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REGULARITY RESULTS FOR LINEAR ELLIPTIC PROBLEMS RELATED TO THE PRIMITIVE EQUATIONS

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(*Dedicated to the memory of Jacques-Louis Lions in recognition of his deep influence on mathematical sciences*)

Abstract

The authors study the regularity of solutions of the GFD-Stokes problem and of some second order linear elliptic partial differential equations related to the Primitive Equations of the ocean. The present work generalizes the regularity results in [18] by taking into consideration the non-homogeneous boundary conditions and the dependence of solutions on the thickness ε of the domain occupied by the ocean and its varying bottom topography. These regularity results are important tools in the study of the PEs (see e.g. [6]), and they seem also to possess their own interest.

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§1. Introduction

We establish regularity results for the GFD-Stokes system and some second order elliptic partial differential equations related to the primitive equations of the ocean in thin domains with varying bottom topography. These equations constitute the core part of the Primitive Equations (PEs for brevity) for the atmosphere and the ocean. Motivated by the smallness of the aspect ratio of the domain occupied by the ocean, we studied the global existence of solutions for the PEs in thin domains, in [6]. As a preliminary step, we study, in this article, the H^2-regularity of solutions of the Stokes-type problem and some second order elliptic partial differential equations related to the primitive equations. Compared to the work [18, 19], an important aspect of our results in this article is that we determine the dependence on

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the thickness parameter ε and the bottom topography for the constants of the H^2 regularity. We also derive as in [16] some Sobolev inequalities with the exact dependence on ε of the constants appearing in the aforementioned inequalities. These results play a crucial role in establishing the global existence of strong solutions of the PEs in thin domains (see [6]).

We present an outline of this article. In the rest of this section we describe the problem and present the main results. Then, in Section 2, we study the H^2 regularity of two non-homogeneous elliptic boundary value problems. In Section 3 we use the results of Section 2 and give the proof of the main Theorem. Finally in Section 4, we derive various Sobolev type inequalities together with some applications of the results of Section 3.

Notations. The domain occupied by the ocean is of the form

$$M_\varepsilon = \left\{(x_1, x_2, x_3)) \in \mathbb{R}^3,\ (x_1, x_2) \in \Gamma_i, -\varepsilon h(x_1, x_2) < x_3 < 0\right\},$$

and its boundary $\partial M_\varepsilon = \Gamma_i \cup \Gamma_b \cup \Gamma_l$, where Γ_i is the interface between the ocean and the atmosphere, assumed to be a bounded smooth open subset of $\mathbb{R}^2$, Γ_b is the bottom boundary of the ocean, and Γ_l is its lateral boundary. Throughout this article , we will assume that h is independent of ε, where ε is the small parameter representing the thickness of the domain. In the final results we assume that h is a positive constant but in several parts of the article we will make the following assumptions concerning h: there exist positive constants $\bar{h}, \underline{h}, h_1$ such that

$$h \in C^2(\overline{\Gamma_i}), \quad 0 < \underline{h} \le h \le \bar{h}, \text{ and } ||h||_{C^2(\bar{\Gamma}_i)} \le h_1. \tag{1.1}$$

We are concerned with the regularity of solutions of the GFD-Stokes problem, namely,

$$\begin{cases} -\triangle v - \dfrac{\partial^2 v}{\partial x_3^2} + \operatorname{grad} p = f_1 \quad \text{in } M_\varepsilon, \\ \displaystyle\int_{-\varepsilon h}^{0} \operatorname{div} v = 0 \quad \text{on}\,\Gamma_i, \\ \dfrac{\partial v}{\partial x_3} + \alpha_v v = g_v \quad \text{on } \Gamma_i, v = 0, \quad \text{on } \Gamma_b \cup \Gamma_l, \end{cases} \tag{1.2}$$

where $v = v(x_1, x_2, x_3) \in \mathbb{R}^2$, and $p = p(x_1, x_2) \in \mathbb{R}$ are the unknown functions, f_1 is the external volume force, $\alpha_v > 0, g_v$ are given. We are also interested in the regularity of solutions of the following elliptic problem related to the equation of the temperature T or the equation of salinity:

$$\begin{cases} -\triangle T - \dfrac{\partial^2 T}{\partial x_3^2} = f_2 \quad \text{in } M_\varepsilon, \\ \dfrac{\partial T}{\partial x_3} + \alpha_{_T} T = g_{_T} \quad \text{on } \Gamma_i, \\ \frac{\partial T}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma_b \cup \Gamma_l, \end{cases} \tag{1.3}$$

where $T = T(x_1, x_2, x_3) \in \mathbb{R}$ is the unknown function, f_2 is the heating source inside the ocean, $\alpha_T > 0, g_T$ is given, $\mathbf{n}$ is the unit outward normal to the boundary. Throughout this article, we use $\nabla, \triangle, \operatorname{div}$ to denote the two dimensional gradient, Laplacian and divergence operators on the horizontal plane, and use $\nabla_3, \triangle_3$ and div_3 for the corresponding 3D differential operators. The spaces $H^s(M_\varepsilon), H_0^s(M_\varepsilon), s \ge 0$, are the usual Sobolev spaces constructed on $L^2(M_\varepsilon)$, and

$$\mathbb{L}^2(M_\varepsilon) = (L^2(M_\varepsilon))^2, \quad \mathbb{H}^s(M_\varepsilon) = (H^s(M_\varepsilon))^2.$$

Furthermore, we define the space (see [8–10])

$$\mathcal{V}_1 = \Big\{ v \in C^\infty(M_\varepsilon) : v \text{ is zero near } \Gamma_l \cup \Gamma_b,\ \operatorname{div} \int_{-\varepsilon h}^0 v\, dz = 0 \Big\}, \tag{1.4}$$

and H_1 is defined to be the closure of $\mathcal{V}_1$ in $\mathbb{L}^2(M_\varepsilon)$, $H_2 = L^2(M_\varepsilon)$, and V_1 is the closure of $\mathcal{V}_1$ in $\mathbb{H}^1(M_\varepsilon)$, $V_2 = H^1(M_\varepsilon)$. The norms and inner products for the spaces H and H_k $(k = 1, 2)$ are the L^2 ones, all denoted by $(\cdot,\cdot)_\varepsilon$ and $|\cdot|_\varepsilon$. Throughout the paper, c_0 will stand for a numerical constant that may vary from line to line.

The main result of this paper is the following

Theorem 1.1. *The hypotheses are those above, and we assume that M_ε is convex and that h is a positive constant. Let $(v, p) \in \mathbb{H}^1(M_\varepsilon) \times L^2(\Gamma_i)$ (resp. $T \in H^1(M_\varepsilon)$) be a weak solution of* (1.2) (*resp.* (1.3)). *Then*

$$(v, p) \in \mathbb{H}^2(M_\varepsilon) \times H^1(M_\varepsilon), \quad T \in H^2(M_\varepsilon). \tag{1.5}$$

Moreover the following inequalities hold:

$$|v|^2_{\mathbb{H}^2(M_\varepsilon)} + \varepsilon |p|^2_{H^1(\Gamma_i)} \le c_0 \big[|f_1|^2_\varepsilon + |g_v|^2_{L^2(\Gamma_i)} + \varepsilon |\nabla g_v|^2_{L^2(\Gamma_i)} \big], \tag{1.6}$$

$$|T|^2_{H^2(M_\varepsilon)} \le c_0 \big(|f_2|^2_\varepsilon + |\nabla g_T|^2_{L^2(\Gamma_i)} + |g_T|^2_{L^2(\Gamma_i)} \big). \tag{1.7}$$

Remark 1.1. We will study in a separate article the case where M_ε is not convex and h satisfies (1.1).

§2. Preliminary Results

A preliminary step in the proof of Theorem 1.1 is the study of the H^2-regularity of the solution of an auxiliary elliptic boundary value problem, which is obtained by setting $p = 0$ in (1.2), and deleting the second equation (compare also to (1.3)).

Lemma 2.1. *Assume that M_ε is convex, and $h \in \mathcal{C}^2(\overline{\Gamma}_i)$. For $f \in L^2(M_\varepsilon)$ and $g \in H^1_0(\Gamma_i)$, there exists a unique $\Psi \in H^2(M_\varepsilon)$ solution of*

$$\begin{cases} -\triangle_3 \Psi = f \quad \text{in } M_\varepsilon, \\ \dfrac{\partial \Psi}{\partial x_3} + \alpha \Psi = g \quad \text{on } \Gamma_i, \\ \Psi = 0 \quad \text{on } \Gamma_b \cup \Gamma_l. \end{cases} \tag{2.1}$$

Furthermore, there exists a constant $c(h, \alpha)$ depending only on α and h (and Γ_i), such that

$$\sum_{k,j=1}^3 \Big| \frac{\partial^2 \Psi}{\partial x_k \partial x_j} \Big|^2_\varepsilon \le c(h, \alpha) [|f|^2_\varepsilon + |g|^2_i + |\nabla g|^2_i].$$

For a function in $L^2(\Gamma_i)$, its norm is denoted by $|\cdot|_i$.

