem, we assume

$$f_\# \in L^2(\Omega), \quad g_\# \in L^2_\ell(\Gamma_-) \ \text{ and } \ \psi_\# \in \widetilde{L}^2_A(\Omega) \ \text{ with } \ \psi_\# \le g_\# \ \text{ on } \Gamma_-, \tag{6.7}$$

and we consider the convex sets

$$K'_{\psi_\#} = \{v \in L^2(\Omega)\colon\ v \ge \psi_\# \ \text{ a.e. in } \Omega\} \qquad \text{and} \qquad \widetilde{K}'_{\psi_\#} = K'_{\psi_\#} \cap \widetilde{L}^2_A(\Omega). \tag{6.8}$$

Under the coercivity assumption (6.3) with $\mu = 0$ all the results in the previous sections also apply to the first order stationary problem:

$$u_\# \in \widetilde{K}'_{\psi_\#}, \quad u_\#|_{\Gamma_-} = g_\#\colon \ \int_\Omega (Au_\# - f_\#)\,(v - u_\#)\, dx \;\ge\; 0, \quad \forall\, v \in K'_{\psi_\#}. \tag{6.9}$$

Let, for $t \in [0,\infty[$,

$$\xi(t) \;=\; \int_t^{t+1} \left\{ \int_\Omega |f(\tau) - f_\#|^2\, dx \;+ \int_{\Gamma_-} |g(\tau) - g_\#|^2\, \ell\, d\Gamma \right\} d\tau, \tag{6.10}$$

$$\eta(t) \;=\; \int_t^{t+1} \left\{ \int_\Omega \Big|(\partial_t + A)\,\psi(t) - A\psi_\#\Big|^2\, dx \;+ \int_{\Gamma_-} |\psi(t) - \psi_\#|^2\, \ell\, d\Gamma \right\} d\tau. \tag{6.11}$$

Theorem 6.1. *Assuming* (6.5) *and* (6.3) *with* $\mu = 0$*, if* $\xi(t) + \eta(t) \to 0$ *as* $t \to \infty$*, then the solution* $u(t)$ *of* (6.1) *is asymptotically stable in the sense*

$$u(t) \to u_\# \ \ \textit{in } L^2(\Omega) \qquad \textit{as} \ \ t \to \infty, \tag{6.12}$$

where $u_\#$ *is the unique strong solution of* (6.9).

Proof. Using the translation argument we may assume that $\psi = \psi_\# = 0$ without loss of generality, since the assumption (6.11) reduces to (6.10) for the corresponding translated data.

Since $u(t)$ and $u_\#$ satisfy the complementary problem (3.1), by integration in Ω first and, afterwards multiplication by $e^{\beta t}$, we easily obtain for $w(t) = u(t) - u_\#$ and $\tau > \sigma \geq 0$,

$$\begin{aligned}
&\int_\sigma^\tau \int_\Omega w(\partial_t w + Aw)\, e^{\beta t}\, dx\, dt \\
&\leq \int_\sigma^\tau \int_\Omega w(f(t) - f_\#)\, e^{\beta t}\, dx\, dt \\
&\leq \frac{\beta}{2} \int_\sigma^\tau \int_\Omega w^2\, e^{\beta t}\, dx\, dt \;+\; \frac{1}{2\beta} \int_\sigma^\tau \int_\Omega |f(t) - f_\#|^2\, e^{\beta t}\, dx\, dt.
\end{aligned} \tag{6.13}$$

On the other hand, using the formula (6.4) with $\lambda = \beta$ and neglecting the nonnegative terms, by (6.3) with $\mu = 0$, we obtain

$$\begin{aligned}
\int_\sigma^\tau \int_\Omega w(\partial_t w + Aw)\, e^{\beta t}\, dx\, dt \geq \frac{\beta}{2} \int_\sigma^\tau \int_\Omega w^2\, e^{\beta t}\, dx\, dt \;-\; \frac{1}{2} \int_\sigma^\tau \int_{\Gamma_-} w^2\, e^{\beta t}\, \ell\, d\Gamma\, dt \\
+ \frac{1}{2} \int_\Omega [w^2(\tau)\, e^{\beta \tau} - w^2(\sigma)\, e^{\beta \sigma}]\, dx.
\end{aligned} \tag{6.14}$$

Hence, combining (6.13) with (6.14) we have for all $\tau > \sigma \geq 0$,

$$\begin{aligned}
&e^{\beta\tau} \int_\Omega w^2(\tau)\, dx \;-\; e^{\beta\sigma} \int_\Omega w^2(\sigma)\, dx \\
&\leq \int_\sigma^\tau \left\{ \frac{1}{\beta} \int_\Omega |f(t) - f_\#|^2\, dx \;+\; \int_{\Gamma_-} |g(t) - g_\#|^2\, \ell\, d\Gamma \right\} e^{\beta t}\, dt,
\end{aligned} \tag{6.15}$$

which implies the estimate for all $t > 0$, $t_0 \geq 0$:

$$\int_\Omega |u - u_\#|^2\, (t + t_0)\, dx \;\leq\; e^{-\beta t} \int_\Omega |u - u_\#|^2\, (t_0)\, dx \;+\; C_\beta \sup_{t_0 < \tau < t + t_0} \xi(\tau), \tag{6.16}$$

where $\beta > 0$ and $C_\beta = (1 \vee \frac{1}{\beta})\, [1 + (1 - e^{-\beta})^{-1}]$, by well-known results (see Remark 6.1, below).

