Chin. Ann. of Math.
23B:2(2002),267-276.

INSTABILITY OF TRAVELING WAVES OF THE KURAMOTO-SIVASHINSKY EQUATION***

W. STRAUSS* WANG GUANXIANG**

(*Dedicated to the memory of Jacques-Louis Lions*)

Abstract

Consider any traveling wave solution of the Kuramoto-Sivashinsky equation that is asymptotic to a constant as $x \to +\infty$. The authors prove that it is nonlinearly unstable under H^1 perturbations. The proof is based on a general theorem in Banach spaces asserting that linear instability implies nonlinear instability.

Keywords Traveling wave, Kuramoto-Sivashinsky equation, Instability
2000 MR Subject Classification 35K55, 35K90
Chinese Library Classification O175.26 **Document Code** A
Article ID 0252-9599(2002)02-0267-10

§1. Introduction

The Kuramoto-Sivashinsky equation

$$u_t + u_{xxxx} + u_{xx} + uu_x = 0 \tag{0.1}$$

was derived by Kuramoto[2] as a model describing phase turbulence in reaction-diffusion systems and independently by Sivashinsky[3] as a model of flame propagation. There are many numerical and some theoretical results showing that some of its solutions engage in very complicated dynamical behavior.

A traveling wave solution $u = \varphi(x - ct)$ satisfies, after one integration, the third-order equation

$$\varphi''' + \varphi' + \frac{1}{2}(\varphi - c)^2 = k \tag{0.2}$$

where k is a constant. A special case is a steady state $c = 0$. This ordinary differential equation has been studied extensively. Numerical studies[6−20] indicate the existence of heteroclinic and homoclinic orbits, as well as periodic and quasiperiodic solutions. Theoretical results include the existence of periodic solutions and heteroclinic orbits[4,5]. In particular,

Manuscript received August 15, 2001
*Department of Mathematics and LCDS, Brown University, Providence RI 02912, USA.
**School of Mathematical Sciences, Peking University, Beijing, 100871, China.
Visiting Brown University from September 2000 to August 2001.
***Project Supported in part by NSF Grant DMS-0071838.

Troy[4] proved that if $k=1$, there exist at least two distinct odd solutions of (0.2) such that $\varphi(x) \to c \mp \sqrt{2}$ as $x \to \pm\infty$. He conjectured that there are an infinite number of different ones. Furthermore for $k \neq 1$ there are probably many others.

In this paper we consider any traveling wave solution $\varphi(x-ct)$ of (0.1) that approaches a constant as $x \to +\infty$. Then we consider solutions $u(x,t)$ of (0.1) with initial data $u(x,0)$ arbitrarily near $\varphi(x)$ in the $H^1(\mathbf{R})$ norm. We prove that there exist such solutions that do not remain near $\varphi(x-ct)$ in the $H^1(\mathbf{R})$ norm at some later times. The instability of the traveling waves is a hint of the complexity of the dynamics of (0.1).

Our proof is based on the principle of linearization. We prove that the essential spectrum of the linearized generator meets the right half-plane and thus generates modes $e^{\lambda t}$ with $\mathcal{R}\lambda > 0$ (Lemma 2.1). Then we invoke a general theorem that asserts that linearized instability implies nonlinear instability (Theorem 1.1).

Theorem 1.1 is a slight generalization of an earlier theorem[1] concerning nonlinear semigroups in a Banach space X. In the present case we have two Banach spaces $X \subset Z$, the linear semigroup is smoothing (mapping Z into X), while the nonlinear term loses regularity (mapping X into Z). The gain and loss of regularity compensate for each other.

§1. The Abstract Theorem

Consider an evolution equation

$$\frac{du}{dt} = Lu + F(u), \tag{1.1}$$

where L is a linear operator that generates a strongly continuous semigroup e^{tL} on a Banach space X, and F is a strongly continuous operator such that $F(0)=0$. We focus on the instability of the zero solution of equation (1.1). About such a problem, the following question was addressed in a previous article[1].

If the spectrum of L meets the right half-plane $\{\mathcal{R}\lambda > 0\}$, does it follow that the zero solution of (1.1) is nonlinearly unstable?

Here, the zero solution is called nonlinearly stable in X if for any $\epsilon > 0 \quad \exists \delta > 0$ such that $\|u_0\|_X < \delta$ implies that the unique solution $u \in C([0,\infty);X)$ of equation (1.1) with $u(0)=u_0$ satisfies $\sup\limits_{0 \le t < \infty} \|u(t)\|_X < \epsilon$. Otherwise, it is called nonlinearly unstable.

In [1], the authors considered the whole problem in only one space X, that is to say, the nonlinear operator maps X into X. However, many equations possess nonlinear terms that include derivatives and therefore F maps into a larger Banach space Z. If, therefore, the linear part is smoothing, mapping Z back into X, then we can recover the nonlinear instability as before. This is the content of the following theorem.