The proof of Lemma 2.1 is given below. We first construct a function Ψ^* satisfying the boundary conditions in (2.1) and with the exact dependence on ε of the L^2-norm of the second order derivatives (see Lemma 2.2 below). Then $\mathrm{e}^{\alpha x_3}(\Psi - \Psi^*) = \widehat{\Phi}$ satisfies the homogeneous Neumann condition on Γ_i and the homogeneous Dirichlet boundary condition on $\Gamma_l \cup \Gamma_b$. By a reflection argument, we extend the force f to $x_3 > 0$ to be an odd function. We then consider a homogeneous Dirichlet problem on the convex domain

$$\widehat{M} = \{ (x_1, x_2, x_3) \in \mathbb{R}^3; (x_1, x_2) \in \Gamma_i, \quad -\varepsilon h(x_1, x_2) < x_3 < \varepsilon h(x_1, x_2) \},$$

the solution of which $\widehat{W}$ coincides with $\widehat{\Phi}$ on M_ε. Finally we use the classical H^2 regularity results in convex domains (see [4]) to obtain that $\widehat{W} \in H^2(\widehat{M}_\varepsilon)$ and thus $\widehat{\Phi}$ and Ψ are in

$H^2(M_\varepsilon)$ together with the bounds on the L^2-norm of their second derivatives. We start with the following lifting lemma.

Lemma 2.2. *Let $h \in \mathcal{C}^2(\overline{\Gamma}_i)$ and $g \in H_0^1(\Gamma_i)$. There exists $\Psi^\star \in H^2(M_\varepsilon)$, such that*

$$\frac{\partial \Psi^\star}{\partial x_3} + \alpha \Psi^\star = g \quad on \ \ \Gamma_i, \quad \Psi^\star = 0 \quad on \ \ \Gamma_l \cup \Gamma_b.$$

Furthermore, there exists a constant $c(h)$ depending only on h, such that for $0 < \varepsilon \le 1$,

$$\sum_{k,j=1}^{3} \Big| \frac{\partial^2 \Psi^\star}{\partial x_k \partial x_j} \Big|_\varepsilon^2 \le (c(h)\varepsilon^2 + 1)(|g|_i^2 + |\nabla g|_i^2). \tag{2.2}$$

Proof. We first construct a function $\widetilde{\Psi}$ as a solution of the heat equation with $-x_3$ corresponding to time:

$$\begin{cases} \dfrac{\partial \widetilde{\Psi}}{\partial x_3} = -\triangle \widetilde{\Psi} \quad \text{in } \Gamma_i \times (-\infty, 0), \\ \widetilde{\Psi} = 0 \quad \text{on } \partial\Gamma_i \times (-\infty, 0), \\ \widetilde{\Psi}(x_1, x_2, 0) = g(x_1, x_2) \text{ on } \Gamma_i. \end{cases} \tag{2.3}$$

We set for $(x_1, x_2, x_3) \in \Gamma_i \times (-\infty, 0)$,

$$\Psi^\star(x_1, x_2, x_3) = \mathrm{e}^{-\alpha x_3} \int_{-\varepsilon h(x_1,x_2)}^{x_3} \widetilde{\Psi}(x_1, x_2, z) dz.$$

Note that $\Psi^\star(x_1, x_2, x_3) = 0$ when $(x_1, x_2, x_3) \in \Gamma_l \cup \Gamma_b$. Furthermore

$$\frac{\partial \Psi^\star}{\partial x_3} + \alpha \Psi^\star = \mathrm{e}^{-\alpha x_3} \widetilde{\Psi}(x_1, x_2, x_3), \tag{2.4}$$

and therefore, if $x_3 = 0$, we have $\dfrac{\partial \Psi^\star}{\partial x_3} + \alpha \Psi^\star = g$; that is, $\Psi^\star$ satisfies the boundary conditions in (2.1). Now we recall the classical energy estimates for $\widetilde{\Psi}$ solution of the heat equation with $-x_3$ corresponding to time.

We have

$$\frac{1}{2} |\widetilde{\Psi}|_i^2(x_3) + \int_{x_3}^0 |\nabla \widetilde{\Psi}|_i^2 = \frac{1}{2} |g|_i^2, \tag{2.5}$$

$$\frac{1}{2} |\nabla \widetilde{\Psi}|_i^2(x_3) + \int_{x_3}^0 \Big| \frac{\partial \widetilde{\Psi}}{\partial x_3} \Big|_i^2 (z) dz = \frac{1}{2} |\nabla g|_i^2, \tag{2.6}$$

$$-\frac{x_3}{2} \Big| \frac{\partial \widetilde{\Psi}}{\partial x_3} \Big|_i^2 (x_3) - \int_{x_3}^0 z \Big| \frac{\partial \nabla \widetilde{\Psi}}{\partial x_3} \Big|_i^2 (z)\, dz = \frac{1}{2} \int_{x_3}^0 \Big| \frac{\partial \widetilde{\Psi}}{\partial x_3} \Big|_i^2 (z)\, dz, \tag{2.7}$$

$$\frac{x_3^2}{2} \Big| \frac{\partial \nabla \widetilde{\Psi}}{\partial x_3} \Big|_i^2 (x_3) + \int_{x_3}^0 z^2 \Big| \frac{\partial^2 \widetilde{\Psi}}{\partial x_3^2} \Big|_i^2 (z)\, dz = -\int_{x_3}^0 z \Big| \frac{\partial \nabla \widetilde{\Psi}}{\partial x_3} \Big|_i^2 (z)\, dz. \tag{2.8}$$

Hence (2.5), (2.6), and (2.3) yield

$$|\widetilde{\Psi}|_i^2(-\varepsilon \bar{h}) \le |g|_i^2, \quad \frac{1}{2} |\nabla \widetilde{\Psi}|_i^2(-\varepsilon \bar{h}) + \frac{1}{2} |\Delta \widetilde{\Psi}|_\varepsilon^2 + \frac{1}{2} \Big| \frac{\partial \widetilde{\Psi}}{\partial x_3} \Big|_\varepsilon^2 \le \frac{1}{2} |\nabla g|_i^2, \tag{2.9}$$

and by integration of (2.5) and (2.6) with respect to x_3 from $-\varepsilon \bar{h}$ to 0, we obtain

$$|\widetilde{\Psi}|_\varepsilon^2 + |\nabla \widetilde{\Psi}|_\varepsilon^2 \le \varepsilon \bar{h} (|g|_i^2 + |\nabla g|_i^2). \tag{2.10}$$

Now, taking $x_3 = -\varepsilon\bar{h}$ in (2.7) and (2.8), we can write, using (2.6),

$$\frac{\varepsilon\bar{h}}{2}\Big|\frac{\partial\widetilde{\Psi}}{\partial x_3}\Big|_i^2(-\varepsilon\bar{h}) - \int_{-\varepsilon\bar{h}}^0 z\Big|\frac{\partial\nabla\widetilde{\Psi}}{\partial x_3}\Big|_i^2(z)\,dz \le \frac{1}{4}|\nabla g|_i^2, \tag{2.11}$$

$$\frac{\varepsilon^2\bar{h}^2}{2}\Big|\frac{\partial\nabla\widetilde{\Psi}}{\partial x_3}\Big|_i^2(-\varepsilon\bar{h}) + \int_{-\varepsilon\bar{h}}^0 z^2\Big|\frac{\partial^2\widetilde{\Psi}}{\partial x_3^2}\Big|_i^2(z)\,dz \le \frac{1}{4}|\nabla g|_i^2, \tag{2.12}$$

and, in particular,

$$\varepsilon^2\underline{h}^2\int_{-\varepsilon\bar{h}}^{-\varepsilon\underline{h}}\Big|\frac{\partial^2\widetilde{\Psi}}{\partial x_3^2}\Big|_i^2(z)\,dz \le \frac{1}{4}|\nabla g|_i^2. \tag{2.13}$$

Therefore, using

$$|\varphi(x_1,x_2,-\varepsilon h(x_1,x_2))| \le |\varphi(x_1,x_2,-\varepsilon\underline{h})| + \Big|\int_{-\varepsilon\bar{h}}^{-\varepsilon\underline{h}}\frac{\partial\varphi}{\partial x_3}(x_1,x_2,z)\,dz\Big|,$$

we obtain

$$\begin{aligned}
\int_{\Gamma_i}|\widetilde{\Psi}(x_1,x_2,-\varepsilon h(x_1,x_2))|^2\,dx_1dx_2 &\le c_0\big[1+\varepsilon\bar{h}\big](|g|_i^2+|\nabla g|_i^2),\\
\int_{\Gamma_i}|\nabla\widetilde{\Psi}(x_1,x_2,-\varepsilon h(x_1,x_2))|^2\,dx_1dx_2 &\le c_0\Big(1+\frac{\bar{h}}{\underline{h}}\Big)(|g|_i^2+|\nabla g|_i^2),\\
\int_{\Gamma_i}\Big|\frac{\partial\widetilde{\Psi}}{\partial x_3}(x_1,x_2,-\varepsilon h(x_1,x_2))\Big|^2\,dx_1dx_2 &\le c_0\Big(\frac{\bar{h}+\underline{h}}{\varepsilon\bar{h}\underline{h}}\Big)(|g|_i^2+|\nabla g|_i^2).
\end{aligned} \tag{2.14}$$