From (6.16) with $t_0 = 0$ we obtain first that $u - u_\#$ is bounded in $L^\infty(0, \infty; L^2(\Omega))$ and, afterwards, that $\xi(t) \to 0$ as $t \to \infty$ implies the conclusion

$$\|u(t) - u_\#\|_{L^2(\Omega)} \to 0 \quad \text{as} \;\; t \to +\infty.$$

Remark 6.1. This type of global behaviour of solutions is similar to other nonlinear evolution equations (see, for instance, [9]), in particular, in monotone parabolic variational inequalities. The passage of (6.15) to (6.16) follows by the elementary standard estimate (after changing variables)

$$\begin{aligned}
&\int_0^t \varphi(t_0 + s)\, e^{\beta(s-t)}\, ds \\
&= \sum_{k=0}^{n-1} \int_k^{k+1} \varphi(t_0 + s)\, e^{-\beta(t-s)}\, ds \;+\; \int_n^t \varphi(t_0 + s)\, e^{-\beta(t-s)}\, ds \\
&\leq M_t \Big(\sum_{k=0}^{n-1} e^{-\beta(t+k-1)} + 1 \Big) \;\leq\; M_t \Big(\sum_{j=1}^{n-1} e^{-j\beta} + 1 \Big) = M_t [(1 - e^{-\beta})^{-1} + 1],
\end{aligned}$$

where $M_t = \sup\limits_{t < \tau < t + t_0} \xi(\tau)$ and $\xi(t) = \int_t^{t+1} \varphi(\tau)\, d\tau$ as in (6.10).

As in Section 4 we can also show that (6.12) with the nondegeneracy condition (4.5) for the stationary problem yields the asymptotic stability of the coincidence sets.

Theorem 6.2. Under the assumptions of Theorem 6.1, let the condition (4.5) be fulfilled for $\psi_\#$ and $f_\#$. Then, if $\chi(t) = \chi_{\{u(t)=\psi(t)\}}$ and $\chi_\# = \chi_{\{u_\#=\psi_\#\}}$ denote respectively the characteristic functions of the coincidence sets of the evolutionary and the stationary problems (6.1) and (6.9), we have

$$\chi(t) \to \chi_\# \quad \text{in } \ L^p(\Omega), \ \ 1 \le p < \infty \ \text{ as } t \to \infty. \tag{6.17}$$

Proof. We can argue as in the proof of Theorem 4.1, by passing to the limit $t \to \infty$ in the equation for $u = u(t)$

$$Hu - (H\psi - f)\,\chi = f \qquad \text{a.e. } \ \Omega, \quad t > 0, \tag{6.18}$$

in the sense of distributions, by noting that $u(t) \to u_\#$ in $L^2(\Omega)$ implies $\partial_t u(t) \rightharpoonup 0$ in a weak sense. Here we can use the argument of Lions (see [6], p. 509) for the translated functions $w(t) = u(t) - \psi(t)$ and $w_\# = u_\# - \psi_\#$, by noting that $\widehat{w}(t) = w(t) - w_\# \to 0$ as $t \to \infty$ in the sense

$$\int_t^{t+1} \|\widehat{w}(\tau)\|^2_{L^2(\Omega)}\, d\tau \ = \int_0^1 \|\widehat{w}(\sigma + t)\|^2_{L^2(\Omega)}\, d\sigma \ \longrightarrow \ 0.$$

Then the argument of Theorem 4.1 shows first that, if we denote by $\widehat{\chi} \in L^\infty(0,\infty; L^\infty(0, 1; L^\infty(\Omega)))$ the function

$$\widehat{\chi}(t): \quad]0,1[\, \ni \sigma \ \longrightarrow \ \chi(\sigma + t) \in L^\infty(0,\infty; L^\infty(\Omega)),$$

we obtain from (6.18) that as $t \to \infty$

$$\widehat{\chi}(t) \to \widehat{\chi}_\# = \chi_\# \qquad \text{first in } \ L^\infty(0,1; L^\infty(\Omega))\text{-weak*},$$

and, since they are characteristic functions, also strongly in $L^p(0,1; L^p(\Omega))$ for any $1 \le p < \infty$, which yields (6.17).

References

[1] Bardos, C., Problèmes aux limites pour les équations aux dérivées partielles du premier ordre [J], *Ann. Scient. Éc.Norm. Sup.,* 4e série, 3(1970), 185–233.

[2] Bensoussan, A. & Lion, J. L., Inéquations variationelles nonlinéaires du premier et du second ordre [J], *C. R. Acad. Sci. Paris,* **276**(1973), 1411–1415.

[3] Friedman, A., Variational principles and free boundary problems [M], Wiley, New York, 1982.

[4] Lévi, L., Obstacle problems for scalar conservation laws [J], *M2AN Math. Model. Numer. Anal.,* **35**(2001), 575–593.

[5] Lévi, L. & Vallet, G., Entropy solutions for first-order quasilinear equations related to a bilateral obstacle condition in a bounded domain [J], *Chin. Ann. Math.,* **22B:**1(2001), 93–114.

[6] Lions, J. L., Quelques Méthodes de Résoution des Problèmes Aux Limites Non-Linéaires [M], Dunod, Gauthier-Villars, Paris, 1969.

[7] Lions, J. L., Partial differential inequalities [J], *Russian Math. Surveys,* **27**(1971), 91–161.

[8] Mignot, F. & Puel J. P., Inéquations variationelles et quasi variationelles hyperboliques du premier ordre [J], *J. Math. Pures Appl.,* **55**(1976), 353–378.

[9] Rodrigues, J. F., Some remarks on the asymptotic behaviour of strong solutions to monotone parabolic variational inequalities [J], *Rendiconti Mat.,* **4:**7(1984), 458–470.

[10] Rodrigues, J. F., Obstacle problems in mathematical physics [M], North-Holland,Amsterdam, 1987.