Theorem 1.1. *Assume the following.*

(i) *X, Z are two Banach spaces with $X \subset Z$ and $\|u\|_Z \le C_1 \|u\|_X$ for $u \in X$.*

(ii) *L generates a strongly continuous semigroup e^{tL} on the space Z, and the semigroup e^{tL} maps Z into X for $t>0$, and $\int_0^1 \|e^{tL}\|_{Z\to X}\, dt = C_4 < \infty$.*

(iii) *The spectrum of L on X meets the right half-plane, $\{\mathcal{R}\lambda > 0\}$.*

(iv) *$F: \quad X \to Z$ is continuous and $\exists \rho_0 > 0, C_3 > 0, \alpha > 1$ such that $\|F(u)\|_Z \le C_3 \|u\|_X^\alpha$ for $\|u\|_X < \rho_0$.*

Then the zero solution of (1.1) *is nonlinearly unstable in the space X.*

Remark 1.1. If $Z = X$, the theorem reduces to the theorem in [1].

To prove the theorem, we need the following two lemmas cited from [1]. For brevity, the proofs of the lemmas are omitted. The first lemma asserts the existence of an approximate eigenvector v corresponding to an eigenvalue of maximal growth.

Lemma 1.1. *Let the spectrum of e^L on X be denoted by $\sigma_X(e^L)$. Let $e^\lambda \in \sigma_X(e^L)$ such that $|e^\lambda|$ equals the spectral radius of e^L on X. For every $\eta > 0$ and every integer $m > 0$, there exists $v \in X$ such that*

$$\|(e^{mL} - e^{m\lambda})v\|_X < \eta\|v\|_X, \tag{1.2}$$

$$\|e^{tL}v\|_X \le 2Ke^{t\mathcal{R}\lambda}\|v\|_X, \quad \forall t,\ 0 \le t \le m, \tag{1.3}$$

where $K = \sup\{\|e^{\theta L}\|_{X\to X} : 0 \le \theta \le 1\}$ and $\mathcal{R}\lambda$ means the real part of λ.

The second lemma asserts that the whole semigroup grows at approximately the same rate as the eigenvalue.

Lemma 1.2. *Under the assumption of Lemma 1.1, for all $\epsilon > 0$, there exists a constant C_ϵ so that for all $0 \le t < \infty$ we have*

$$e^{\mathcal{R}\lambda t} \le \|e^{tL}\|_{X\to X} \le C_\epsilon e^{(\mathcal{R}\lambda+\epsilon)t}.$$

Proof of Theorem 1.1. If $u \in C([0,T);X) \quad (T \le \infty)$ is a solution of (1.1) with initial data $u(0) = v \in X$, then it formally satisfies the associated integral equation

$$u(t) = e^{tL}v + \int_0^t e^{(t-\tau)L}F(u(\tau))\,d\tau, \qquad 0 \le t < T. \tag{1.4}$$

We are going to prove that there exists a universal constant $\epsilon_0 > 0$ such that $\sup\limits_{0\le t<T}\|u(t)\|_X > \epsilon_0$ no matter how small $\|v\|_X$ may be.

Let us first define some quantities used below. Let

$$\mu = e^\lambda \tag{1.5}$$

as in Lemma 1.1. Choose

$$\epsilon = \frac{(\alpha-1)\mathcal{R}\lambda}{2} \tag{1.6}$$

and C_ϵ as in Lemma 1.2. Let $C_2 = \|e^L\|_{Z\to X}$. Define k by

$$k^{\alpha-1} = 2|\mu|^\alpha C_3\Big[2K + \frac{1}{2|\mu|}\Big]^\alpha\Big[\frac{2C_\epsilon C_2}{(\alpha-1)\mathcal{R}\lambda}e^{-\alpha\mathcal{R}\lambda} + C_4\Big]. \tag{1.7}$$

Let δ be free to remain arbitrarily small within the interval $(0,\delta_0)$ with

$$\delta_0 \equiv \min\Big\{\frac{1}{k}, 1, \frac{\rho_0}{2}\Big\}. \tag{1.8}$$

Let T^* be the integer in the interval $(b, b+1]$ where

$$b = \ln\Big(\frac{1}{\delta k}\Big)/\ln|\mu| > 0. \tag{1.9}$$

Note that (1.5) and (1.9) imply that

$$\frac{1}{k} < \delta e^{T^*\mathcal{R}\lambda} \le \frac{|\mu|}{k}, \tag{1.10}$$

and T^* is dependent on δ as well as on μ and k. We may assume that the zero solution is stable. Thus there exists $\delta' > 0$ such that if $\|v\|_X = \delta < \delta'$, then there exists a unique solution $u \in C([0,\infty);X)$ of the integral equation (1.4). Let v be given by Lemma 1.1 with $m = T^*$ and $\eta = \frac{1}{4k}$, we take $\|v\|_X = \delta$. Now define

$$T = \sup\Big\{t : \|u(\tau) - e^{\tau L}v\|_X < \frac{\delta e^{\mathcal{R}\lambda\tau}}{2|\mu|} \text{ and } \|u(\tau)\|_X < \frac{\rho_0}{2} \quad \text{for} \quad 0 < \tau \le t\Big\}. \tag{1.12}$$