We now derive estimates on $\Psi^\star$. Since $\Psi^\star = \mathrm{e}^{-\alpha x_3}\int_{-\varepsilon h(x_1,x_2)}^{x_3}\widetilde{\Psi}(x_1,x_2,z)dz$, we have, by the Cauchy-Schwarz inequality and (2.10),

$$|\Psi^\star|_\varepsilon^2 \le \varepsilon^2 e^{2\bar{h}\alpha}\bar{h}^2|\widetilde{\Psi}|_\varepsilon^2 \le \varepsilon^3 e^{2\bar{h}\alpha}\bar{h}^3(|g|_i^2+|\nabla g|_i^2).$$

Furthermore,

$$\mathrm{e}^{\alpha x_3}\frac{\partial\Psi^\star}{\partial x_k} = \int_{-\varepsilon h(x_1,x_2)}^{x_3}\frac{\partial\widetilde{\Psi}}{\partial x_k}(x_1,x_2,z)dz + \varepsilon\frac{\partial h}{\partial x_k}\widetilde{\Psi}(x_1,x_2,-\varepsilon h(x_1,x_2)), \quad k=1,2,$$

and for $k,j=1,2$,

$$\begin{aligned}
\mathrm{e}^{\alpha x_3}\frac{\partial^2\Psi^\star}{\partial x_k\partial x_j} &= \int_{-\varepsilon h(x_1,x_2)}^{x_3}\frac{\partial^2\widetilde{\Psi}}{\partial x_k\partial x_i}(x_1,x_2,z)dz + \varepsilon\frac{\partial h}{\partial x_j}\frac{\partial\widetilde{\Psi}}{\partial x_k}(x_1,x_2,-\varepsilon h(x_1,x_2))\\
&\quad + \varepsilon\frac{\partial^2 h}{\partial x_k\partial x_j}\widetilde{\Psi}(x_1,x_2,-\varepsilon h(x_1,x_2)) - \varepsilon^2\frac{\partial h}{\partial x_k}\frac{\partial h}{\partial x_j}\frac{\partial\widetilde{\Psi}}{\partial x_3}(x_1,x_2,-\varepsilon h(x_1,x_2)).
\end{aligned}$$

Therefore using (2.14), we have

$$\begin{aligned}
\Big|\frac{\partial\Psi^\star}{\partial x_k}\Big|_\varepsilon^2 &\le \varepsilon^2 e^{2\bar{h}\alpha}\bar{h}^2(\varepsilon\bar{h}+h_1(1/2+\bar{h}+\sqrt{\varepsilon\bar{h}})(|g|_i^2+|\nabla g|_i^2)\\
&\le c(h)\varepsilon^2(|g|_i^2+|\nabla g|_i^2),
\end{aligned} \tag{2.15}$$

$$\begin{aligned}
\Big|\frac{\partial^2\Psi^\star}{\partial x_k\partial x_j}\Big|_\varepsilon^2 &\le c_0\varepsilon^2\Big(\bar{h}^2+h_1\bar{h}\Big(1+\frac{\bar{h}}{\underline{h}}\Big)+\bar{h}h_2(1+\sqrt{\varepsilon\bar{h}})+\bar{h}h_1^2\frac{\bar{h}+\underline{h}}{\bar{h}\underline{h}}\Big)(|g|_i^2+|\nabla g|_i^2)\\
&\le c(h)\varepsilon^2(|g|_i^2+|\nabla g|_i^2).
\end{aligned} \tag{2.16}$$

Furthermore, since $\dfrac{\partial\Psi^\star}{\partial x_3} = -\alpha\Psi^\star + \mathrm{e}^{-\alpha x_3}\widetilde{\Psi}$, we have

$$\begin{aligned}\Big|\frac{\partial\Psi^\star}{\partial x_3}\Big|_\varepsilon^2 &\le 2\alpha|\Psi^\star|_\varepsilon^2 + 2e^{2\alpha\bar{h}}|\widetilde{\Psi}|_\varepsilon^2 \le c(h)\varepsilon^2(|g|_i^2 + |\nabla g|_i^2),\\ \Big|\nabla\frac{\partial\Psi^\star}{\partial x_3}\Big|_\varepsilon^2 &\le 2\alpha|\nabla\Psi^\star|_\varepsilon^2 + 2|\nabla\widetilde{\Psi}|_\varepsilon^2 \le c(h)\varepsilon^2(|g|_i^2 + |\nabla g|_i^2).\end{aligned} \tag{2.17}$$

Finally, since

$$\frac{\partial^2\Psi^\star}{\partial x_3^2} = -\alpha\frac{\partial\Psi^\star}{\partial x_3} - \alpha\mathrm{e}^{-\alpha x_3}\widetilde{\Psi} + \mathrm{e}^{-\alpha x_3}\frac{\partial\widetilde{\Psi}}{\partial x_3}, \tag{2.18}$$

we have

$$\Big|\frac{\partial^2\Psi^\star}{\partial x_3^2}\Big|_\varepsilon^2 \le (c(h)\varepsilon^2 + 1)(|g|_i^2 + |\nabla g|_i^2). \tag{2.19}$$

Proof of Lemma 2.1. Back to the proof of Lemma 2.1, we construct Ψ by setting $\Psi = \Phi + \Psi^\star$, where Ψ^* is the H^2-function constructed in Lemma 2.2 and Φ is the unique solution of

$$\begin{cases} -\triangle_3\Phi = \bar{f} \quad \text{in } M_\varepsilon,\\ \dfrac{\partial\Phi}{\partial x_3} + \alpha\Phi = 0 \quad \text{on } \Gamma_i,\\ \Phi = 0 \quad \text{on } \Gamma_b\cup\Gamma_l, \end{cases} \tag{2.20}$$

where $\bar{f} = f - \Delta_3\Psi^\star \in L^2(M_\varepsilon)$, and $|\bar{f}|_\varepsilon \le |f|_\varepsilon + c(h)(|g|_i + |\nabla g|_i)$. Note that

$$\frac{1}{2}|\nabla_3\Phi|_\varepsilon^2 + \alpha^2|\Phi|_{L^2(\Gamma_i\times\{0\})} \le \frac{\varepsilon^2}{2}|\bar{f}|^2. \tag{2.21}$$

In order to prove that $\Phi \in H^2(M_\varepsilon)$, we define

$$\widehat{\Phi}(x_1, x_2, x_3) = e^{\alpha x_3}\Phi(x_1, x_2, x_3). \tag{2.22}$$

Note that

$$\frac{\partial\widehat{\Phi}}{\partial x_3} = e^{\alpha x_3}\Big(\frac{\partial\Phi}{\partial x_3} + \alpha\Phi\Big) \quad \text{and} \quad \frac{\partial^2\widehat{\Phi}}{\partial x_3^2} = e^{\alpha x_3}\Big(\frac{\partial^2\Phi}{\partial x_3^2} + 2\alpha\frac{\partial\Phi}{\partial x_3} + \alpha^2\Phi\Big). \tag{2.23}$$

The function $\widehat{\Phi}$ satisfies the boundary conditions

$$\frac{\partial\widehat{\Phi}}{\partial x_3} = 0 \ \text{ on } \ \Gamma_i, \quad \widehat{\Phi} = 0 \ \text{ on } \ \Gamma_l\cup\Gamma_b. \tag{2.24}$$

Moreover

$$\Delta_3\widehat{\Phi} = e^{\alpha x_3}\Delta_3\Phi + e^{\alpha x_3}\Big(2\alpha\frac{\partial\Phi}{\partial x_3} + \alpha^2\Phi\Big). \tag{2.25}$$

Hence

$$-\Delta_3\widehat{\Phi} = e^{\alpha x_3}\bar{f} - e^{\alpha x_3}\Big(2\alpha\frac{\partial\Phi}{\partial x_3} + \alpha^2\Phi\Big) = \hat{f}. \tag{2.26}$$

It is easy to see that $\hat{f} \in L^2(M_\varepsilon)$, and

$$|\hat{f}|_\varepsilon \le |f|_\varepsilon + c(h,\alpha)(|g|_i + |\nabla g|_i).$$

Therefore, we write the equation satisfied by $\widehat{\Phi}$ in the following form:

$$\begin{cases} -\triangle_3 \widehat{\Phi} = \hat{f} \in L^2(M_\varepsilon), \\ \dfrac{\partial \widehat{\Phi}}{\partial x_3} = 0 \quad \text{on } \Gamma_i, \\ \widehat{\Phi} = 0 \quad \text{on } \Gamma_b \cup \Gamma_l. \end{cases} \tag{2.27}$$

