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SOME REMARKS ON THE NULL CONDITION FOR NONLINEAR ELASTODYNAMIC SYSTEM

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Abstract

The author derives the same null condition as in [1] for the nonlinear elastodynamic system in a simpler way and proves the equivalence of the null conditions introduced in [1] and [7] respectively.

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§1. Introduction

It is well-known that, if there is no external force, the displacement $u = (u^1, u^2, u^3) = u(t, x)$ of an isotropic, homogeneous hyperelastic material is governed by the following quasi-linear hyperbolic system (cf. [9])

$$\partial_t^2 u^i + \sum_{j,k,l=1}^{3} a_{ij}^{kl}(\nabla u)\partial_k\partial_l u^j = 0 \quad (i = 1, 2, 3),$$

where $\nabla = (\partial_1, \partial_2, \partial_3)$ and (a_{ij}^{kl}) stands for the elastic tensor.

As in the theory of 3D nonlinear wave equations , the global existence of classical solutions hinges on two basic assumptions. First, the initial deformation must be small. Second, the nonlinear terms must obey a type of null condition. The omission of either of these two assumptions may lead to the breakdown of classical solutions in finite time. For example, the formation of singularities for large displacements was illustrated by Tahvildar-Zadeh[8]. Moreover, F. John[4] proved that in the spherically symmetric case, a genuine nonlinear condition leads to the formation of singularities even for small initial data.

Klainerman[5] introduced the null condition for quasilinear wave equations and proved the global existence of classical solutions with small initial data. Agemi[1] and Sideris[7] introduced their null conditions respectively for nonlinear elastodynamic system in different ways and proved the global existence of classical solutions to the initial value problem with small initial data.

The first purpose of the present paper is to derive the null condition for the nonlinear elastodynamic system by a simpler method. The second purpose is to prove the equivalence of null conditions introduced in [1] and [7] respectively.

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§2. Null Conditions for the Nonlinear Elastodynamic System

We now consider the following more general system

$$\sum_{j=1}^{3}\sum_{\alpha,\beta=0}^{3} a_{ij}^{\alpha\beta}(\partial u)\partial_\alpha\partial_\beta u^j = 0 \quad (i=1,2,3), \tag{2.1}$$

where $\partial = (\partial_0, \nabla), \partial_0 = \partial_t, a_{ij}^{\alpha\beta}(\partial u) = a_{ij}^{\beta\alpha}(\partial u)$, and

$$\begin{aligned} &a_{ii}^{00}(0) = 1, a_{ii}^{ii}(0) = -c_1^2, a_{ii}^{jj}(0) = -c_2^2 \ (i \neq j),\\ &a_{ij}^{ij}(0) = -(c_1^2 - c_2^2)/2 \ (i \neq j), \quad i,j = 1,2,3,\\ &a_{ij}^{\alpha\beta}(0) = 0, \quad \text{otherwise}. \end{aligned} \tag{2.2}$$

For infinitesimal u , equation (2.1) reduces to the linear elastic system

$$\frac{\partial^2 u}{\partial t^2} - c_2^2 \triangle u - (c_1^2 - c_2^2)\nabla \mathrm{div} u = 0,$$

where the material constants c_1 and c_2 ($c_1 > c_2 > 0$), which correspond to the propagation speeds of longitudinal and transverse waves, respectively, are connected with the Lame constants λ and μ by

$$c_1^2 = \lambda + 2\mu, \quad c_2^2 = \mu.$$

Agemi[1] introduced $v = (v^1, v^2, v^3)$, where $v^i = (\partial u^i), i = 1,2,3$. Then $v \in \mathbf{R}^{12}$ satisfies a first order system containing 12 equations.

As in [4], we consider the plane wave solution of (2.1):

$$u(t,x) = w(t,s),$$

where $s = \zeta \cdot x$ stands for the inner product of $\zeta, x \in \mathbf{R}^3$ and $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ with $|\zeta| = 1$.