Clearly $T > 0$. By (1.4) we have, for $0 < t \le \min\{T^*, T\}$,

$$\|u(t) - e^{tL}v\|_X \le \int_0^{t-1} \|e^{(t-\tau-1)L}\|_{X\to X} \|e^L\|_{Z\to X}\|F(u(\tau))\|_Z \,d\tau + \int_{t-1}^{t} \|e^{(t-\tau)L}\|_{Z\to X} \quad \|F(u(\tau))\|_Z \,d\tau. \tag{1.13}$$

Taking $\epsilon = \frac{(\alpha-1)}{2}\mathcal{R}\lambda$ in Lemma 1.2 and using the assumptions of the theorem, we have

$$\|u(t) - e^{tL}v\|_X \le \int_0^{t-1} C_\epsilon e^{\frac{\alpha+1}{2}\mathcal{R}\lambda(t-\tau-1)}C_2C_3\|u(\tau)\|_X^\alpha \,d\tau + \int_{t-1}^{t} \|e^{(t-\tau)L}\|_{Z\to X}C_3\|u(\tau)\|_X^\alpha \,d\tau. \tag{1.14}$$

Within these integrals we use (1.3) and (1.12) to obtain

$$\|u(\tau)\|_X \le \|e^{\tau L}v\|_X + \|u(\tau) - e^{\tau L}v\|_X \le (2K + \frac{1}{2|\mu|})\delta e^{\tau\mathcal{R}\lambda}. \tag{1.15}$$

Substituting (1.15) into (1.14), we have

$$\begin{aligned}\|u(t) - e^{tL}v\|_X &\le C_2C_3C_\epsilon\Big(2K + \frac{1}{2|\mu|}\Big)^\alpha \delta^\alpha e^{\frac{\alpha+1}{2}\mathcal{R}\lambda(t-1)} \int_0^{t-1} e^{\frac{\alpha-1}{2}\mathcal{R}\lambda\tau}\,d\tau \\ &\quad + C_3\Big(2K + \frac{1}{2|\mu|}\Big)^\alpha \delta^\alpha \int_{t-1}^{t} \|e^{(t-\tau)L}\|_{Z\to X}e^{\tau\alpha\mathcal{R}\lambda}\,d\tau \\ &< C_2C_3C_\epsilon\Big(2K + \frac{1}{2|\mu|}\Big)^\alpha \delta^\alpha e^{\frac{\alpha+1}{2}\mathcal{R}\lambda(t-1)} \frac{2}{(\alpha-1)\mathcal{R}\lambda} e^{\frac{(\alpha-1)}{2}\mathcal{R}\lambda(t-1)} \\ &\quad + C_4C_3\Big(2K + \frac{1}{2|\mu|}\Big)^\alpha \delta^\alpha e^{\alpha\mathcal{R}\lambda t} \\ &= \frac{k^{\alpha-1}}{2|\mu|^\alpha}(\delta e^{\mathcal{R}\lambda t})^\alpha\end{aligned}$$

by (1.7). Then we have

$$\|u(t) - e^{tL}v\|_X < \frac{k^{\alpha-1}}{2|\mu|^\alpha}(\delta e^{\mathcal{R}\lambda t})^\alpha \quad \text{for} \qquad 0 \le t \le \min(T, T^*). \tag{1.16}$$

Now if $T \le T^*$, then we claim that $\|u(T)\|_X \ge \frac{\rho_0}{2}$. Indeed, if $T \le T^*$ and $\|u(T)\|_X < \frac{\rho_0}{2}$, then by definition (1.12) of T, we have

$$\|u(T) - e^{TL}v\|_X = \frac{\delta e^{\mathcal{R}\lambda T}}{2|\mu|}.$$

Combining it with (1.16) for $t = T$, we obtain

$$\frac{\delta e^{\mathcal{R}\lambda T}}{2|\mu|} < \frac{k^{\alpha-1}}{2|\mu|^\alpha}(\delta e^{\mathcal{R}\lambda T})^\alpha,$$

that is,

$$\delta e^{\mathcal{R}\lambda T} > \frac{|\mu|}{k},$$

which means $T > T^*$ by (1.11) and leads to a contradiction. Thus the claim is proven. Next, if $\|u(T)\|_X \neq \frac{\rho_0}{2}$, we have $T > T^*$. Choose $t = T^*$ so that by (1.11) we have

$$\|u(T^*) - e^{T^*L}v\|_X < \frac{k^{\alpha-1}}{2|\mu|^\alpha}(\delta e^{\mathcal{R}\lambda T^*})^\alpha \leq \frac{k^{\alpha-1}}{2|\mu|^\alpha}\Big(\frac{|\mu|}{k}\Big)^\alpha = \frac{1}{2k}. \tag{1.19}$$

