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ON THE UNIQUENESS OF THE WEAK SOLUTIONS OF A QUASILINEAR HYPERBOLIC SYSTEM WITH A SINGULAR SOURCE TERM**

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(*Dedicated to the memory of Jacques-Louis Lions*)

Abstract

This paper is a continuation of the authors' previous paper [1]. In this paper the authors prove, assuming additional conditions on the initial data, some results about the existence and uniqueness of the entropy weak solutions of the Cauchy problem for the singular hyperbolic system

$$\begin{cases} a_t + (au)_x + \frac{2au}{x} = 0, \\ u_t + \frac{1}{2}(a^2+u^2)_x = 0, \end{cases} \quad x > 0,\ t \geq 0.$$

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§1. Introduction and Main Results

We consider the Cauchy problem for the quasilinear hyperbolic system

$$\begin{cases} a_t + (au)_x + \frac{2au}{x} = 0, \\ u_t + \frac{1}{2}(a^2+u^2)_x = 0, \end{cases} \quad x > 0,\ t \geq 0, \tag{1.1}$$

with the initial data

$$(a(x,0), u(x,0)) = (a_0(x), u_0(x)), \ \ x > 0. \tag{1.2}$$

The system (1.1) appears in the study of the radial symetric solutions in $\mathbf{R}^3 \times \mathbf{R}_+$ for a conservative system modelling the isentropic flow introduced by G.B.Whitham in [7, Chap.9] where a is the sound speed and u is the radial velocity. If $f : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ is defined by $f(a,u) = (au, \frac{1}{2}(a^2+u^2))$, then two eingenvalues of ∇f are

$$\lambda_1 = u - a, \quad \lambda_2 = u + a \tag{1.3}$$

and so the strict hyperbolicity fails if $a = 0$, but the system is genuinely nonlinear with Riemann invariants

$$l = -u + a, \quad r = u + a \tag{1.4}$$

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which satisfy the equivalent system (for classical solutions) :

$$\begin{cases} r_t + \frac{1}{2}\,(r^2)_x + \frac{r^2-l^2}{2x} = 0, \\ \\ l_t - \frac{1}{2}\,(l^2)_x + \frac{r^2-l^2}{2x} = 0, \end{cases} \quad x > 0,\ t \ge 0, \tag{1.5}$$

with initial data

$$(r(x,0), l(x,0)) = (r_0(x),\, l_0(x)),\ \ x > 0, \tag{1.6}$$

with $r_0 = u_0 + a_0$, $l_0 = -u_0 + a_0$.

Following [1], if $a_0, u_0 \in H_0^1(\mathbf{R}_+)$, we will say that

$$v = (a,u) \in (L^\infty_{\text{loc}}([0,+\infty[\times[0,+\infty[))^2$$

is a weak solution for the Cauchy problem (1.1), (1.2) in $\mathbf{R}_+ \times [0,+\infty[$ if, for each pair $\varphi \in C_0^\infty(\mathbf{R}_+ \times [0,+\infty[)$, $\psi \in C_0^\infty([0,+\infty[\times[0,+\infty[)$,

$$\int_{\mathbf{R}_+\times[0,+\infty[} \Big(a\varphi_t + au\varphi_x - \frac{2au}{x}\,\varphi\Big)\, dx\, dt + \int_{\mathbf{R}_+\times[0,+\infty[} \Big(u\psi_t + \frac{1}{2}(a^2+u^2)\psi_x\Big)\, dx\, dt$$
$$+ \int_{\mathbf{R}_+} a_0(x)\varphi(x,0)\, dx + \int_{\mathbf{R}_+} u_0(x)\psi(x,0)\, dx = 0. \tag{1.7}$$

A weak notion of null boundary condition for v (at $x = 0$) is contained in (1.7). Moreover, we will say that $v = (a,u)$ verifying (1.7) is an entropy weak solution if, for every pair of smooth functions $\eta, q : \mathbf{R}^2 \longrightarrow \mathbf{R}$, η convex (entropy/entropy flux pair) such that $\nabla\eta \cdot \nabla f = \nabla q$ in $\mathbf{R}^2$, we have

$$\frac{\partial}{\partial t}\,\eta(v) + \frac{\partial}{\partial t}\,q(v) + \nabla\eta(v)\cdot\Big(\frac{2au}{x}, 0\Big) \le 0 \tag{1.8}$$

in $\mathcal{D}'(\mathbf{R}_+ \times \mathbf{R}_+)$. By applying the compensated compacteness method of Tartar, Murat and DiPerna (cf. [2] and [3]) and some ideas of M.E.Schonbek in [6] we have proved in [1] the following result:

Theorem 1.1. *Assume* $a_0, u_0 \in H_0^1(\mathbf{R}_+)$, $u_0(x) \ge a_0(x) \ge 0$, $x \in \mathbf{R}_+$. *Then, there exists* $v = (a,u) \in (L^\infty(\mathbf{R}_+ \times [0,+\infty[))^2$, *with* $u \ge a \ge 0$ *a.e. in* $\mathbf{R}_+ \times [0,+\infty[$, *which is an entropy weak solution for the Cauchy problem* (1.1), (1.2) *in* $\mathbf{R}_+ \times [0,+\infty[$. *Moreover there exists a sequence* $v_\varepsilon = (a_\varepsilon, u_\varepsilon) \in \big(C([0,+\infty[; H^3 \cap H_0^1) \cap C^1([0,+\infty[; H^1) \cap L^\infty(\mathbf{R}^2_+)\big)^2$ *such that* $0 \le a_\varepsilon \le u_\varepsilon \le M$, $v_\varepsilon \longrightarrow v$ *a.e. in* $\mathbf{R}_+ \times [0,+\infty[$ *and in* $\big(L^\infty(\mathbf{R}^2_+)\big)^2$ *weak* $*$, $v_\varepsilon(.,0) \longrightarrow v(.,0)$ *in* $(H_0^1(\mathbf{R}_+))^2$ *and* v_ε *is the solution of the approximate parabolic system*

$$\begin{cases} a_{\varepsilon t} + (a_\varepsilon u_\varepsilon)_x + \frac{2a_\varepsilon u_\varepsilon}{x+\varepsilon} = \varepsilon\, a_{\varepsilon xx}, \\ \\ u_{\varepsilon t} + \frac{1}{2}\,(a_\varepsilon^2 + u_\varepsilon^2)_x = \varepsilon\, u_{\varepsilon xx}, \end{cases} \quad x > 0,\ t \ge 0, \tag{1.9}$$

with initial data

$$v_{0\varepsilon} = (a_{0\varepsilon}, u_{0\varepsilon}) \in \big(H^3(\mathbf{R}_+) \cap H_0^2(\mathbf{R}_+)\big)^2,\ u_{0\varepsilon} \ge a_{0\varepsilon} \ge 0.$$

In the framework of Theorem 2.1, we have, for

$$l_\varepsilon = -u_\varepsilon + a_\varepsilon, \quad r_\varepsilon = u_\varepsilon + a_\varepsilon,$$
$$l = -u + a, \quad r = u + a : \tag{1.10}$$
$$l_\varepsilon \le 0, \quad 0 \le r_\varepsilon \le M_1, \quad r_\varepsilon^2 - l_\varepsilon^2 \ge 0, \quad l \le 0,\ 0 \le r \le M_1, \quad r^2 - l^2 \ge 0, \text{ a.e.,}$$

$$\begin{cases} r_{\varepsilon t} + \frac{1}{2}\,(r_\varepsilon^2)_x + \frac{r_\varepsilon^2 - l_\varepsilon^2}{2(x+\varepsilon)} = \varepsilon\, r_{\varepsilon xx}, \\ \\ l_{\varepsilon t} - \frac{1}{2}\,(l_\varepsilon^2)_x + \frac{r_\varepsilon^2 - l_\varepsilon^2}{2(x+\varepsilon)} = \varepsilon\, l_{\varepsilon xx}, \end{cases} \quad x > 0,\ t \ge 0, \tag{1.11}$$

with $(r_\varepsilon, l_\varepsilon) \in \big(C([0,+\infty[; H^3 \cap H_0^1) \cap C^1([0,+\infty[; H^1)\big)^2$ and

$$\begin{aligned} r_\varepsilon(\cdot\,, 0) &= r_{0\varepsilon} = u_{0\varepsilon} + a_{0\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} r_0, \\ l_\varepsilon(\cdot\,, 0) &= l_{0\varepsilon} = -u_{0\varepsilon} + a_{0\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} l_0 \quad \text{in } H_0^1(\mathbf{R}_+). \end{aligned} \tag{1.12}$$

We will prove the following estimate:

Theorem 1.2. *In the framework of Theorem* 1.1, *assume* $\frac{a_0}{x}, \frac{u_0}{x} \in L^\infty(\mathbf{R}_+)$. *Then, we have, for each* $t \geq 0$,

$$\Big\|\frac{r_\varepsilon(\cdot,t)}{x+\varepsilon}\Big\|_\infty \leq \Big\|\frac{r_{0\varepsilon}}{x+\varepsilon}\Big\|_\infty \leq \Big\|\frac{r_0}{x}\Big\|_\infty. \tag{1.13}$$

Hence,

$$\Big\|\frac{l_\varepsilon(\cdot,t)}{x+\varepsilon}\Big\|_\infty \leq \Big\|\frac{r_\varepsilon(\cdot,t)}{x+\varepsilon}\Big\|_\infty \leq \Big\|\frac{r_0}{x}\Big\|_\infty$$

and, a.e. in $t \in [0,+\infty[$, $\dfrac{l(\cdot,t)}{x}, \dfrac{r(\cdot,t)}{x} \in L^\infty(\mathbf{R}_+)$ *with*

$$\Big\|\frac{l(\cdot,t)}{x}\Big\|_\infty \leq \Big\|\frac{r(\cdot,t)}{x}\Big\|_\infty \leq \Big\|\frac{r_0}{x}\Big\|_\infty.$$

Now, by an adaptation of Kruzkov's methods (cf. [5]), we can prove the following theorem.

Theorem 1.3. *Under the hypothesis of Theorems* 1.1 *and* 1.2 *assume*

$$a_0, u_0 \in BV(\mathbf{R}_+) = \{w \in L^1(\mathbf{R}_+) \mid TV(w) < +\infty\}$$

where TV *denotes the total variation in* $\mathbf{R}_+$. *Then, we have*

$$\Big\|\frac{\partial r_\varepsilon(\cdot,t)}{\partial t}\Big\|_1 + \Big\|\frac{\partial l_\varepsilon(\cdot,t)}{\partial t}\Big\|_1 \leq c\, e^{c_1 t}, \quad t \geq 0, \tag{1.14}$$

where $c = c\Big(\|a_0\|_\infty, \|u_0\|_\infty, \big\|\frac{a_0}{x}\big\|_\infty, \big\|\frac{u_0}{x}\big\|_\infty, TV(a_0), TV(u_0)\Big) > 0$ *and* $c_1 = c_1\Big(\big\|\frac{a_0}{x}\big\|_\infty, \big\|\frac{u_0}{x}\big\|_\infty\Big) > 0$ *do not depend on* ε.

From Theorem 1.3 it is easy to derive

Corollary 1.1. *Under the assumptions of Theorem* 1.3, *the weak entropy solution* (a, u) *of the Cauchy problem* (1.1), (1.2) *obtained by the vanishing viscosity method verifies* $a(\cdot,t), u(\cdot,t) \in L^1(\mathbf{R}_+)$ *a.e. in* t *and there exists* $E \subset [0,+\infty[$ *such that* $m([0,+\infty[\backslash E) = 0$ *and*

$$\lim_{\substack{t\to 0^+\\ t\in E}} \int_{\mathbf{R}_+} (|a(x,t) - a_0(x)| + |u(x,t) - u_0(x)|)\, dx = 0.$$

Finally, also by adaptation of Kruzkov's method (cf. [5]), we will prove the following theorem.

Theorem 1.4. *Let the initial conditions* a_0, u_0 *be in* $H_0^1(\mathbf{R}_+)$ *and such that* $\frac{a_0}{x}, \frac{u_0}{x} \in L^\infty(\mathbf{R}_+)$, *and let* $(a_1, u_1), (a_2, u_2)$ *be two weak entropy solutions of the Cauchy problem* (1.1), (1.2) *such that, for* $i = 1, 2$,

$$\frac{a_i}{x}, \frac{u_i}{x} \in L^\infty_{\mathrm{loc}}([0,+\infty[\times[0,+\infty[)$$

and there exists $E \subset [0,+\infty[$, *with* $m([0,+\infty[\backslash E) = 0$, *such that for each* $R > 0$,

$$\lim_{\substack{t\to 0^+\\ t\in E}} \int_{0<x<R} (|a_i(x,t) - a_0(x)| + |u_i(x,t) - u_0(x)|)\, dx = 0. \tag{1.15}$$

Then, $(a_1, u_1) = (a_2, u_2)$ *a.e. in* $\mathbf{R}_+ \times [0,+\infty[$.