Let

$$\widehat{F}(x_1, x_2, x_3) = \begin{cases} \hat{f}(x_1, x_2, x_3), & -\varepsilon h(x_1, x_2) < x_3 < 0, \\ \hat{f}(x_1, x_2, -x_3), & 0 < x_3 < \varepsilon h(x_1, x_2). \end{cases}$$

Let $\widehat{M_\varepsilon} = \big\{(x_1, x_2, x_3) \in \mathbb{R}^3,\ (x_1, x_2) \in \Gamma_i,\ -\varepsilon h(x_1, x_2) < x_3 < \varepsilon h(x_1, x_2)\big\}$, and consider the following Laplace problem:

$$\begin{cases} -\triangle_3 \widehat{W} = \widehat{F} \in L^2(\widehat{M_\varepsilon}), \\ \widehat{W} = 0 \quad \text{on } \partial\widehat{M_\varepsilon}. \end{cases}$$

The convexity of M_ε implies that $\widehat{M_\varepsilon}$ is convex. Moreover since $\widehat{F} \in L^2(\widehat{M_\varepsilon})$, we obtain, thanks to [4], $\widehat{W} \in H^2(\widehat{M_\varepsilon})$, and

$$\sum_{k,j=1}^{3} \Big| \frac{\partial^2 \widehat{W}}{\partial x_k \partial x_j} \Big|^2_{L^2(\widehat{M_\varepsilon})} \le |\widehat{F}|^2_{L^2(\widehat{M_\varepsilon})} \le 2|\widehat{f}|^2_\varepsilon \le c(h, \alpha)[|f|^2_\varepsilon + |g|^2_i + |\nabla g|^2_i].$$

Since $\widehat{F}$ is even in x_3, the solution $\widehat{W}$ is also even in x_3. Therefore, $\dfrac{\partial \widehat{W}}{\partial x_3} = 0$ at $x_3 = 0$. By the uniqueness of solutions of (2.27), we obtain $\widehat{W}|_{M_\varepsilon} = \widehat{\Phi}$, and therefore

$$\sum_{k,j=1}^{3} \Big| \frac{\partial^2 \widehat{\Phi}}{\partial x_k \partial x_j} \Big|^2_{\varepsilon} \le c(h, \alpha)[|f|^2_\varepsilon + |g|^2_i + |\nabla g|^2_i]. \tag{2.28}$$

Finally, since $\Psi = \Phi + \Psi^\star$, we obtain, thanks to Lemma 2.2 and (2.28),

$$\sum_{k,j=1}^{3} \Big| \frac{\partial^2 \Psi}{\partial x_k \partial x_j} \Big|^2_{\varepsilon} \le c(h, \alpha)[|f|^2_\varepsilon + |g|^2_i + |\nabla g|^2_i].$$

Remark 2.1. As a corollary of Lemma 2.1, by an interpolation argument (see [7]), we have for $g = 0$: if $f \in H^{-1/2-\delta}(M_\varepsilon)$ with $0 < \delta < \frac{1}{2}$, then $\widehat{\Psi} \in H^{3/2-\delta}(M_\varepsilon)$, and if $f \in H^{-1/2+\delta}(M_\varepsilon)$ with $0 < \delta < \frac{1}{2}$, then $\widehat{\Psi} \in H^{2-\delta}(M_\varepsilon)$.

We will also need, in the proof of the main result, the following regularity result:

The next lemma establishes the H^2 regularity of the temperature.

Lemma 2.3. *Assume that M_ε is convex, and that h is constant. For $f \in L^2(M_\varepsilon)$ and $g \in H_0^1(\Gamma_i)$, there exists a unique $T \in H^2(M_\varepsilon)$ solution of*

$$\begin{cases} -\triangle_3 T = f_2 \quad \text{in } M_\varepsilon, \\ \dfrac{\partial T}{\partial x_3} + \alpha_T T = g_T \quad \text{on } \Gamma_i, \\ \dfrac{\partial T}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma_b \cup \Gamma_l. \end{cases} \tag{2.29}$$

Furthermore, there exists a constant $c(h, \alpha_T)$ depending only on α_T and h, such that for all

$\varepsilon > 0$,

$$\sum_{k,j=1}^{3} \Big|\frac{\partial^2 T}{\partial x_j \partial x_k}\Big|_\varepsilon^2 \le c(h,\alpha_T)[|f_2|_\varepsilon^2 + |g_T|_i^2 + |\nabla g_T|_i^2].$$

Proof. We start with the case $g_T = 0$. Thus let $\widetilde{T}$ be the unique solution of

$$\begin{cases} -\triangle_3 \widetilde{T} = f_2 \quad \text{in } M_\varepsilon, \\ \dfrac{\partial \widetilde{T}}{\partial x_3} + \alpha_T \widetilde{T} = 0 \quad \text{on } \Gamma_i, \\ \dfrac{\partial \widetilde{T}}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma_b \cup \Gamma_l. \end{cases} \tag{2.30}$$

We note that

$$|\nabla \widetilde{T}|_\varepsilon^2 + \Big|\frac{\partial \widetilde{T}}{\partial x_3}\Big|_\varepsilon^2 + \alpha_T |\widetilde{T}|_i^2 \le |f_2|_\varepsilon |T|_\varepsilon, \tag{2.31}$$

but since (see Appendix)

$$|\widetilde{T}|_\varepsilon \le \sqrt{2\varepsilon\bar{h}}|\widetilde{T}|_i + \varepsilon\bar{h}\Big|\frac{\partial \widetilde{T}}{\partial x_3}\Big|_\varepsilon, \tag{2.32}$$

we obtain

$$|\nabla \widetilde{T}|_\varepsilon^2 + \frac{1}{2}\Big|\frac{\partial \widetilde{T}}{\partial x_3}\Big|_\varepsilon^2 + \frac{\alpha_T}{2}|\widetilde{T}|_i^2 \le \frac{\varepsilon\bar{h}}{\alpha_T}|f_2|_\varepsilon^2 + \varepsilon^2\bar{h}^2|f_2|_\varepsilon^2. \tag{2.33}$$

Therefore

$$|\widetilde{T}|_\varepsilon^2 \le 4\varepsilon^2\bar{h}^2\Big[\frac{1}{\alpha_T} + \varepsilon\bar{h}\Big]^2 |f_2|_\varepsilon^2. \tag{2.34}$$

For the sake of simplicity, we will assume now that the function h is constant and let $\eta(x_3)$ and $T^\star$ be defined by

$$\eta(x_3) = \exp\Big[\frac{\alpha_T}{\varepsilon h}\Big(\varepsilon h x_3 + \frac{1}{2}x_3^2\Big)\Big] \quad \text{and } T^\star = \eta\widetilde{T}. \tag{2.35}$$

Since h is constant, it is easy to check that

$$\begin{aligned} -\Delta_3 T^\star &= 2\frac{\eta'}{\eta}\frac{\partial \widetilde{T}}{\partial x_3} + \frac{\eta''}{\eta}\widetilde{T} + f_2, \\ \frac{\partial T^\star}{\partial x_3} &= 0 \text{ on } \Gamma_b \cup \Gamma_i \quad \text{and} \quad \frac{\partial T^\star}{\partial \mathbf{n}} = 0 \text{ on } \Gamma_l. \end{aligned} \tag{2.36}$$

Noting that $\eta'(x_3) = \frac{\alpha_T}{\varepsilon h}(\varepsilon h + x_3)\eta$ and $\eta''(x_3) = \frac{\alpha_T}{\varepsilon h}\eta + \frac{\alpha_T^2}{\varepsilon^2 h^2}(\varepsilon h + x_3)^2\eta$, we can write

$$\begin{cases} -\Delta_3 T^\star = 2\dfrac{\alpha_T}{\varepsilon h}(\varepsilon h + x_3)\dfrac{\partial \widetilde{T}}{\partial x_3} + \dfrac{\alpha_T^2}{\varepsilon^2 h^2}(\varepsilon h + x_3)^2\widetilde{T} + \dfrac{\alpha_T}{\varepsilon h}\widetilde{T} + f_2 = \tilde{f}_2, \\ \dfrac{\partial T^\star}{\partial x_3} = 0 \quad \text{on } \Gamma_b \cup \Gamma_i, \\ \dfrac{\partial T^\star}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma_l. \end{cases} \tag{2.37}$$

Since, by assumption, the domain M_ε is convex, we can apply the classical results of the H^2-regularity in convex domains (see (3,1,2,2) in [4]) and obtain

$$\sum_{k,j=1}^{3} \Big|\frac{\partial^2 T^\star}{\partial x_k \partial x_j}\Big|_\varepsilon^2 \le |\Delta T^*|_\varepsilon^2 = |\tilde{f}_2|_\varepsilon^2.$$

Now, using (1.34), there exists a constant $c(h,\alpha_T)$ independent of ε, such that for $0<\varepsilon\le 1$, we have $|\tilde f_2|_\varepsilon^2 \le c(h,\alpha_T)|f_2|_\varepsilon^2$ and

$$\sum_{k,j=1}^{3}\Big|\frac{\partial^2 T^\star}{\partial x_k\partial x_j}\Big|_\varepsilon^2 \le c(h,\alpha_T)|\tilde f_2|_\varepsilon^2.$$

Finally, noting that $\exp(-\alpha_T\varepsilon h)\le \eta(x_3)\le 1$, we have

$$|T^\star|_\varepsilon^2 \le \exp(2\alpha_T h)|\widetilde T|_\varepsilon^2 \le c(h,\alpha_T)\varepsilon^2|f_2|_\varepsilon^2$$

and we obtain easily

$$\sum_{k,j=1}^{3}\Big|\frac{\partial^2 \widetilde T}{\partial x_k\partial x_j}\Big|_\varepsilon^2 \le c(h,\alpha_T)|\tilde f_2|_\varepsilon^2.$$

Now we treat the general case with $g_T\ne 0$. We use a lemma similar to Lemma 2.2.