So

$$\|u(T^*)\|_X \geq \|e^{T^*L}v\|_X - \frac{1}{2k}. \tag{1.20}$$

On the other hand, taking $m = T^*$ and $\eta = \frac{1}{4k}$ in Lemma 1.1, (1.2) implies

$$\|e^{T^*L}v\|_X \geq \|e^{T^*\lambda}v\|_X - \frac{1}{4k}\|v\|_X.$$

Since $\|e^{T^*\lambda}v\|_X = e^{T^*\mathcal{R}\lambda}\|v\|_X$ and $\|v\|_X = \delta$, we get by (1.11)

$$\|e^{T^*L}v\|_X \geq e^{\mathcal{R}\lambda T^*}\delta - \frac{\delta}{4k} > \frac{1}{k} - \frac{\delta}{4k}.$$

Hence (1.20) implies

$$\|u(T^*)\|_X \geq \frac{1}{k} - \frac{\delta}{4k} - \frac{1}{2k} > \frac{1}{4k}, \tag{1.21}$$

since $\delta < 1$.

Therefore, there exists a time t (either T or T^*) at which

$$\|u(t)\|_X \geq \min\Big\{\frac{1}{4k}, \frac{\rho}{2}\Big\} \equiv \epsilon_0,$$

and ϵ_0 is a universal constant independent of the size of the initial data v.

Remark 1.2. The proof shows that there exist $C > 0$ and $\epsilon_0 > 0$ such that for all sufficiently small positive δ, there is a solution u that satisfies $\|u(0)\|_X < \delta$ but

$$\sup_{0\leq t\leq C|\log\delta|} \|u(t)\| \geq \epsilon_0.$$

Thus the escape time occurs logarithmically soon.

§2. Application to the Kuramoto-Sivashinsky Equation

The Kuramoto-Sivashinsky equation in one dimension is

$$v_t + v_{x^4} + v_{x^2} + \frac{1}{2}(v_x)^2 = 0, \quad -\infty < x < \infty. \tag{2.1}$$

With $u = v_x$ it can be written as

$$u_t + u_{x^4} + u_{x^2} + uu_x = 0, \quad -\infty < x < \infty. \tag{2.2}$$

If $\varphi(x - ct)$ is a traveling wave solution of (2.2), then φ satisfies the ordinary differential equation

$$\varphi''' + \varphi' + \frac{1}{2}(\varphi - c)^2 = k \tag{2.3}$$

where k is a constant. If $c = 0$, then φ is a steady-state solution. Troy[4] proved that if $k = 1$, then equation (2.3) admits at least two odd solutions satisfying

$$\lim_{x\to\infty}\varphi(x) = c - \sqrt{2}, \qquad \lim_{x\to-\infty}\varphi(x) = c + \sqrt{2}.$$

The goal of this section is to prove the following theorem.

Theorem 2.1. *All the traveling waves $\varphi(x-ct)$ of the Kuramoto-Sivashinsky equation satisfying $\varphi \in L^\infty(\mathbf{R})$, $\quad \varphi_x, \varphi_{xx} \in L^2(\mathbf{R})$ and $\varphi - b_+ \in L^2([0,\infty))$ are nonlinearly unstable in the space $H^1(\mathbf{R})$.*

This means that there exist positive ϵ_0 and C_0, a sequence $\{u_n\}$ of solutions of the K-S equation, and a sequence of times $0 \le t_n \le C_0 \log n$ such that $\|u_n(0)-\varphi\|_{H^1(\mathbf{R})} \to 0$ but $\|u_n(t_n) - \varphi(\cdot - ct_n)\|_{H^1(\mathbf{R})} \ge \epsilon_0$.

If $\varphi(x-ct) \in H^1(\mathbf{R})$ is a traveling-wave solution of the K-S equation (2.2), then letting $w(x,t) = u(x,t) - \varphi(x-ct)$, we have

$$w_t + w_{x^4} + w_{x^2} + \varphi w_x + \varphi_x w + w w_x = 0, \quad -\infty < x < \infty, \tag{2.5}$$

with initial value

$$w(x,0) = w_0(x) \equiv u_0(x) - \varphi(x). \tag{2.6}$$

So the stability of traveling-wave solutions of (2.2) is translated into the stability of the zero solution of (2.5). In order to prove Theorem 2.1, taking $Z \equiv L^2(\mathbf{R})$, $\quad X = H^1(\mathbf{R})$, we need to prove that the four conditions of Theorem 1.1 are satisfied by the associated equation (2.5).

Denote the linear partial differential operator in (2.5) by $L = -(\partial_x^4 + \partial_x^2 + \varphi\partial_x + \varphi_x) \equiv L_0 - [(\varphi - b_+)\partial_x + \varphi_x]$ with $L_0 = -(\partial_x^4 + \partial_x^2 + b_+\partial_x)$. Then (2.5) may be rewritten in the form (1.1),

$$w_t = Lw + F(w), \tag{2.7}$$

where $F(w) = -ww_x$. Note that F maps $H^1(\mathbf{R})$ into $L^2(\mathbf{R})$ and satisfies

$$\|F(w)\|_{L^2} \le \|w\|_{H^1}^2. \tag{2.8}$$

This proves Condition (iv) of Theorem 1.1 with $C_3 = 1$ and $\alpha = 2$.