§2. Proof of Theorem 1.2.

With $r = r_\varepsilon$, $0 < \varepsilon < 1$, let us consider the first equation of (1.11):

$$r_t + r\,r_x + \frac{r^2 - l^2}{2(x+\varepsilon)} = \varepsilon\, r_{xx}. \tag{2.1}$$

With $v = v_\varepsilon = \dfrac{r_\varepsilon^2}{(x+\varepsilon)^2}$ we obtain, multiplying (2.1) by $\dfrac{2r}{(x+\varepsilon)^2}$, and since $r^2 - l^2 \geq 0$,

$$v_t + 2vr_x - 8\,\varepsilon\frac{r_x r}{(x+\varepsilon)^3} + 6\,\varepsilon\frac{r^2}{(x+\varepsilon)^4} + 2\,\varepsilon\frac{r_x^2}{(x+\varepsilon)^2} \leq \varepsilon v_{xx}. \tag{2.2}$$

If we multiply (2.2) by v^p, $p \geq 1$, and integrate in $\mathbf{R}_+$, we obtain with $\int \cdot = \int_{\mathbf{R}_+} \cdot\, dx$,

$$\begin{aligned}&\frac{1}{p+1}\frac{\partial}{\partial t}\int v^{p+1} + 2\int v^{p+1} r_x - 8\,\varepsilon\int\frac{r_x r v^p}{(x+\varepsilon)^3}\\ &+ 6\,\varepsilon\int\frac{r^2 v^p}{(x+\varepsilon)^4} + 2\,\varepsilon\int\frac{r_x^2 v^p}{(x+\varepsilon)^2} + p\,\varepsilon\int (v_x)^2 v^{p+1} \leq 0.\end{aligned} \tag{2.3}$$

We have

$$\begin{aligned}2\int v^{p+1} r_x &= 2\int v^p r_x \left(\frac{r}{x+\varepsilon}\right)^2 = \frac{2}{3}\int v^p \frac{1}{(x+\varepsilon)^2}\frac{\partial}{\partial x}(r^3)\\ &= -\frac{2}{3}\int \frac{\partial}{\partial x}(v^p)\frac{1}{(x+\varepsilon)^2}\, r^3 + \frac{4}{3}\int v^p \frac{1}{(x+\varepsilon)^3}\, r^3\\ &= -\frac{2}{3}\int \frac{\partial}{\partial x}(v^p) vr + \frac{4}{3}\int v^p v^{3/2}\\ &= -\frac{2}{3}\,p\int v^p v_x r + \frac{4}{3}\int v^{p+3/2}\\ &= -\frac{2}{3}\frac{p}{p+1}\int \frac{\partial}{\partial x}(v^{p+1})\, r + \frac{4}{3}\int v^{p+3/2}\\ &= \frac{2}{3}\frac{p}{p+1}\int (v^{p+1})\, r_x + \frac{4}{3}\int v^{p+3/2}\end{aligned}$$

and so

$$2\int v^{p+1} r_x = \frac{4(p+1)}{2p+3}\int v^{p+3/2}. \tag{2.4}$$

Moreover,

$$\begin{aligned}-8\int\frac{r_x r v^p}{(x+\varepsilon)^3} &= -4\int\frac{\frac{\partial}{\partial x}(r^2) v^p}{(x+\varepsilon)^3} = -12\int\frac{r^2 v^p}{(x+\varepsilon)^4} + 4\int\frac{r^2\frac{\partial}{\partial x}(v^p)}{(x+\varepsilon)^3}\\ &= -12\int\frac{v^{p+1}}{(x+\varepsilon)^2} + 4\,p\int\frac{r^2 v^{p-1} v_x}{(x+\varepsilon)^3}\\ &= -12\int\frac{v^{p+1}}{(x+\varepsilon)^2} + 4\frac{p}{p+1}\int\frac{\frac{\partial}{\partial x}(v^{p+1})}{(x+\varepsilon)}\\ &= -12\int\frac{v^{p+1}}{(x+\varepsilon)^2} + 4\frac{p}{p+1}\int\frac{v^{p+1}}{(x+\varepsilon)^2},\end{aligned}$$

and so

$$-8\,\varepsilon\int\frac{r_x r v^p}{(x+\varepsilon)^3} = \Big(-12 + 4\,\frac{p}{p+1}\Big)\varepsilon\int\frac{v^{p+1}}{(x+\varepsilon)^2}. \tag{2.5}$$