Lemma 2.4. *Assume that h is constant and $g_T\in H^1(\Gamma_i)$. There exists $\bar T\in H^2(M_\varepsilon)$, such that*

$$\frac{\partial \overline T}{\partial x_3}+\alpha_T\overline T=g_T \quad on\ \ \Gamma_i,\qquad \frac{\partial\overline T}{\partial \mathbf n}=0\quad on\quad \Gamma_l\cup\Gamma_b.$$

Furthermore, there exists a constant $c(h)$ depending only on h, such that for $0<\varepsilon\le 1$,

$$\sum_{k,j=1}^{3}\Big|\frac{\partial^2 \overline T}{\partial x_k\partial x_j}\Big|_\varepsilon^2 \le \frac{(c(h)\varepsilon^2+1)}{\alpha^2}(|g|_i^2+|\nabla g|_i^2). \tag{2.38}$$

Proof. We proceed as in the proof of Lemma 2.2, and construct a function $\overline\Psi$ as a solution of the heat equation with $-x_3$ corresponding to time:

$$\begin{cases}\dfrac{\partial\overline\Psi}{\partial x_3}=-\triangle\overline\Psi \quad \text{in } \Gamma_i\times(-\infty,0),\\ \dfrac{\partial\overline\Psi}{\partial\mathbf n}=0 \quad \text{on } \partial\Gamma_i\times(-\infty,0),\\ \overline\Psi(x_1,x_2,0)=g_T(x_1,x_2) \text{ on } \Gamma_i.\end{cases} \tag{2.39}$$

We then set for $(x_1,x_2,x_3)\in\Gamma_i\times(-\infty,0)$,

$$\overline T(x_1,x_2,x_3)=\mathrm e^{-\alpha x_3}\int_0^{x_3}\overline\Psi(x_1,x_2,z)\,dz-\Big(x_3-\frac1\alpha\Big)\mathrm e^{\alpha\varepsilon h}\overline\Psi(x_1,x_2,-\varepsilon h). \tag{2.40}$$

We can easily check that $\dfrac{\partial\overline T}{\partial x_3}+\alpha_T\overline T=g_T$ for $x_3=0$, $\dfrac{\partial\overline T}{\partial x_3}=0$ for $x_3=-\varepsilon h$. Following the same lines of the proof of Lemma 2.2, we have

$$|\overline\Psi|_i^2(-\varepsilon\bar h)\le|g_T|_i^2,\quad \frac12|\nabla\overline\Psi|_i^2(-\varepsilon\bar h)+\frac12|\Delta\overline\Psi|_\varepsilon^2+\frac12\Big|\frac{\partial\overline\Psi}{\partial x_3}\Big|_\varepsilon^2\le\frac12|\nabla g_T|_i^2, \tag{2.41}$$

$$|\overline\Psi|_\varepsilon^2+|\nabla\overline\Psi|_\varepsilon^2\le\varepsilon\bar h(|g_T|_i^2+|\nabla g_T|_i^2). \tag{2.42}$$

From this we can easily check that $\overline T\in H^2(M_\varepsilon)$ and that (2.38) holds.

§3. Proof of the Main Result

In this section, we prove the main result of this paper as stated in Theorem 1.1. We have already established in Lemma 2.4 the H^2 regularity of the temperature T and the estimates (the dependence on ε) of the L^2-norms of its second derivatives. Thus we need only to study the regularity of the velocity, governed by the problem (1.2), i.e.,

$$\begin{cases} -(\Delta v + \dfrac{\partial^2 v}{\partial x_3^2}) + \nabla p = f_1 \quad \text{in} \quad M_\varepsilon, \\ \operatorname{div} \displaystyle\int_{-\varepsilon h}^{0} v\, dz = 0 \quad \text{in} \quad \Gamma_i, \\ v = 0 \quad \text{on} \quad \Gamma_l \cup \Gamma_b, \quad \dfrac{\partial v}{\partial x_3} + \alpha_v v = g_v \quad \text{on} \quad \Gamma_i. \end{cases} \tag{3.1}$$

This result has been proven in [18] where $\varepsilon = 1$ and $g_v = 0$. As indicated in the Introduction we study here the general case where $g_v \neq 0$ and we carefully investigate the dependence on ε of the constants.

Our approach to obtain the H^2 regularity is the same as in [18] and is based on the following observation: assume that the solution v of (3.1) satisfies $\dfrac{\partial v}{\partial x_3}\Big|_{\Gamma_i} \in \mathbb{L}^2(\Gamma_i)$ and $\dfrac{\partial v}{\partial x_3}\Big|_{\Gamma_b} \in \mathbb{L}^2(\Gamma_b)$, then integrating (3.1) in x_3 over $(-\varepsilon h, 0)$ yields a standard Stokes in 2-D with homogeneous boundary condition on Γ_i. By the classical regularity theory of the 2D-Stokes problem in smooth domains (see for instance [15] and [3]), p belongs to $H^1(\Gamma_i)$. Then, by moving the pressure term to the right hand side, the problem (3.1) reduces to an elliptic problem of the type studied in Lemma 2.1, and the H^2 regularity of v follows. The estimates on the L^2 norms of the second derivatives are then obtained using the trace theorem and the estimates in Lemma 2.1. Therefore, we start with proving that

$$\frac{\partial v}{\partial x_3}\Big|_{\Gamma_i} \in \mathbb{L}^2(\Gamma_i), \qquad \frac{\partial v}{\partial x_3}\Big|_{\Gamma_b} \in \mathbb{L}^2(\Gamma_b).$$

Lemma 3.1. *Assume that $h \in C^2(\overline{\Gamma}_i)$ and M_ε is convex. Let (v, p) be the weak solution of* (3.1), *then*

$$\frac{\partial v}{\partial x_3}\Big|_{\Gamma_i} \in \mathbb{L}^2(\Gamma_i), \qquad \frac{\partial v}{\partial x_3}\Big|_{\Gamma_b} \in \mathbb{L}^2(\Gamma_b). \tag{3.2}$$

Proof. By integration by parts, we have

$$|\nabla_3 v|_\varepsilon^2 + \alpha_v |v|_i^2 = (f, v)_\varepsilon + (g_v, v)_i, \tag{3.3}$$

and therefore the existence and uniqueness of the weak solution (v, p) follows from the Lax-Milgram theorem and De Rham's theorem. Hence $\nabla p \in H^{-1}(M_\varepsilon)$ and thus $\nabla p \in H^{-1}(\Gamma_i)$ since p is independent of x_3.

Let v_i be the unique solution of

$$\begin{cases} \Delta v_i = \nabla p \quad \text{in } \Gamma_i, \\ v_i = 0 \quad \text{on } \partial\Gamma_i. \end{cases} \tag{3.4}$$

Here v_i satisfies a 2D Laplace equation on Γ_i. Hence $v_i \in H_0^1(\Gamma_i)$. Let $\tilde{v} = v - v_i$, then $\tilde{v}$ satisfies

$$\begin{cases} \Delta_3 \tilde{v} = f_1, \\ \tilde{v} = 0 \quad \text{on } \Gamma_l, \\ \tilde{v} = -v_i \quad \text{on } \Gamma_b, \\ \dfrac{\partial \tilde{v}}{\partial x_3} + \alpha_v \tilde{v} = g_v - \alpha v_i \quad \text{on } \Gamma_i. \end{cases} \tag{3.5}$$

Thanks to Lemma 3.2, with $\psi_i = v_i$ and $\gamma = -\delta$ for some $0 < \delta < \frac{1}{2}$, we have $\tilde{v} \in H^{3/2-\delta}(M_\varepsilon)$.

$$g_i = -\frac{1}{\varepsilon h}\int_{-\varepsilon h}^{0} \operatorname{div} \tilde{v}\, dx_3 \in H^{1/2-\delta}(\Gamma_i). \tag{3.6}$$