To prove Condition (ii) in Theorem 1.1, we need the following two lemmas.

Lemma 2.1. *Let $L_0 = -(\partial_x^4 + \partial_x^2 + b_+\partial_x)$ for any real constant b_+. Then*

$$\|e^{tL_0}\|_{H^m \to H^m} \le e^{\frac{t}{4}} \quad \textit{for} \quad m \in \mathbf{R}, \quad 0 \le t < \infty, \tag{2.8}$$

$$\|e^{tL_0}\|_{L^2 \to H^1} \le a(t) \equiv 4t^{-\frac{1}{4}} \quad \textit{for} \quad 0 < t \le 1. \tag{2.9}$$

Proof. We write $u(x,t) = e^{tL_0}u_0(x)$. By Fourier transformation,

$$\hat{u}(\xi,t) = e^{-t(\xi^4 - \xi^2 + i\xi)}\hat{u}_0(\xi).$$

$$\|u(t)\|_{H^m}^2 = \int_{-\infty}^{\infty} (1+\xi^2)^m |\hat{u}(\xi,t)|^2 \, d\xi = \int_{-\infty}^{\infty} (1+\xi^2)^m e^{-2t(\xi^4-\xi^2)} |\hat{u}_0(\xi)|^2 \, d\xi$$
$$\le \sup_{\xi \in \mathbf{R}} e^{-2t(\xi^4 - \xi^2)} \int_{-\infty}^{\infty} (1+\xi^2)^m |\hat{u}_0(\xi)|^2 \, d\xi = e^{\frac{t}{2}} \|u_0\|_{H^m}^2.$$

Hence

$$\|e^{tL_0}\|_{H^m \to H^m} \le e^{\frac{t}{4}}.$$

On the other hand, letting $s = \xi^2$, we have

$$\|u(t)\|_{H^1}^2 \le \sup_{s \in \mathbf{R}^+} f(s) \int_{-\infty}^{\infty} |\hat{u}_0(\xi)|^2 \, d\xi$$

with $f(s) = (1+s)e^{-2t(s^2-s)},\ t > 0$. Elementary computation shows that

$$\sup_{s>0} f(s) \le \Big(\frac{3}{2} + \frac{1}{2}t^{-\frac{1}{2}}\Big)e^{\frac{t}{2}}.$$

Thus

$$\|u(x,t)\|_{H^1} \le \Big(\frac{3}{2} + \frac{1}{2}t^{-\frac{1}{2}}\Big)^{\frac{1}{2}} e^{\frac{t}{4}} \|u_0\|_{L^2},$$

and

$$\|e^{tL_0}\|_{L^2\to H^1} \le \Big(\frac{3}{2} + \frac{1}{2}t^{-\frac{1}{2}}\Big)^{\frac{1}{2}} e^{\frac{t}{4}} \le 4t^{-\frac{1}{4}} \quad \text{for} \quad 0 < t \le 1,$$

since $e^{\frac{t}{4}} \le e^{\frac{1}{4}} < 2$. Thus Lemma 2.1 has been proved.

The following lemma proves Condition (ii).

Lemma 2.2. *Let* $L = -[\partial_x^4 + \partial_x^2 + \varphi(x)\partial_x + \varphi'(x)] \equiv L_0 - (\varphi - b_+)\partial_x - \varphi_x$ *with* $\varphi \in L^\infty(\mathbf{R}), \quad \varphi_x \in L^2(\mathbf{R}), \quad \chi_{[0,\infty)}(\varphi - b_+) \in L^2(\mathbf{R})$. *Then*

$$\|e^{tL}\|_{L^2\to H^1} \le C_5 t^{-\frac{1}{4}} \quad for \quad 0 < t \le 1, \tag{2.10}$$

$$\|e^{tL}\|_{H^1\to H^1} \le C_6 < \infty \quad for \quad 0 \le t \le 1, \tag{2.11}$$

where the constants C_5, C_6 *are defined by* (2.17), (2.20) *below.*

Proof. Consider the initial value problem

$$u_t = Lu = L_0 u - [\varphi(x) - b_+]\partial_x u - \varphi'(x)u,$$
$$u(x,0) = u_0(x), \quad x \in \mathbf{R}.$$

Then $u(x,t) = e^{tL}u_0(x), \quad t \ge 0, \quad x \in \mathbf{R}$. Thus

$$u(t) = e^{tL_0}u_0 - \int_0^t e^{(t-\tau)L_0}[(\varphi - b_+)\partial_x u + \varphi' u]\, d\tau.$$