We have also

$$v_x = \frac{2rr_x}{(x+\varepsilon)^2} - 2\,\frac{r^2}{(x+\varepsilon)^3}$$

and so, with $p \geq 2$,

$$\begin{aligned}
v_x v^{p/2-1} r &= 2\,v^{p/2} r_x - 2\,v^{\frac{p+1}{2}},\\
v^{p/2} r_x &= \frac{1}{2}\,v_x v^{\frac{p-2}{2}} r + v^{\frac{p+1}{2}},\\
v^p r_x^2 &= \frac{1}{4}\,v_x^2 v^{p-1}(x+\varepsilon)^2 + v^{p+1} + v_x v^p (x+\varepsilon),\\
2\int \frac{r_x^2 v^p}{(x+\varepsilon)^2} &= \frac{1}{2}\int v^{p-1} v_x^2 + 2\int \frac{v^{p+1}}{(x+\varepsilon)^2} + \frac{2}{p+1}\int \frac{\frac{\partial}{\partial x}(v^{p+1})}{(x+\varepsilon)},\\
2\,\varepsilon \int \frac{r_x^2 v^p}{(x+\varepsilon)^2} &= \frac{1}{2}\varepsilon \int v^{p-1} v_x^2 + \Big(2 + \frac{2}{p+1}\Big)\varepsilon \int \frac{v^{p+1}}{(x+\varepsilon)^2}. \qquad (2.6)
\end{aligned}$$

By (2.3), $\cdots$, (2.6), we obtain

$$\frac{1}{p+1}\,\frac{\partial}{\partial t}\int v^{p+1} + \frac{4(p+1)}{2p+3}\int v^{p+3/2} \leq \frac{2}{p+1}\,\varepsilon \int \frac{v^{p+1}}{(x+\varepsilon)^2}. \qquad (2.7)$$

But, since

$$\frac{v^{p+1}}{(x+\varepsilon)^2} = \frac{v^{\frac{p+1}{2}}}{(x+\varepsilon)} \cdot v^{\frac{p+1}{2}} \cdot \frac{1}{(x+\varepsilon)}$$

and $\frac{1}{2} + \frac{p+1}{2p+3} + \frac{1}{4p+6} = 1$, we derive by Hölder's inequality,

$$\begin{aligned}
&\frac{2\varepsilon}{p+1}\int \frac{v^{p+1}}{(x+\varepsilon)^2}\\
&\leq \frac{2\varepsilon^{1/2}}{(p+1)^{1/2}}\Big(\int \frac{v^{p+1}}{(x+\varepsilon)^2}\Big)^{\frac{1}{2}}\Big(\int v^{p+3/2}\Big)^{\frac{p+1}{2p+3}}\frac{\varepsilon^{1/2}}{(p+1)^{1/2}}\Big(\int \frac{1}{(x+\varepsilon)^{4p+6}}\Big)^{\frac{1}{4p+6}}
\end{aligned}$$

and so

$$\frac{2\varepsilon}{p+1}\int \frac{v^{p+1}}{(x+\varepsilon)^2} \leq 2\Big(\int v^{p+3/2}\Big)^{\frac{2p+2}{2p+3}}\,\frac{1}{p+1}\,\frac{1}{(4p+5)^{\frac{2}{2p+3}}}\ \varepsilon^{2\left(\frac{1}{2}-\frac{4p+5}{4p+6}\right)}.$$

We derive, by the inequality $b^{1/q}c^{1/q'} \leq \frac{1}{q}b + \frac{1}{q'}c$, with $q = \frac{2p+3}{2p+2}$, $q' = 2p+3$,

$$\frac{2\varepsilon}{p+1}\int \frac{v^{p+1}}{(x+\varepsilon)^2} \leq 2\Big[\frac{2p+2}{2p+3}\int v^{p+3/2} + \frac{1}{2p+3}\,\frac{1}{(p+1)^{2p+3}}\,\frac{1}{4p+5}\varepsilon^{-2(p+1)}\Big]. \qquad (2.8)$$

If we fix $n \geq 1$, let p be such that

$$\frac{\varepsilon^{-2(p+1)}}{(p+1)^{2p+3}} \leq \varepsilon^{n(p+1)}\Big(\text{that is, } -\log\varepsilon \leq \frac{2p+3}{(p+1)\,(n+2)}\ \log(p+1)\Big).$$