It is easy to see that $w(t,s)$ satisfies the following system in one space dimension

$$\sum_{j=1}^{3}(a_{ij}^{00}\partial_0^2 w^j + 2\sum_{k=1}^{3} a_{ij}^{k0}\zeta_k\partial_k\partial_0 w^j + \sum_{k,l=1}^{3} a_{ij}^{kl}\zeta_k\zeta_l\partial_1^2 w^j) = 0 \quad (i=1,2,3).$$

Let

$$v = \partial w = (\partial_t w, \partial_s w) = (\partial_0 w, \partial_1 w) = (\partial_0 w^1, \partial_0 w^2, \partial_0 w^3, \partial_1 w^1, \partial_1 w^2, \partial_1 w^3).$$

Thus, $v \in \mathbf{R}^6$ satisfies the following first order system

$$\partial_0 v + a(v)\partial_1 v = 0, \tag{2.3}$$

where $a(v) = a_0(v)^{-1}a_1(v)$ with

$$a_0(v) = \begin{pmatrix} (a_{ij}^{00})_{3\times3} & 0 \\ 0 & I_{3\times3} \end{pmatrix}_{6\times6},$$

$$a_1(v) = \begin{pmatrix} \Big(2\sum\limits_{k=1}^{3} a_{ij}^{k0}\zeta_k\Big)_{3\times3} & \Big(\sum\limits_{k=1}^{3} a_{ij}^{kl}\zeta_k\zeta_l\Big)_{3\times3} \\ -I_{3\times3} & 0 \end{pmatrix}_{6\times6}.$$

Since $c_1 > c_2 > 0$, we have $\lambda + \mu > 0$ and $\mu > 0$. Therefore, the linear elastic tensor $(a_{ij}^{\alpha\beta}(0))$ satisfies the strong-ellipticity condition by [10]. Since $|\partial u|$ is small, by continuity, the elastic tensor $(a_{ij}^{\alpha\beta}(\partial u))$ satisfies the strong-ellipticity condition in a neighborhood of $\partial u = 0$. From the proof of Theorem 5.1 in [10], we can get that System (2.3) is hyperbolic in a neighborhood of $v = 0$, that is, $a(v)$ has 6 real eigenvalues and 6 linearly independent eigenvectors in a neighborhood of $v = 0$.

We shall investigate the eigenvalues $\lambda = \lambda(v)$ of $a(v)$ and the corresponding right eigenvector $r = r(v)$ in a neighborhood of $v = 0$. Since

$$\det(a_0(v)\lambda - a_1(v)) = \det\Big(a_{ij}^{00}\lambda^2 - 2\lambda\sum_{k=1}^{3} a_{ij}^{k0}\zeta_k + \sum_{k,l=1}^{3} a_{ij}^{kl}\zeta_k\zeta_l\Big)_{3\times3}, \tag{2.4}$$

by assumption (2.2) on $a_{ij}^{\alpha\beta}(0)$, we get, at $v = 0$,

$$\det(a_0(0)\lambda - a_1(0)) = (c_1^2 - \lambda^2)(\lambda^2 - c_2^2)^2.$$

Therefore, $a(v)$ has 6 eigenvalues : $\lambda_1^{\pm}(v), \lambda_{2,1}^{\pm}(v)$, and $\lambda_{2,2}^{\pm}(v)$ such that

$$\lambda_1^{\pm}(0) = \pm c_1,\ \lambda_{2,1}^{\pm}(0) = \lambda_{2,2}^{\pm}(0) = \pm c_2.$$

Moreover, the eigenvectors $r_1^{\pm}$ and $r_2^{\pm}$ corresponding to $\pm c_1$ and $\pm c_2$ are

$$r_1^{\pm} = (\mp c_1\zeta_1, \mp c_1\zeta_2, \mp c_1\zeta_3, \zeta_1, \zeta_2, \zeta_3), \tag{2.5}$$

$$r_2^{\pm} = (\mp c_2\xi_1, \mp c_2\xi_2, \mp c_2\xi_3, \xi_1, \xi_2, \xi_3), \text{for any } \xi \in \mathbf{R}^3 \text{ satisfying } \xi\cdot\zeta = 0. \tag{2.6}$$

From (2.4), the same calculation as in [1] gives

$$\pm 2c_1(D\lambda_1^{\pm})(0) = -\sum_{i,j}\zeta_i\zeta_j\Big\{c_1^2(Da_{ij}^{00}(0) \mp 2c_1\sum_k (Da_{ij}^{k0})(0)\zeta_k + \sum_{k,l}(Da_{ij}^{kl})(0)\zeta_k\zeta_l\Big\}, \tag{2.7}$$

$$\begin{aligned} &\pm 2c_2 D(\lambda_{2,1}^{\pm}(v) + \lambda_{2,2}^{\pm}(v))|_{v=0} \\ &= -\sum_i (1-\zeta_i^2)\Big\{c_2^2(Da_{ii}^{00})(0) \mp c_2\sum_k (Da_{ii}^{k0})(0)\zeta_k + \sum_{k,l}(Da_{ii}^{kl})(0)\zeta_k\zeta_l\Big\} \\ &\quad -\sum_{i\neq j}\zeta_i\zeta_j\Big\{c_2^2(Da_{ij}^{00})(0) \mp c_2\sum_k (Da_{ij}^{k0})(0)\zeta_k + \sum_{k,l}(Da_{ij}^{kl})(0)\zeta_k\zeta_l\Big\}, \end{aligned} \tag{2.8}$$

where $D = \dfrac{\partial}{\partial v_\alpha^i} = \dfrac{\partial}{\partial(\partial_\alpha w^i)}$ ($\alpha = 0, 1$ and $i = 1, 2, 3$).