Denote $A = \|\varphi - b_+\|_{L^\infty}, \quad B = \|\varphi'\|_{L^2}$. Then for $0 < t \le 1$,

$$\begin{aligned}\|u(t)\|_{H^1} \le{}& \|e^{tL_0}\|_{L^2\to H^1}\|u_0\|_{L^2} + \int_0^t \|e^{(t-\tau)L_0}\|_{L^2\to H^1}\|\varphi - b_+\|_{L^\infty}\|\partial_x u\|_{L^2}\, d\tau \\ &+ \int_0^t \|e^{(t-\tau)L_0}\|_{L^2\to H^1}\|\varphi'\|_{L^2}\|u\|_{L^\infty}\, d\tau \qquad (2.12)\\ \le{}& a(t)\|u_0\|_{L^2} + (A+B)\int_0^t a(t-\tau)\|u(\tau)\|_{H^1}\, d\tau \quad \text{for} \quad 0 < t \le 1. \qquad (2.13)\end{aligned}$$

By iteration,

$$\begin{aligned}\|u(t)\|_{H^1} \le{}& a(t)\|u_0\|_{L^2} + (A+B)\int_0^t a(t-\tau)\Big[a(\tau)\|u_0\|_{L^2} \\ &+ (A+B)\int_0^\tau a(\tau - s)\|u(s)\|_{H^1}\, ds\Big]\, d\tau \\ ={}& a(t)\|u_0\|_{L^2} + (A+B)\int_0^t a(t-\tau)a(\tau)\|u_0\|_{L^2}\, d\tau \\ &+ (A+B)^2\int_0^t\int_0^\tau a(t-\tau)a(\tau - s)\|u(s)\|_{H^1}\, ds d\tau. \qquad (2.14)\end{aligned}$$

The second term on the right side of (2.14) is

$$\begin{aligned}&(A+B)\int_0^t a(t-\tau)a(\tau)\|u_0\|_{L^2}\,d\tau\\&=(A+B)\|u_0\|_{L^2}\int_0^t 4(t-\tau)^{-\frac14}4\tau^{-\frac14}\,d\tau\\&=16(A+B)C_8t^{\frac12}\|u_0\|_{L^2}\quad\text{for}\quad 0<t\le 1,\end{aligned}\tag{2.15}$$

where $C_8=\int_0^1(1-r)^{-\frac14}r^{-\frac14}\,dr$. By exchanging the order of integration, we get from the third term on the right side of (2.14),

$$\int_0^t\int_0^\tau a(t-\tau)a(\tau-s)\|u(s)\|_{H^1}\,dsd\tau=\int_0^t\Big[\int_s^t a(t-\tau)a(\tau-s)\,d\tau\Big]\|u(s)\|_{H^1}\,ds.$$

Now

$$\begin{aligned}\int_s^t a(t-\tau)a(\tau-s)\,d\tau&=16\int_s^t(t-\tau)^{-\frac14}(\tau-s)^{-\frac14}\,d\tau\\&=16C_8(t-s)^{\frac12}\le 16C_8\ \text{ for }\ 0<s\le t\le 1.\end{aligned}\tag{2.16}$$

Therefore (2.13)–(2.16) imply

$$\|u\|_{H^1}\le[a(t)+16C_8(A+B)]\|u_0\|_{L^2}+\int_0^t 16C_8(A+B)^2\|u(s)\|_{H^1}\,ds\quad\text{for}\quad 0<t\le 1.$$

By Gronwall's inequality, we get

$$\|u\|_{H^1}\le[4t^{-\frac14}+16C_8(A+B)]\exp[16C_8(A+B)^2t]\|u_0\|_{L^2}\quad\text{for}\quad 0<t\le 1.$$

So with the constant

$$C_5=[4+16C_8(A+B)]e^{16C_8(A+B)^2},\tag{2.17}$$

we have

$$\|u\|_{H^1}\le C_5t^{-\frac14}\|u_0\|_{L^2}\quad\text{for}\quad 0<t\le 1.\tag{2.18}$$

Thus (2.10) has been proven. To prove (2.11), replacing the first term on the right side of (2.12) by $\|e^{tL_0}\|_{H^1\to H^1}\|u_0\|_{H^1}$ and using (2.8), we have

$$\|u(t)\|_{H^1}\le e^{\frac t4}\|u_0\|_{H^1}+(A+B)\int_0^t a(t-\tau)\|u(\tau)\|_{H^1}\,d\tau\quad\text{for}\quad 0<t\le 1.\tag{2.19}$$

Similarly iterating and computing as above, we obtain

$$\|u(t)\|_{H^1}\le[2+16(A+B)]e^{16C_8(A+B)^2}\|u_0\|_{H^1}\equiv C_6\|u_0\|_{H^1}\quad\text{for}\quad 0<t\le 1.\tag{2.20}$$

Hence (2.11) is proven and the proof of Lemma 2.2 is finished.

We now proceed to verify Condition (iii) of Theorem 1.1. Formula (2.11) in Lemma 2.2 means that L generates a strongly continuous semigroup on the Banach space $H^1(\mathbf{R})$ (see [22]). By Fourier transformation, the essential spectrum of L_0 on $H^1(\mathbf{R})$ is

$$\sigma_e(L_0)\supset\{-\xi^4+\xi^2-ib_+\xi|\xi\in\mathbf{R}\}.\tag{2.21}$$

This curve meets the vertical lines $Re\lambda=\alpha$ for $-\infty<\alpha\le\frac14$ because $-\infty<-\xi^4+\xi^2\le\frac14$. We now prove that the same curve belongs to the essential spectrum of L.