We deduce, from (2.8),

$$\frac{2\varepsilon}{p+1}\int \frac{v^{p+1}}{(x+\varepsilon)^2} \leq \frac{4(p+1)}{2p+3}\int v^{p+3/2} + \frac{2}{(4p+5)(2p+3)}\ \varepsilon^{n(p+1)}. \qquad (2.9)$$

From (2.7) and (2.9) we derive

$$\frac{\partial}{\partial t}\int v^{p+1} \leq \frac{2(p+1)}{(4p+5)(2p+3)}\ \varepsilon^{n(p+1)}. \qquad (2.10)$$

Hence, for $t \geq 0$, we obtain

$$\|v_\varepsilon(\cdot,t)\|_{p+1}^{p+1} \;\leq\; \|v_{0\varepsilon}\|_{p+1}^{p+1} + \frac{2(p+1)t}{(4p+5)(2p+3)}\,\varepsilon^{n(p+1)}$$

and so, by the inequality $\big(b^{p+1}+a^{p+1}\big)^{1/(p+1)} \leq b+c$, we deduce

$$\|v_\varepsilon(\cdot,t)\|_{p+1} \;\leq\; \|v_{0\varepsilon}\|_{p+1} + \Big[\frac{2(p+1)t}{(4p+5)(2p+3)}\Big]^{\frac{1}{p+1}}\varepsilon^{n}. \tag{2.11}$$

Letting $p \to +\infty$ we obtain from (2.11), $\|v_\varepsilon(\cdot,t)\|_\infty \leq \|v_{0\varepsilon}\|_\infty + \varepsilon^n$. Now we let $n \to +\infty$ and we derive

$$\Big\|\frac{r_\varepsilon(\cdot,t)}{x+\varepsilon}\Big\|_\infty^2 = \|v_\varepsilon(\cdot,t)\|_\infty \leq \|v_{0\varepsilon}\|_\infty = \Big\|\frac{r_{0\varepsilon}}{x+\varepsilon}\Big\|_\infty^2 \leq \Big\|\frac{r_0}{x}\Big\|_\infty^2.$$

Finally we let $\varepsilon \to 0$ and we deduce a.e. on $t \in [0,+\infty[$, $\frac{r(\cdot,t)}{x} \in L^\infty(\mathbf{R}_+)$ and

$$\Big\|\frac{r(\cdot,t)}{x}\Big\|_\infty \leq \Big\|\frac{r_0}{x}\Big\|_\infty$$

and so

$$\Big\|\frac{l(\cdot,t)}{x}\Big\|_\infty \leq \Big\|\frac{r(\cdot,t)}{x}\Big\|_\infty \leq \Big\|\frac{r_0}{x}\Big\|_\infty,$$

and this achieves the proof of Theorem 1.2.

§3. Proofs of Theorems 1.3 and 1.4

3.1. Sketch of the proof of Theorem 1.3

We follow the lines of the proofs of Theorems 2.3 and 3.1 of Chap.II in [4], applying the method of Kruzkov for the case of scalar conservation laws (cf. [5]). We take the t derivative in both equations of the approximate system (1.11) in r_ε and l_ε; we multiply the first equation by $\mathrm{sgn}(r_{\varepsilon t})$, the second equation by $\mathrm{sgn}(l_{\varepsilon t})$ and both equations by $\psi_R(x) = \chi(x/R)$, $R > 0$, where sgn denotes the usual sign function and χ is the cut function introduced in (2.25) of Chap.II in [4]. We integrate in $\mathbf{R}_+$ and we add the two equations. If we point out the estimate

$$\Big|\int_{\mathbf{R}_+}\frac{2(r_\varepsilon r_{\varepsilon t} - l_\varepsilon l_{\varepsilon t})}{2(x+\varepsilon)}\,[\mathrm{sgn}(r_{\varepsilon t}) + \mathrm{sgn}(l_{\varepsilon t})]\,\psi_R(x)\,dx\Big| \leq c_1\int_{\mathbf{R}_+}(|r_{\varepsilon t}| + |l_{\varepsilon t}|)\,\psi_R(x)\,dx,$$

by Theorem 1.2, we easily deduce, with the help of Lemma 3.1 in Chap.II of [4],

$$\begin{aligned}&\int_{\mathbf{R}_+}(|r_{\varepsilon t}(x,t)| + |l_{\varepsilon t}(x,t)|)\,\psi_R(x)\,dx\\ &\leq c + \Big(\frac{c_0}{R} + c_1\Big)\int_0^t\int_{\mathbf{R}_+}(|r_{\varepsilon t}(x,\tau)| + |l_{\varepsilon t}(x,\tau)|)\,\psi_R(x)\,dx\,d\tau,\end{aligned}$$