Definition 2.1. *For any fixed $\zeta \neq 0$, $|\zeta| = 1$, System (2.3) is not genuinely nonlinear if*

$$r_1^{\pm}\cdot\nabla_v\lambda_1^{\pm}|_{v=0} = 0, \quad r_2^{\pm}\cdot\nabla_v(\lambda_{2,1}^{\pm} + \lambda_{2,2}^{\pm})|_{v=0} = 0. \tag{2.9}$$

Then we have the following

Theorem 2.1. *For any fixed $\zeta \neq 0$, $|\zeta| = 1$, System (2.3) is not genuinely nonlinear if and only if*

$$(\mathrm{N})_1 \qquad \sum_{i,j,k=1}^{3}\sum_{\alpha,\beta,\gamma=0}^{3}\frac{\partial a_{ij}^{\alpha\beta}(\partial u)}{\partial(\partial_\gamma u^k)}|_{\partial u=0}X_iX_jX_kX_\alpha X_\beta X_\gamma = 0,$$

$$\forall\ (X_0, X_1, X_2, X_3) \text{ satisfying } X_0^2 - c_1^2|X|^2 = 0,$$

$$\begin{aligned}(\mathrm{N})_2 \qquad &\sum_{i,k=1}^{3}\sum_{\alpha,\beta,\gamma=0}^{3}\frac{\partial a_{ii}^{\alpha\beta}(\partial u)}{\partial(\partial_\gamma u^k)}|_{\partial u=0}(|X|^2 - X_i^2)\xi_kX_\alpha X_\beta X_\gamma \\ &-\sum_{i,j,k=1,i\neq j}^{3}\sum_{\alpha,\beta,\gamma=0}^{3}\frac{\partial a_{ij}^{\alpha\beta}(\partial u)}{\partial(\partial_\gamma u^k)}|_{\partial u=0}X_iX_j\xi_kX_\alpha X_\beta X_\gamma = 0,\end{aligned}$$

$$\forall\ \xi, (X_0, X_1, X_2, X_3) \text{ satisfying } X_0^2 - c_2^2|X|^2 = 0,\ \xi\cdot X = 0.$$

Proof. Let

$$\zeta_i = \frac{X_i}{|X|}, X_0 = \mp c_1|X| \quad (i = 1, 2, 3).$$

By (2.5), we rewrite $r_1^{\pm}$ as

$$r_1^{\pm} = (X_0X_1, X_0X_2, X_0X_3, |X|X_1, |X|X_2, |X|X_3).$$

Note

$$\begin{aligned}
&\frac{\partial a_{ij}^{\alpha\beta}(v)}{\partial v_0^k} = \frac{\partial a_{ij}^{\alpha\beta}(\partial u)}{\partial u_0^k},\\
&\frac{\partial a_{ij}^{\alpha\beta}(v)}{\partial v_1^k} = \sum_{h=1}^{3} \frac{\partial a_{ij}^{\alpha\beta}(\partial u)}{\partial u_h^k}\zeta_h = \sum_{h=1}^{3} \frac{\partial a_{ij}^{\alpha\beta}(\partial u)}{\partial u_h^k}\frac{X_h}{|X|} \quad (k = 1, 2, 3),
\end{aligned}$$

where $u_\alpha^k = \partial_\alpha u^k$ ($\alpha = 0, 1, 2, 3$ and $k = 1, 2, 3$). It follows from (2.7) and the first condition of (2.9) that