Therefore, since $\operatorname{div} v_i = g_i$, with this new information on v_i, we rewrite the equation for v_i in the form of a 2D Stokes problem:

$$\begin{cases} -\Delta v_i + \nabla p = 0 \quad \text{in } \Gamma_i, \\ \text{div } v_i = g_i \in H^{\frac{1}{2}-\delta}(\Gamma_i) \end{cases} \tag{3.7}$$

and thanks the classical regularity on the non-homogeneous Stokes problem on the Γ_i (see [15]), we have $v_i \in H^{\frac{3}{2}-\delta}(\Gamma_i) \cap H_0^1(\Gamma_i) = H_0^{\frac{3}{2}-\delta}(\Gamma_i)$. With this new information on the regularity of v_i, we go back to the problem (3.5) and using Lemma 3.2 with $\psi_i = v_i$ and $\gamma = \frac{1}{2} - \delta$, we conclude that $\tilde{v} \in H^{2-\delta}(M_\varepsilon)$ and thus

$$\frac{\partial \tilde{v}}{\partial x_3}\Big|_{\Gamma_i} \in H^{1/2-\delta}(\Gamma_i), \quad \frac{\partial \tilde{v}}{\partial x_3}\Big|_{\Gamma_b} \in H^{1/2-\delta}(\Gamma_b). \tag{3.8}$$

Lemma 3.2. *Assume that M_ε is convex, $h \in \mathcal{C}^2(\overline{\Gamma}_i)$. For $f \in L^2(M_\varepsilon)$ and $g \in H_0^1(\Gamma_i)$, and $\psi_i \in H_0^{1+\gamma}(\Gamma_b)$, $-\frac{1}{2} < \gamma < \frac{1}{2}$, $\gamma \neq 0$, there exists a unique $\Psi_i \in H^{3/2+\gamma}(M_\varepsilon)$ solution of*

$$\begin{cases} -\triangle_3 \Psi_i = f \quad \textit{in } M_\varepsilon, \\ \dfrac{\partial \Psi_i}{\partial x_3} + \alpha_v \Psi_i = g - \alpha_v \psi_i \quad \textit{on } \Gamma_i, \\ \Psi_i = -\psi_i \quad \textit{on } \Gamma_b, \\ \Psi_i = 0 \quad \textit{on } \Gamma_l. \end{cases} \tag{3.9}$$

Proof. Thanks to Lemma 2.1, the problem is reduced to the case $f = 0$ and $g = 0$, by replacing Ψ_i with $\Psi_i - \Psi$, where Ψ is the function constructed in Lemma 2.1. Thus, without loss of generality, we may assume that $f = 0$ and $g = 0$.

Let Q_ε be the cylinder $Q_\varepsilon = \Gamma_i \times (-\varepsilon, 0)$, and let v_p be the unique solution of

$$\begin{cases} \Delta_3 v_p = 0, \quad \text{in } Q_\varepsilon, \\ v_p = 0 \quad \text{on } \partial\Gamma_i \times (-\varepsilon, 0), \\ v_p = -\psi_i \quad \text{on } \Gamma_i \times \{-\varepsilon\}, \\ v_p = \varepsilon h \alpha_v \psi_i \quad \text{on } \Gamma_i \times \{0\}. \end{cases} \tag{3.10}$$

We will show that $v_p \in H^{3/2+\gamma}(Q_\varepsilon)$ for all $-\frac{1}{2} < \gamma < \frac{1}{2}$, $\gamma \neq 0$. To this end, let $\widehat{Q}_\varepsilon$ be any C^2-domain containing Q_ε such that $\Gamma_i \times \{-\varepsilon, 0\} \subset \partial\widehat{Q}_\varepsilon$. Since ψ_i (resp. $h\alpha_v\psi_i$) is in $H_0^{1+\gamma}(\Gamma_i \times \{-\varepsilon\})$ (resp. $H_0^{1+\gamma}(\Gamma_i \times \{0\})$), we can define a function $V_i \in H^1(\partial\widehat{Q}_\varepsilon)$ by setting $V_i = -\psi_i$ on $\Gamma_i \times \{-\varepsilon\}$, $V_i = \varepsilon h \alpha_v \psi_i$ on $\Gamma_i \times \{0\}$, and $V_i = 0$ on $\partial\widehat{Q}_\varepsilon - \Gamma_i \times \{-\varepsilon, 0\}$. Now let V_p be the unique solution of $\Delta_3 V_p = 0$ in $\widehat{Q}_\varepsilon$ and $V_p = V_i$ on $\partial\widehat{Q}_\varepsilon$. Since $\partial\widehat{Q}_\varepsilon$ is of class C^2, the classical regularity results (see [7]) yield $V_p \in H^{3/2+\gamma}(\widehat{Q}_\varepsilon)$ for $-\frac{1}{2} < \gamma < \frac{1}{2}$, $\gamma \neq 0$. Now let $\widetilde{V}_i$ be the trace of V_p on $\partial\Gamma_i \times (-\varepsilon, 0)$. It is easy to see that $\widetilde{V}_i \in H_0^{1+\gamma}(\partial\Gamma_i \times (-\varepsilon, 0))$. Let $\widetilde{V}_p = V_p - v_p$, we have

$$\begin{cases} \Delta_3 \widetilde{V}_p = 0 \quad \text{in } Q_\varepsilon, \\ \widetilde{V}_p = 0 \quad \text{on } \Gamma_i \times \{-\varepsilon, 0\}, \\ \widetilde{V}_p = \widetilde{V}_i \quad \text{on } \partial\Gamma_i \times (-\varepsilon, 0). \end{cases} \tag{3.11}$$

Using a reflection argument around $x_3 = 0$ (resp. $x_3 = -\varepsilon$) by extending $\widetilde{V}_i$ in a "symmetrically" odd function defined on $\partial\Gamma_i \times (-\varepsilon, \varepsilon)$ (resp. $\partial\Gamma_i \times (-2\varepsilon, 0)$) , and using the classical local regularity theory (see [7]), we conclude that $\widetilde{V}_p \in H^{3/2+\gamma}(Q_\varepsilon)$ for $-\frac{1}{2} < \gamma < \frac{1}{2}$, $\gamma \neq 0$. Hence since $V_p \in H^{3/2+\gamma}(Q_\varepsilon)$, we have $v_p = V_p - \widetilde{V}_p \in H^{3/2+\gamma}(Q_\varepsilon)$.

Now let

$$\tilde{v}_p(x_1, x_2, x_3) = -\frac{x_3}{\varepsilon h(x_1, x_3)} v_p\Big(x_1, x_2, \frac{x_3}{h(x_1, x_2)}\Big) \quad \text{for } (x_1, x_2, x_3) \in M_\varepsilon. \tag{3.12}$$

It is obvious that $\tilde{v}_p \in H^{3/2+\gamma}(M_\varepsilon)$,

$$\tilde{v}_p(x_1, x_2, -\varepsilon h(x_1, x_2)) = -\psi_i(x_1, x_2) \quad \text{and} \quad \frac{\partial \tilde{v}_p}{\partial x_3} + \alpha_v \tilde{v}_p = -\alpha_v v_i \quad \text{on } \Gamma_i.$$

Therefore setting $\widetilde{V} = \tilde{v} - \tilde{v}_p$, we have

$$\begin{cases} \Delta_3 \widetilde{V} = f_1 - \Delta_3 \tilde{v}_p \in H^{-1/2+\gamma}(M_\varepsilon), \\ \widetilde{V} = 0 \quad \text{on } \Gamma_l \cup \Gamma_b, \\ \dfrac{\partial \widetilde{V}}{\partial x_3} + \alpha_v \widetilde{V} = 0 \quad \text{on } \Gamma_i. \end{cases} \tag{3.13}$$

Hence, thanks to Lemma 2.1 and Remark 2.1, we see that $\widetilde{V}$ and thus $\tilde{v}$ are in $H^{3/2+\gamma}(M_\varepsilon)$ for $-\frac{1}{2} < \gamma < \frac{1}{2}$, $\gamma \neq 0$.

Proof of Theorem 1.1. The proof is divided into two steps. In Step 1, we prove the H^2 regularity of solutions, i.e., $v \in H^2(M_\varepsilon)$ and $p \in H^1(\Gamma_i)$. Then, in Step 2, we establish the Cattabriga-Solonnikov type inequality on the solutions, i.e., prove the bounds on the L^2-norms of the second derivative of v and the H^1-norm on the pressure, in particular we establish their (non) dependence on ε.