where

$$\begin{aligned}c_0 &= c_0(\|a_0\|_\infty, \|u_0\|_\infty),\\ c &= c\Big(\|a_0\|_\infty,\, \|u_0\|_\infty,\, \Big\|\frac{a_0}{x}\Big\|_\infty,\, \Big\|\frac{u_0}{x}\Big\|_\infty,\, TV(a_0),\, TV(u_0)\Big),\\ c_1 &= c_1\Big(\Big\|\frac{a_0}{x}\Big\|_\infty,\, \Big\|\frac{u_0}{x}\Big\|_\infty\Big)\end{aligned}$$

are positive constants not depending on ε. The result follows if we apply Gronwall's inequality and then let $R \to \infty$.

Before proving Theorem 1.4, we need the following lemma that can be proved like the Lemma 4.2 in Chap.II of [4] (cf. also the inequality (3.12) in [5]).

Lemma 3.1. *Let* $(a_i, u_i) \in \left(L^\infty_{\rm loc}([0,+\infty[\times[0,+\infty[)^2\right.$, $i = 1,2$, *be two solutions in* $\mathcal{D}'(\mathbf{R}_+\times\mathbf{R}_+)$ *of the system* (1.1) *verifying the entropy condition* (1.8) *and let* $r_i = u_i + a_i$, $l_i = -u_i + a_i$, $i = 1,2$. *Then, we have, in* $\mathcal{D}'(\mathbf{R}_+\times\mathbf{R}_+)$,

$$\begin{aligned}&\frac{\partial}{\partial t}|r_1 - r_2| + \frac{\partial}{\partial x}\Big[\mathrm{sgn}(r_1 - r_2)\frac{1}{2}(r_1^2 - r_2^2)\Big]\\ &+ \mathrm{sgn}(r_1 - r_2)\frac{1}{2x}[(r_1^2 - l_1^2) - (r_2^2 - l_2^2)] \ \le\ 0,\\ &\frac{\partial}{\partial t}|l_1 - l_2| - \frac{\partial}{\partial x}\Big[\mathrm{sgn}(l_1 - l_2)\frac{1}{2}(l_1^2 - l_2^2)\Big]\\ &+ \mathrm{sgn}(l_1 - l_2)\frac{1}{2x}[(r_1^2 - l_1^2) - (r_2^2 - l_2^2)] \ \le\ 0.\end{aligned}$$

3.2. Sketch of the proof of Theorem 1.4

We follow the lines of the proof of Theorem 4.1 in Chap.II of [4], applying the method of Kruzkov for the case of scalar conservation laws (cf. [5]). With $T > 0$, $R > 0$ and $W = L^\infty\big(]0, R + MT + 1[\times[0, T + 1[\big)$, let us put

$$M = M(T,R) = \max_{1\le i\le 2}\left(\|r_i\|_W\,,\ \|l_i\|_W\right),$$

where $r_i = u_i + a_i$, $l_i = -u_i + a_i$, $i = 1,2$. For $\delta, \varepsilon, \theta > 0$, $\varepsilon < \delta < \min(1, T/3)$, $\theta < \min(1, R/2)$ and $x > 0$, we set

$$\varphi(x,t) = (1 - h(x/\theta))\,(Y_\varepsilon(t-\delta) - Y_\varepsilon(t-T))\,(1 - Y_\theta(x - R - M(T-t)))$$

where $Y_\varepsilon(t) = \int_{-\infty}^t \zeta_\varepsilon(s)\,ds$, $\zeta_\varepsilon \in \mathcal{D}(\mathbf{R})$ is a positive cut-off function with support in $[-\varepsilon,\varepsilon]$, $h \in C^\infty([0,+\infty[)$, $0 \le h \le 1$, $h(x) = 0$ if $x \ge 2$ and $h(x) = 1$ if $x \le 1$. With this choice of test function we deduce from the inequalities in Lemma 3.1:

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}(|r_1 - r_2| + |l_1 - l_2|)\,(1 - h(x/\theta))\,(\zeta_\varepsilon(t-\delta) - \zeta_\varepsilon(t-T))\\ &\cdot Y_\theta(x - R - M(T-t))\,dx\,dt\\ &-\int_0^{+\infty}\int_0^{+\infty}(1 - h(x/\theta))\,(Y_\varepsilon(t-\delta) - Y_\varepsilon(t-T))\Big[M(|r_1 - r_2| + |l_1 - l_2|)\\ &+\mathrm{sgn}(r_1 - r_2)\,\frac{1}{2}\,(r_1^2 - r_2^2) - \mathrm{sgn}(l_1 - l_2)\,\frac{1}{2}(l_1^2 - l_2^2)\Big]\\ &\cdot\zeta_\theta(x - R - M(T-t))\,dx\,dt\\ &-\int_0^{+\infty}\int_0^{+\infty}\frac{1}{\theta}h'(x/\theta)\,(Y_\varepsilon(t-\delta) - Y_\varepsilon(t-T))\,(1 - Y_\theta(x - R - M(T-t)))\\ &\cdot\Big[\mathrm{sgn}(r_1 - r_2)\,\frac{1}{2}\,(r_1^2 - r_2^2) - \mathrm{sgn}(l_1 - l_2)\,\frac{1}{2}\,(l_1^2 - l_2^2)]\,dx\,dt\\ &-\int_0^{+\infty}\int_0^{+\infty}(1 - h(x/\theta))\,(Y_\varepsilon(t-\delta) - Y_\varepsilon(t-T))\,(1 - Y_\theta(x - R - M(T-t)))\\ &\cdot[\mathrm{sgn}(r_1 - r_2) + \mathrm{sgn}(l_1 - l_2)]\,\frac{1}{2x}\,[(r_1^2 - l_1^2) - (r_2^2 - l_2^2)]\,dx\,dt\ \ge 0.\end{aligned}$$

Now, we point out that in the third integral in the previous inequality we can put $r_1^2 - r_2^2 = (r_1 - r_2)\,\dfrac{(r_1 + r_2)}{x}\,x$, $l_1^2 - l_2^2 = (l_1 - l_2)\,\dfrac{(l_1 + l_2)}{x}\,x$. Hence, by the assumptions, this integral can be estimated in modulus by

$$c\int_0^{T+1}\int_\theta^{2\theta}\frac{x}{\theta}\,dx\,dt \le c(T+1)\frac{3}{2}\,\theta \underset{\theta\to 0^+}{\longrightarrow} 0.$$

Moreover, regarding the last integral in the same inequality, we observe that

$$\begin{aligned}\Big|\frac{1}{x}\left[(r_1^2-l_1^2)-(r_2^2-l_2^2)\right]\Big| &\le \frac{|r_1|+|r_2|}{x}\,|r_1-r_2|+\frac{|l_1|+|l_2|}{x}\,|l_1-l_2|\\ &\le c\,(|r_1-r_2|+|l_1-l_2|).\end{aligned}$$

We can now continue as in the proof of Theorem 4.1 in Chap.II of [4] to deduce, for almost T and δ,

$$\begin{aligned}&\int_{|x|\le R}(|r_1(x,T)-r_2(x,T)|+|l_1(x,T)-l_2(x,T)|)\,dx\\ \le&\int_{|x|\le R+M(T-\delta)}(|r_1(x,\delta)-r_2(x,\delta)|+|l_1(x,\delta)-l_2(x,\delta)|)\,dx\\ &+c\int_0^T\int_{|x|\le R+M(T-\tau)}(|r_1(x,\tau)-r_2(x,\tau)|+|l_1(x,\tau)-l_2(x,\tau)|)\,dx\end{aligned}$$

with $c=c(T,R)>0$, increasing function of T and R. Hence, by the assumption (1.15), we derive, a.e. in $T>0$, since the two solutions have the same initial data,

$$\begin{aligned}&\int_{|x|\le R}(|r_1(x,T)-r_2(x,T)|+|l_1(x,T)-l_2(x,T)|)\,dx\\ \le&\ c\int_0^T\int_{|x|\le R+M(T-\tau)}(|r_1(x,\tau)-r_2(x,\tau)|+|l_1(x,\tau)-l_2(x,\tau)|)\,dx\end{aligned}$$

and this implies

$$\int_{|x|\le R}(|r_1(x,T)-r_2(x,T)|+|l_1(x,T)-l_2(x,T)|)\,dx=0$$

for all $R>0$ and a.e. in $T>0$, which achieves the proof of Theorem 1.4.

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