$$\begin{aligned}
0 &= r_1^{\pm} \cdot \nabla_v \lambda_1^{\pm}|_{v=0}\\
&= \sum_{k=1}^{3} X_0X_k \frac{\partial \lambda_1^{\pm}(v)}{\partial v_0^k}(0) + \sum_{k=1}^{3} |X|X_k \frac{\partial \lambda_1^{\pm}(v)}{\partial v_1^k}(0)\\
&= \frac{1}{\mp 2c_1}\Big\{ \sum_{k=1}^{3} X_0X_k \sum_{i,j=1}^{3} \frac{X_i}{|X|}\frac{X_j}{|X|} \sum_{\alpha,\beta=0}^{3} \frac{\partial a_{ij}^{\alpha\beta}(v)}{\partial v_0^k}(0)\frac{X_\alpha}{|X|}\frac{X_\beta}{|X|}\\
&\quad + \sum_{k=1}^{3} |X|X_k \sum_{i,j=1}^{3} \frac{X_i}{|X|}\frac{X_j}{|X|} \sum_{\alpha,\beta=0}^{3} \frac{\partial a_{ij}^{\alpha\beta}(v)}{\partial v_1^k}(0)\frac{X_\alpha}{|X|}\frac{X_\beta}{|X|}\Big\}\\
&= \frac{1}{|X|^4}\Big\{ \sum_{i,j,k=1}^{3} \sum_{\alpha,\beta=0}^{3} \frac{\partial a_{ij}^{\alpha\beta}(\partial u)}{\partial u_0^k}(0)X_\alpha X_\beta X_0 X_i X_j X_k\\
&\quad + \sum_{i,j,k=1}^{3} \sum_{\alpha,\beta=0}^{3} \sum_{h=1}^{3} \frac{\partial a_{ij}^{\alpha\beta}(\partial u)}{\partial u_h^k}(0)X_\alpha X_\beta X_h X_i X_j X_k\Big\}\\
&= \frac{1}{|X|^4} \sum_{i,j,k=1}^{3} \sum_{\alpha,\beta,\gamma=0}^{3} \frac{\partial a_{ij}^{\alpha\beta}(\partial u)}{\partial u_\gamma^k}(0)X_i X_j X_k X_\alpha X_\beta X_\gamma.
\end{aligned}$$

So

$$r_1^{\pm} \cdot \nabla_v \lambda_1^{\pm}|_{v=0} = 0 \text{ if and only if } (N)_1 \text{ holds.}$$

Similarly, it comes from (2.8) and the second condition of (2.9) that

$$r_2^{\pm} \cdot \nabla_v (\lambda_{2,1}^{\pm} + \lambda_{2,2}^{\pm})|_{v=0} = 0 \text{ if and only if } (N)_2 \text{ holds.}$$

In [1], $(\mathrm{N})_1$ and $(\mathrm{N})_2$ are referenced as the null condition for System (2.1). We now get the same result in the previous simpler way.

§3. Conditions $(\mathbf{N})_1$ and $(\mathbf{N})_2$ are Equivalent to $(\mathbf{N})_1$ for the Nonlinear Elastodynamic System

We now prove that for the nonlinear elastodynamic system, condition $(\mathrm{N})_2$ always holds.

Let $\varphi(t, x)$ be a smooth deformation . The unknown u is a displacement from the reference configuration:

$$u(t, x) = \varphi(t, x) - x.$$

Then, the deformation gradient $F = \nabla\varphi = (\partial_l \varphi^i)$ and the displacement gradient $G = F - I = \nabla u = (G_{il})$. Thus,the stored energy function can be written as

$$W = W(F) = V(G).$$

The displacement u then satisfies the following system

$$\partial_t^2 u - \mathrm{div}\frac{\partial V}{\partial G} = 0, \tag{3.1}$$

that is,

$$\partial_t^2 u^i - \sum_{l=1}^{3} \frac{\partial}{\partial x_l}\frac{\partial V}{\partial G_{il}} = 0 \quad (i = 1, 2, 3).$$

System (3.1) can be written as

$$\partial_t^2 u^i - c_2^2 \triangle u^i - (c_1^2 - c_2^2)\partial_i \mathrm{div} u = F^i(\nabla u, \nabla^2 u) + \text{higher terms of } \partial u, \tag{3.2}$$

where

$$F^i(\nabla u, \nabla^2 u) = \sum_{j,k,l,m,n} B_{ijk}^{lmn} F^i(\nabla u, \nabla^2 u)\partial_l \partial_m u^j \partial_n u^k.$$

It is easy to get the following

Proposition 3.1. *The null condition for System* (3.2) *holds if and only if*

$$(\mathrm{N})_1' \qquad \sum_{ijklmn=1}^{3} B_{ijk}^{lmn} X_i X_j X_k X_l X_m X_n = 0, \quad \forall\ X \in \mathbf{R}^3,$$

$$(\mathrm{N})_2' \qquad \sum_{iklmn=1}^{3} B_{ijk}^{lmn}(|X|^2 - X_i^2)\xi_k X_l X_m X_n - \sum_{i\neq j,klmn=1}^{3} B_{ijk}^{lmn} X_i X_j \xi_k X_l X_m X_n = 0,$$

$$\forall\ \xi, X \in \mathbf{R}^3, \xi \cdot X = 0.$$

Let k_1, k_2, k_3 be the principal invariants of the strain matrix $C = G + G^T + GG^T$, where G^T denotes the transpose of G. Then the stored energy function can be given by

$$V(G) = \sigma(k_1, k_2, k_3).$$

Theorem 3.1. $(\mathrm{N})_2'$ *is always satisfied for the nonlinear elastodynamic system. In other words,* $(\mathrm{N})_1'$ *is the only null condition for the nonlinear elastodynamic system.*