Lemma 2.3. *The essential spectrum of L on $H^1(\mathbf{R})$ contains that of L_0.*

Proof. Let $\xi \in \mathbf{R}$ and let $\lambda = P(\xi) = -\xi^4 + \xi^2 - ib_+\xi$. Following Schechter[23], $\lambda \in \sigma_e(L)$ if there exists a sequence $\{\zeta_n\} \subset H^1(\mathbf{R})$ with

$$\|\zeta_n\|_{H^1} = 1, \quad \|(L-\lambda)\zeta_n\|_{H^1} \to 0,$$

and $\{\zeta_n\}$ does not have a strongly convergent subsequence in $H^1(\mathbf{R})$. (Here we use the definition: $\lambda \notin \sigma_e(L)$ if and only if $L - \lambda$ is Fredholm with index zero.) Now let $\zeta_0 \not\equiv 0$ be a C^∞ function with compact support in $(0,\infty)$. Define

$$\zeta_n(x) = c_n e^{-i\xi x}\zeta_0(x/n)/\sqrt{n}, \quad n = 1, 2, \cdots,$$

where c_n is chosen so that $\|\zeta_n\|_{H^1} = 1$. In fact,

$$\|\zeta_n\|_{L^2} = c_n\|\zeta_0\|_{L^2} \ \text{ and } \ 1 = \|\zeta_n\|_{H^1} \le kc_n$$

for some positive constant k. Hence $c_n \ge \frac{1}{k} > 0$. Since $\|\zeta_n\|_{L^\infty} \to 0$ but $\|\zeta_n\|_{L^2}$ is bounded away from zero, $\{\zeta_n\}$ can have no convergent subsequence in $L^2(\mathbf{R})$.

It remains to show that $\|(L-\lambda)\zeta_n\|_{H^1} \to 0$. We write

$$L - \lambda = L_0 - \lambda + (\varphi - b_+)\partial_x - \varphi_x.$$

Now elementary computations show

$$(L_0-\lambda)\zeta_n(x) = e^{i\xi x}\sum_{1\le s\le 4} P^{(s)}(\xi)c_n\zeta_0^{(s)}\Big(\frac{x}{n}\Big)/(s!n^{\frac{1}{2}+s}),$$

$$\partial(L_0-\lambda)\zeta_n(x) = i\xi(L_0-\lambda)\zeta_n(x) + e^{i\xi x}\sum_{1\le s\le 4} P^{(s)}(\xi)c_n\zeta_0^{(s+1)}\Big(\frac{x}{n}\Big)/(s!n^{\frac{3}{2}+s}).$$

Thus

$$\begin{aligned}\|(L_0-\lambda)\zeta_n(x)\|_{H^1} &\le (1+|\xi|)\sum_{1\le s\le 4}|P^{(s)}(\xi)|c_n\|\zeta_0^{(s)}\Big(\frac{x}{n}\Big)\|_{L^2}/(s!n^{\frac{1}{2}+s}) \\ &\quad + \sum_{1\le s\le 4}|P^{(s)}(\xi)|c_n\Big\|\zeta_0^{(s+1)}\Big(\frac{x}{n}\Big)\Big\|_{L^2}/(s!n^{\frac{3}{2}+s}) \to 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

Moreover, for any positive integer m, $\|\partial_x^m\zeta_n\|_{L^\infty} \to 0$ as $n \to \infty$, so we have

$$\|(\varphi-b_+)\partial_x\zeta_n\|_{L^2}^2 \le \|\partial_x\zeta_n\|_{L^\infty}^2\|\chi_{[0,\infty)}(\varphi-b_+)\|_{L^2}^2 \to 0,$$

and

$$\begin{aligned}\|\partial_x[(\varphi-b_+)\partial_x\zeta_n]\|_{L^2}^2 &\le 2\int_R \varphi_x^2[\partial_x\zeta_n(x)]^2\,dx + 2\int_R(\varphi-b_+)^2[\partial_{xx}\zeta_n(x)]^2\,dx \\ &\le 2\|\partial_x\zeta_n\|_{L^\infty}^2\|\varphi_x\|_{L^2}^2 + 2\|\partial_{xx}\zeta_n\|_{L^\infty}^2\|\chi_{[0,\infty)}(\varphi-b_+)\|_{L^2}^2 \to 0.\end{aligned}$$

In addition,

$$\|\varphi_x\zeta_n\|_{L^2} \le \|\zeta_n\|_{L^\infty}\|\varphi_x\|_{L^2} \to 0$$

and

$$\|\varphi_x\partial_x\zeta_n + \varphi_{xx}\partial_x\zeta_n\|_{L^2} \le \|\partial_x\zeta_n\|_{L^\infty}\|\varphi_x\|_{L^2} + \|\partial_x\zeta_n\|_{L^\infty}\|\varphi_{xx}\|_{L^2} \to 0.$$

Thus

$$\|(\varphi-b_+)\partial_x\zeta_n + \varphi_x\zeta_n\|_{H^1} \to 0 \quad \text{as} \quad n\to\infty.$$

So from the estimates above,

$$\|(L-\lambda)\zeta_n\|_{H^1} \to 0 \quad \text{as} \quad n \to \infty.$$

The proof of Lemma 2.3 is completed.