Step 1. The H^2-Regularity of Solutions

Let $\bar{v} = \int_{-\varepsilon h}^0 v\, dz$, we have

$$\frac{\partial^2 \bar{v}(x_1, x_2, x_3)}{\partial x_i^2} = \int_{-\varepsilon h}^0 \frac{\partial^2 v(x_1, x_2, z)}{\partial x_i^2} dz + \varepsilon \frac{\partial h}{\partial x_i} \frac{\partial v(x_1, x_2, -\varepsilon h(x_1, x_2)}{\partial x_i}, \quad i = 1, 2. \tag{3.14}$$

Integrating the first equation in (3.1) with respect to x_3 we obtain the 2D Stokes problem:

$$\begin{cases} -\Delta \bar{v} + \varepsilon h \nabla p = \tilde{f} \quad \text{in} \quad \Gamma_i, \\ \text{div } \bar{v} = 0 \quad \text{in} \quad \Gamma_i, \quad v = 0 \quad \text{on} \quad \partial\Gamma_i, \end{cases} \tag{3.15}$$

where

$$\tilde{f} = \int_{-\varepsilon h}^0 f\, dz + \frac{\partial v}{\partial x_3}\Big|_{x_3=0} - \frac{\partial v}{\partial x_3}\Big|_{x_3=-\varepsilon h} + \varepsilon \sum_{i=1}^2 \frac{\partial h}{\partial x_i} \frac{\partial v(x_1, x_2, -\varepsilon h(x_1, x_2))}{\partial x_i}. \tag{3.16}$$

We have, thanks to Lemma 3.1, $\tilde{f} \in L^2(\Gamma_i)$. Therefore from the classical regularity theory of the 2D Stokes problem, we conclude that $\nabla p \in L^2(\Gamma_i)$. We return to the problem (3.1), and move the gradient of the pressure to the right hand side and obtain, thanks to Lemma 2.1, $v \in H^2(M_\varepsilon)$ and

$$\sum_{k,j=1}^3 \Big|\frac{\partial^2 v}{\partial x_k \partial x_j}\Big|_\varepsilon^2 \le c(h, \alpha)[|f|_\varepsilon^2 + |g|_i^2 + |\nabla g|_i^2] + c(h, \alpha)\varepsilon |\nabla p|_i^2.$$

Furthermore,

$$|\tilde{f}|_i^2 \le \varepsilon \bar{h} |f_1|_\varepsilon^2. \tag{3.17}$$

Step 2. The Cattabriga-Solonnikov Type Inequality

First we homogenize the boundary condition. Let $v_l = (\Psi_1, \Psi_2)$ where Ψ_1 and Ψ_2 are constructed by using Lemma 2.1, i.e.,

$$\begin{cases} -\triangle_3 \Psi_k = f_{1,k} \quad \text{in } M_\varepsilon, \quad k=1,2, \\ \dfrac{\partial \Psi_k}{\partial x_3} + \alpha \Psi_k = g_{v,k} \quad \text{on } \Gamma_i, \quad k=1,2, \\ \Psi_k = 0 \quad \text{on } \Gamma_b \cup \Gamma_l, \quad k=1,2, \end{cases}$$

where $f_1 = (f_{1,1}, f_{1,2})$ $g_v = (g_{v,1}, g_{v,2})$. Thanks to Lemma 2.1, we have

$$\sum_{k,j=1}^{3} \Big| \frac{\partial^2 v_l}{\partial x_k \partial x_j} \Big|_\varepsilon^2 \le c(h, \alpha_v)(|f_1|_\varepsilon^2 + |g_v|_i^2 + |\nabla g_v|_i^2). \tag{3.18}$$

Let $v^* = v - v_l$, we have

$$\begin{cases} -\Big(\Delta v^* + \dfrac{\partial^2 v^*}{\partial x_3^2}\Big) + \nabla p = 0 \quad \text{in} \quad M_\varepsilon, \\ \text{div} \displaystyle\int_{-\varepsilon h}^0 v^* \, dz = g^* \quad \text{in} \quad \Gamma_i, \\ v^* = 0 \quad \text{on} \quad \Gamma_l \cup \Gamma_b, \\ \frac{\partial v^*}{\partial x_3} + \alpha_v v^* = 0 \quad \text{on} \quad \Gamma_i, \end{cases} \tag{3.19}$$

where $g* = -\text{div} \int_{-\varepsilon h}^0 v_l \, dx_3$. Note that inequality (3.18) implies

$$||g^*||_{H^1(\Gamma_i)}^2 \le c(h, \alpha_v)\varepsilon[|f_1|_\varepsilon^2 + |g_v|_i^2 + |\nabla g_v|_i^2]. \tag{3.20}$$

Define $V^* = \frac{1}{h}\int_{-\varepsilon h}^0 v^* \, dx_3$, which satisfies the 2D-Stokes problem

$$\begin{cases} -\Delta V^* + \nabla(\varepsilon p) = F^* \quad \text{in } \Gamma_i, \\ \text{div}\, V^* = G^* \\ V^* = 0 \quad \text{on } \partial\Gamma_i, \end{cases} \tag{3.21}$$

where

$$\begin{aligned} F^* = \sum_{i=1}^{2} \Big[& \frac{\partial}{\partial x_i}\Big(\frac{1}{h^2}\frac{\partial h}{\partial x_i}\Big) \int_{-\varepsilon h}^0 v^* \, dx_3 + \frac{2}{h^2}\frac{\partial h}{\partial x_i} \int_{-\varepsilon h}^0 \frac{\partial v^*}{\partial x_i} \, dx_3 \\ & - \frac{\varepsilon}{h}\frac{\partial h}{\partial x_i}\frac{\partial v^*}{\partial x_i}(x_1, x_2, -\varepsilon h(x_1, x_2))\Big] + \frac{1}{h}\frac{\partial v^*}{\partial x_3}\Big|_{x_3=0} - \frac{1}{h}\frac{\partial v^*}{\partial x_3}\Big|_{x_3=-\varepsilon h}, \\ G^* = & \frac{1}{h}\text{div} \int_{-\varepsilon h}^0 v_l \, dx_3 + \nabla\Big(\frac{1}{h}\Big) \cdot \int_{-\varepsilon h}^0 v^* \, dx_3. \end{aligned} \tag{3.22}$$

Clearly

$$|F^*|_{L^2(\Gamma_i)}^2 \le c(h)\varepsilon\Big[|v^*|_\varepsilon^2 + \Big|\frac{\partial v^*}{\partial x_i}\Big|_\varepsilon^2 + \Big|\frac{\partial v^*}{\partial x_i}\Big|_{L^2(\Gamma_b)}^2\Big] + c(h)\Big[\Big|\frac{\partial v^*}{\partial x_3}\Big|_{L^2(\Gamma_i)}^2 + \Big|\frac{\partial v^*}{\partial x_3}\Big|_{L^2(\Gamma_b)}^2\Big]. \tag{3.23}$$

Now, since $v^* = 0$ on Γ_b, we have $\dfrac{\partial v^*}{\partial x_i} = \varepsilon \dfrac{\partial h}{\partial x_i}\dfrac{\partial v^*}{\partial x_3}$ on γ_b and by the Poincaré inequality, we also have

$$\Big|\frac{\partial v^*}{\partial x_3}\Big|_{L^2(\Gamma_i)}^2 \le 2\alpha_v^2 \varepsilon \Big|\frac{\partial v^*}{\partial x_3}\Big|_\varepsilon^2.$$

Now since

$$\begin{aligned}\Big|\frac{\partial v^*}{\partial x_3}\Big|^2_{L^2(\Gamma_b)} &\le \Big|\frac{\partial v^*}{\partial x_3}\Big|^2_{L^2(\Gamma_i)} + 2\Big|\frac{\partial v^*}{\partial x_3}\Big|_\varepsilon\Big|\frac{\partial^2 v^*}{\partial x_3^2}\Big|_\varepsilon \\ &\le 2\alpha_v^2\varepsilon\Big|\frac{\partial v^*}{\partial x_3}\Big|^2_\varepsilon + \theta\varepsilon\Big|\frac{\partial^2 v^*}{\partial x_3^2}\Big|^2_\varepsilon + \frac{c_0}{\theta}\varepsilon\Big|\frac{\partial v^*}{\partial x_3}\Big|^2_\varepsilon,\end{aligned} \tag{3.24}$$

where θ is a small positive constant independent of ε, we have

$$\Big|\frac{\partial v^*}{\partial x_i}\Big|^2_{L^2(\Gamma_b)} \le c(h,\theta)\Big|\frac{\partial v^*}{\partial x_3}\Big|^2_\varepsilon + c(h)\varepsilon^2\Big|\frac{\partial^2 v^*}{\partial x_3^2}\Big|^2_\varepsilon. \tag{3.25}$$

Thus

$$|F^*|^2_{L^2(\Gamma_i)} \le c(h)\varepsilon\Big[|v^*|^2_\varepsilon + \Big|\frac{\partial v^*}{\partial x_i}\Big|^2_\varepsilon\Big] + c(h)\theta\varepsilon\left|\frac{\partial^2 v^*}{\partial x_3^2}\right|^2_\varepsilon. \tag{3.26}$$

We estimate the H^1-norm of v^*, using $v^* = v - v_l$ and the H^1-estimates of v and v_l. We obtain easily

$$|F^*|^2_{L^2(\Gamma_i)} \le c(h)\varepsilon\big[|f_1|^2_\varepsilon + |g_v|^2_i + |\nabla g_v|^2_i\big] + c(h)\theta\varepsilon\Big|\frac{\partial^2 v^*}{\partial x_3^2}\Big|^2_\varepsilon. \tag{3.27}$$