Proof. By Proposition 3.2 in [1], for the nonlinear term $F(\nabla u, \nabla^2 u)$ in (3.2) we have

$$\begin{aligned} F(\nabla u, \nabla^2 u) = {} & 2(2\sigma_{111} + 3\sigma_{11})\nabla(\mathrm{div}\, u)^2 \\ & + 2(\sigma_{11} - \sigma_{12})(\nabla|\mathrm{rot}\, u|^2 - 2\mathrm{rot}(\mathrm{div}\, u\, \mathrm{rot}\, u)) + Q(u, \nabla u), \end{aligned}$$

where $\sigma_{11} = \frac{\partial^2}{\partial k_1^2}\sigma(0,0,0), \sigma_{12} = \frac{\partial^2}{\partial k_1 \partial k_2}\sigma(0,0,0)$.

From Lemma 3.1 (i) in [1], we can verify that $Q(u, \nabla u)$ satisfies $(\mathrm{N})_2'$. From (iii) of the same lemma, we can also verify that $\nabla u|\mathrm{rot} u|^2$ satisfies $(\mathrm{N})_2'$. On the other hand, it is easy to see that $\mathrm{rot}(\mathrm{div}\, u\, \mathrm{rot}\, u)$ and $\nabla(\mathrm{div}\, u)^2$ satisfy $(\mathrm{N})_2'$. Thus, $F(\nabla u, \nabla^2 u)$ satisfies $(\mathrm{N})_2'$, namely, $(\mathrm{N})_2'$ is always satisfied for the nonlinear elastodynamic system.

§4. Equivalence of the Null Conditions Introduced by Agemi[1] and Sideris[7]

We impose $\sigma_1 = \frac{\partial}{\partial k_1}\sigma(0,0,0) = 0$, which implies that the reference configuration is a stress free state. Then from Theorem 3.1 and Theorem 3.1 in [1], we have

Corollary 4.1. $(\mathrm{N})_1'$ *holds if and only if*

$$2\sigma_{111}+3\sigma_{11}=0, \tag{4.1}$$

where $\sigma_{111}=\frac{\partial^3}{\partial k_1^3}\sigma(0,0,0)$.

The deformation is given in [7] as $\varphi(t,x)=\lambda x+u(t,x)$, where $\lambda>0$. The corresponding deformation gradient and displacement gradient are $F(\lambda)=\nabla\varphi$ and $G(\lambda)$, respectively. Then, the stored energy function can be expressed as

$$W=W(F(\lambda))=V(G(\lambda))=\tau(\lambda,s)=\tau(\lambda,s_1,s_2,s_3),$$

where s_1,s_2,s_3 are the principal invariants of the matrix $\sqrt{F^TF}-\lambda I$. The null condition introduced by [7] is

$$\tau_{111}(\lambda,0)=0, \tag{4.2}$$

where $\tau_{111}(\lambda,0)=\frac{\partial^3}{\partial s_1^3}\tau(\lambda,0,0,0)$.

Theorem 4.1. *The null condition given by Agemi is equivalent to that given by Sideris.*

Proof. From (2.2c) in [7], we know that when $\lambda=1$,

$$\begin{aligned}
k_1&=2s_1+s_1^2-2s_2,\\
k_2&=4s_2+2s_1s_2-6s_3+s_2^2-2s_1s_3,\\
k_3&=8s_3+4s_1s_3+2s_2s_3+s_3^2.
\end{aligned}$$

Then

$$\tau_{111}(1,0)=\frac{\partial^3\tau}{\partial s_1^3}\Big|_{s=0}=\frac{\partial^3\sigma(k_1,k_2,k_3)}{\partial s_1^3}\Big|_{s=0}=8\frac{\partial^3\sigma}{\partial k_1^3}\Big|_{k=0}+12\frac{\partial^2\sigma}{\partial k_1^2}\Big|_{k=0}=4(2\sigma_{111}+3\sigma_{11}).$$

So (4.1) is equivalent to (4.2).

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