Therefore all the four conditions of Theorem 1.1 are satisfied by the linearized equation (2.5) and Theorem 2.1 has been proved.

References

[1] Shatah, J. & Strauss, W., Spectral condition for instability [J], *Contemporary Mathematics,* **255**(2000), 189-198.

[2] Kuramoto, Y., Instability and turbulence of wave fronts in reaction-diffusion systems [J], *Prog. Theoret. Phys.,* **63**(1980), 1885–1903.

[3] Sivashinsky, G., Nonlinear analysis of hydrodynamic instability in laminar flames, Part I, Derivation of basic equations [J], *Acta Astronaut,* **4**(1977), 1117–1206.

[4] Troy, W. C., The existence of steady solutions of Kuramoto-Sivashinsky equation [J], *J. Diff. Eq.,* **82**(1989), 269–313.

[5] Jones, J. & Troy, W. C., Steady solutions of the Kuramoto-Sivashinsky equation for small wave speed [J], *J. Diff. Eq.,* **96**(1992), 28–55.

[6] Yang, T. S., On traveling wave solutions of the Kuramoto-Sivashinsky equation [J], *Physica D,* **110:**1–2(1997), 25–42.

[7] Michelson, D., Steady solutions of the Kuramoto-Sivashinsky equation [J], *Physica D,* **19**(1986), 89–111.

[8] Michelson, D., Bunsen flames as steady solutions of the Kuramoto-Sivashinsky equation [J], *SIAM J. Math. Anal.,* **23:**2(1992), 364–386.

[9] Michelson, D., Stability of the bunsen flame profiles in the Kuramoto-Sivashinsky equation [J], *SIAM J. Math. Anal.,* **27:**3(1996), 765–781.

[10] Green, J. M. & Kim, J. S., The steady states of the Kuramoto-Sivashinsky equation [J], *Progress in Chaotic dynamics, Physica D,* **33:**1–3(1988), 99–120.

[11] Hooper, A. P. & Grimshaw, R., Traveling wave solutions of the Kuramoto-Sivashinsky equation [J], *Wave Motion,* **10:**5(1988), 405–420.

[12] Kevrekidis, I. G., Nicolaenko, B. & Scovel, J. C., Back in the saddle again: a computer assisted study of the Kuramoto-Sivashinsky equation [J], *SIAM J. Appl. Math.,* **50:**3(1990), 760–790.

[13] Trifonor, Yu. Ya., Two-periodical and quasi-periodical wave solutions of the Kuramoto-Sivashinsky equation and their stability and bifurcations [J], *Physica D,* **54:**4(1992), 311–330.

[14] Kent, P. & Elgin, J., Travelling waves of the Kuramoto-Sivashinsky equation: Period-multiplying bifurcations [J], *Nonlinearity,* **5:**4(1992), 899–919.

[15] Elgin, J. N. & Wu, X. S., Stability of cellular states of the Kuramoto-Sivashinsky equation [J], *SIAM J. Appl. Math.,* **56:**6(1996), 1621–1638.

[16] Lan, Y. T., The "cocoon" bifurcations in three-dimensional systems with two fixed points [J], *Internat. J. Bifur. Chaos Appl. Sci. Engrg.,* **2:**3(1992), 543–558.

[17] Feudel, F., Feudel, U. & Brandenburg, A., On the Bifurcation phenomena of the Kuramoto-Sivashinsky equation [J], *Internat. J. Bifur. Chaos Appl. Sci. Engrg.,* **3:**5(1993), 1299–1303.

[18] Kukavica, I., Oscillations of solutions of the Kuramoto-Sivashinsky equation [J], *Physica D,* **76:**4(1994), 369–374.

[19] Dawson, S. P. & Mancho, A. M., Collections of heteroclinic cycles in the Kuramoto-Sivashinsky equation [J], *Physica D,* **100:**3–4(1997), 231–256.

[20] Amdjadi, F., Aston, P. J. & Plechac, P., Symmetry breaking Hopf bifurcations in equations with $O(2)$ symmetry with application to the Kuramoto-Sivashinsky equation [J], *J. Comput. Phys.,* **131:**1(1997), 181–192.

[21] Tadmor, E., The well-posedness of the Kuramoto-Sivashinsky equation [J], *SIAM J. Math. Anal.,* **17:**4(1986), 884–893.

[22] Goldstein, J. A., Semigroups of linear operators and applications [M], Oxford University Press, New York, 1985.

[23] Schechter, M., Spectra of partial differential operators [M], American Elsevier Publishing Company, INC.-New York, 1971.