Similarly, we have

$$|G^*|^2_{H^1(\Gamma_i)} \le c(h)\varepsilon\big[|f_1|^2_\varepsilon + |g_v|^2_i + |\nabla g_v|^2_i\big]. \tag{3.28}$$

Now using the Cattabriga-Solonnikov inequality for the 2D Stokes problem (3.21), we see that there exists a constant c_0 independent of ε such that

$$|V^*|^2_{H^2(\Gamma_i)} + \varepsilon^2|\nabla p|^2_{L^2(\Gamma_i)} \le c_0[|F^*|^2_{L^2(\Gamma_i)} + |G^*|^2_{H^1(\Gamma_i)}]. \tag{3.29}$$

Therefore

$$\varepsilon^2|\nabla p|^2_{L^2(\Gamma_i)} \le c(h,\theta)\varepsilon\big[|f_1|^2_\varepsilon + |g_v|^2_i + |\nabla g_v|^2_i\big] + c(h)\theta\varepsilon\Big|\frac{\partial^2 v^*}{\partial x_3^2}\Big|^2_\varepsilon. \tag{3.30}$$

Since $\Delta_3 v^* = \nabla p$, in M_ε, $v^* = 0$ on $\Gamma_b \cup \Gamma_l$ and $\dfrac{\partial v^*}{\partial x_3} + \alpha_v v^* = 0$ on Γ_i, we have thanks to Lemma 2.1

$$\begin{aligned}\sum_{k,j=1}^3 \Big|\frac{\partial^2 v^*}{\partial x_k \partial x_j}\Big|^2_\varepsilon &\le c(h,\alpha)\varepsilon|\nabla p|^2_{L^2(\Gamma_i)} \\ &\le c(h,\alpha)\big[|f_1|^2_\varepsilon + |g_v|^2_i + |\nabla g_v|^2_i\big] + c(h,\alpha_v)\theta\Big|\frac{\partial^2 v^*}{\partial x_3^2}\Big|^2_\varepsilon,\end{aligned} \tag{3.31}$$

and therefore for θ small enough, so that $c(h,\alpha_v)\theta \le \frac{1}{2}$, we conclude that

$$\sum_{k,j=1}^3 \Big|\frac{\partial^2 v^*}{\partial x_k \partial x_j}\Big|^2_\varepsilon \le c(h,\alpha)\varepsilon|\nabla p|^2_{L^2(\Gamma_i)} \le c(h,\alpha)\big[|f_1|^2_\varepsilon + |g_v|^2_i + |\nabla g_v|^2_i\big]. \tag{3.32}$$

The proof of the main result is now complete.

§4. Appendix

We present in this appendix some Sobolev type inequalities satisfied by solutions of (1.2) and (1.3).

Lemma 4.1. *For v satisfying the boundary condition in* (1.2), *we have*

$$|v|_\varepsilon \le 2\varepsilon\Big|\frac{\partial v}{\partial x_3}\Big|_\varepsilon, \quad v \in H^1(M_\varepsilon), \tag{4.1}$$

$$\Big|\frac{\partial v}{\partial x_3}\Big|_\varepsilon^2 + 2\alpha_v \int_{\Gamma_i} v^2\, dx_1 dx_2 \le 2\varepsilon^2\Big|\frac{\partial^2 v}{\partial x_3^2}\Big|_\varepsilon^2 + 4\varepsilon|g_v|^2, \quad v \in H^2(M_\varepsilon). \tag{4.2}$$

Proof. Inequality (4.1) is the classical Poincaré inequality for v, we omit its proof (note that $v = 0$ on Γ_b but not necessarily on Γ_i). To prove (4.2) we establish this inequality for v smooth, and the result follows by density in the general case; when v is smooth the proof consists in integrating by parts $\int_{M_\varepsilon} v\dfrac{\partial^2 v}{\partial x_3^2}dx$, and using the Cauchy-Schwarz inequality and the boundary condition in (1.2).

Lemma 4.2. *For any $T \in H^1(M_\varepsilon)$, we have*

$$|T|_\varepsilon^2 \le 2\bar{h}\varepsilon|T|^2_{L^2(\Gamma_i\times\{0\})} + 4\bar{h}^2\varepsilon^2\left|\frac{\partial T}{\partial x_3}\right|_\varepsilon^2, \tag{4.3}$$

$$|T|^2_{L^2(\Gamma_i\times\{0\})} \le \frac{2}{\varepsilon\underline{h}}|T|_\varepsilon^2 + \frac{2\bar{h}^2\varepsilon}{\underline{h}}\left|\frac{\partial T}{\partial x_3}\right|_\varepsilon^2. \tag{4.4}$$

The proof of (4.3) and (4.4) is similar to the proof of the Poincaré inequality.

Lemma 4.3 (Agmon's inequality). *For $v \in \mathbb{H}^2(M_\varepsilon)$ satisfying the boundary condition in (1.2),*

$$|v|_{L^\infty} \le c_0\big[\varepsilon^{\frac12}\,|\Delta_3 v|_\varepsilon + |g_v|_i + |\nabla g_v|_i\big]. \tag{4.5}$$

Proof. The proof is an easy extension of the following anisotropic inequality established in [16]:

$$|\bar{v}|_{L^\infty(\Gamma_i\times(-\varepsilon,0))} \le |c_0\bar{v}|_\varepsilon^{\frac14}\Big(\Big|\frac{\partial^2\bar{v}}{\partial x_3'^2}\Big|_\varepsilon + \frac{1}{\varepsilon}\Big|\frac{\partial\bar{v}}{\partial x_3'}\Big|_\varepsilon + \frac{1}{\varepsilon^2}|\bar{v}|\Big)^{\frac14}\cdot\prod_{i=1}^{2}\Big(\Big|\frac{\partial^2\bar{v}}{\partial x_i'^2}\Big|_\varepsilon + \Big|\frac{\partial\bar{v}}{\partial x_i'}\Big|_\varepsilon + |\bar{v}|_\varepsilon\Big)^{\frac14},$$

where $\bar{v}$ is the function corresponding to v via the change of variables to flatten the boundary, and the inequality

$$|\bar{v}|_{L^2(\Gamma_i\times(-\varepsilon,0))} \le \varepsilon\Big|\frac{\partial\bar{v}}{\partial x_3'}\Big|_\varepsilon \le c_0\varepsilon^2\Big[\Big|\frac{\partial^2\bar{v}}{\partial x_3'^2}\Big|_\varepsilon + \varepsilon^{-\frac12}|g_v|_{L^2(\Gamma_i)}\Big].$$

We skip the details.

We recall the following version of Ladyzhenskaya's inequality established in [16]:

$$|u|_{L^6(\Omega)} \le c_0\prod_{i=1}^{3}\Big(\frac{1}{b_i-a_i}|u|_{L^2(\Omega)} + \Big|\frac{\partial u}{\partial x_i}\Big|_{L^2(\Omega)}\Big)^{\frac13}, \quad \forall u \in H^1(\Omega), \tag{4.6}$$

where $\Omega = \prod_{i=1}^{3}(a_i,b_i)$, and c_0 is a numerical constant. As a corollary of (4.6), we prove the following

Lemma 4.4. *There exists a constant c_0 independent of ε such that*

$$|v|_{L^6(M_\varepsilon)} \le c_0\|v\|_\varepsilon, \quad \forall v \in V_1, \tag{4.7}$$

$$|\nabla v|_{L^6(M_\varepsilon)} \le c_0\bar{h}^2|\Delta_3 v|_\varepsilon, \quad \forall v \in D(A_1). \tag{4.8}$$

Proof. Inequality (4.7) is an easy consequences of (4.6) and the fact that v satisfies the Poincaré inequality $|v|_\varepsilon \le \varepsilon||v||_\varepsilon$. The second inequality follows from the fact that $v = 0$ on

Γ_b, which implies that $\nabla v = \varepsilon \nabla h \dfrac{\partial v}{\partial x_3}$. We skip the details.

Now we derive some inequalities concerning a scalar function T which satisfies the following boundary condition on ∂M_ε:

$$\frac{\partial T}{\partial n} = 0 \quad \text{on} \quad \Gamma_l \cup \Gamma_b, \quad \frac{\partial T}{\partial x_3} + \alpha_{_T} T = g_{_T} \quad \text{on} \quad \Gamma_i. \tag{4.9}$$

Lemma 4.5. *For $T \in H^2(M_\varepsilon)$ satisfying the boundary conditions* (4.9), *we have*

$$\sum_{i=1}^{3} \Big| \frac{\partial T}{\partial x_i} \Big|_\varepsilon^2 + \frac{\alpha_{_T}}{2} \int_{\Gamma_i} T^2 dx_1 dx_2 \le |T|_\varepsilon \, |\Delta_3 T|_\varepsilon + \frac{1}{2\alpha_{_T}} |g_{_T}|^2_{L^2(\Gamma_i)}. \tag{4.10}$$

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