

REITERATED HOMOGENIZATION OF DEGENERATE NONLINEAR ELLIPTIC EQUATIONS

J. BYSTRÖM* J. ENGSTRÖM* P. WALL*

(Dedicated to the memory of Jacques-Louis Lions)

Abstract

The authors study homogenization of some nonlinear partial differential equations of the form $-\operatorname{div}(a(hx, h^2x, Du_h)) = f$, where a is periodic in the first two arguments and monotone in the third. In particular the case where a satisfies degenerated structure conditions is studied. It is proved that u_h converges weakly in $W_0^{1,1}(\Omega)$ to the unique solution of a limit problem as $h \rightarrow \infty$. Moreover, explicit expressions for the limit problem are obtained.

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§1. Introduction

This paper is devoted to homogenization of partial differential operators including several periodically oscillating length scales. This type of equations appear in many fields of physics and engineering sciences where the physical phenomena occur in highly heterogeneous media. One example is heat conduction in composite materials involving two different materials which are periodically distributed. The local characteristics are then described by rapidly oscillating functions. A direct numerical treatment of such problems is often impossible due to the rapidly oscillating functions and one has to apply some type of asymptotic analysis. The branch of mathematics developed for the analysis of these types of problems is known as homogenization. For more information concerning the homogenization theory, the reader is referred to [1, 2, 7, 10] and [12].

We will now give a short overview of previous results connected to this work and explain what our contribution is. Let us consider the class of partial differential equations of the form

$$-\operatorname{div}(a_h(x, Du_h)) = f \quad \text{on } \Omega, \quad u_h \in W_0^{1,p}(\Omega), \quad (1.1)$$

where a_h is increasingly oscillating as $h \rightarrow \infty$, Ω is an open bounded subset of $\mathbf{R}^n$, $1 < p < \infty$, $1/p + 1/q = 1$ and $f \in W^{-1,q}(\Omega)$. The homogenization problem for (1.1) consists of the study of the asymptotic behavior of solutions u_h as $h \rightarrow \infty$. In many important cases u_h converges weakly in $W_0^{1,p}(\Omega)$ to the solution u of the homogenized problem

$$-\operatorname{div}(b(Du)) = f \quad \text{on } \Omega, \quad u \in W_0^{1,p}(\Omega).$$

In [6] and [11] the following situation was studied: a_h is of the form $a_h(x, \xi) = a(hx, \xi)$, where a is monotone, continuous and satisfies suitable coerciveness and growth conditions in

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*Department of Mathematics, Luleå University of Technology, SE-971 87 Luleå, Sweden.

E-mail: wall@sm.luth.se

the second argument and is periodic in the first argument. A corresponding homogenization result, with the difference that a only satisfies degenerate structure conditions, was obtained in [9]. In this situation it is natural to work with weighted spaces which means that instead of (1.1) we have

$$-\operatorname{div}(a(hx, Du_h)) = f \quad \text{on } \Omega, \quad u_h \in W_0^{1,p}(\Omega, \lambda_h),$$

where (λ_h) is a sequence of periodic weights.

In the case when $a_h(x, \xi) = a(hx, h^2x, \xi)$, where a is periodic in the first two variables, one speaks about reiterated homogenization. This concept was introduced by Bensoussan, Lions and Papanicolaou in [1], where it was stated a result for linear operators. Concerning reiterated homogenization of nonlinear problems we refer to [13] and [14]. One important application of reiterated homogenization is that it has been an indispensable tool in the construction of structures with extreme effective material properties. Concerning this topic we refer to the collection of classical papers in [5], where the introduction gives a good selection of references. We remark that some of the homogenization problems above also have been studied by Γ -convergence for the corresponding variational problems and by two-scale convergence, but leave out this discussion since it is out of the scope of this work.

In this paper we study reiterated homogenization where a only satisfies degenerate structure conditions. More precisely we prove that the solutions u_h of

$$-\operatorname{div}(a(hx, h^2x, Du_h)) = f \quad \text{on } \Omega, \quad u_h \in W_0^{1,p}(\Omega, \lambda_h),$$

converges weakly to u in $W_0^{1,1}(\Omega)$, where u is the solution of a homogenized problem

$$-\operatorname{div}(b(Du)) = f \quad \text{on } \Omega, \quad u \in W_0^{1,p}(\Omega).$$

This paper is organized in the following way: In Section 2 we fix some notation and present necessary preliminary results. Section 3 contains the homogenization result described above, which also is the main result of this paper. In Section 4 we derive a homogenization result for an auxiliary problem. A key ingredient in the proof of the main result is that the solutions of the auxiliary problem are used to define a special type of test function. Finally, in Section 5 we give some properties of the homogenized operator b .

§2. Preliminaries and Notation

Let Ω be a regular bounded open subset of $\mathbf{R}^n$ and $|E|$ denote the Lebesgue measure of the set E in $\mathbf{R}^n$. Moreover let $\langle \cdot, \cdot \rangle$ denote the Euclidean scalar product on $\mathbf{R}^n$ and χ_E the characteristic function of the set E . Let p be a real constant $1 < p < \infty$ and let q be its conjugate exponent, $1/p + 1/q = 1$. We will denote by C and C_i constants that may change from one place to another.

Furthermore, let $Y = Z = (0, 1)^n$ be the unit cube in $\mathbf{R}^n$. Let $\{\Omega_i \subset Y : i = 1, \dots, N\}$ be a family of disjoint open sets such that $|Y \setminus \bigcup_{i=1}^N \Omega_i| = 0$ and $|\partial\Omega_i| = 0$.

Let λ be a weight on $\mathbf{R}^n$, i.e. λ is measurable and

$$\lambda > 0 \text{ a.e., } \lambda \text{ and } \lambda^{-1/(p-1)} \text{ are in } L_{\text{loc}}^1(\mathbf{R}^n). \quad (2.1)$$

We denote by $L^p(\Omega, \lambda)$ the set of real functions u in $L_{\text{loc}}^1(\Omega)$ such that $u\lambda^{1/p}$ is in $L^p(\Omega)$, by $W^{1,p}(\Omega, \lambda)$ the set of the functions u in $W_{\text{loc}}^{1,1}(\Omega)$ such that $u \in L^p(\Omega, \lambda)$ and $Du \in [L^p(\Omega, \lambda)]^n$. Moreover, we denote by $W_0^{1,p}(\Omega, \lambda)$ the completion of $C_0^1(\Omega)$ with respect to the norm in $W^{1,p}(\Omega, \lambda)$, i.e.

$$\|u\|_{W^{1,p}(\Omega, \lambda)} = \left(\int_{\Omega} (|u|^p + |Du|^p) \lambda \, dx \right)^{1/p}.$$

By $C_{\text{per}}^1(Y)$ we mean the set of all Y -periodic functions in $C^1(\mathbf{R}^n)$ with mean value zero. We also define $W_{\text{per}}^{1,p}(\Omega, \lambda)$ as the set of real functions u in $W_{\text{loc}}^{1,1}(\mathbf{R}^n)$ with mean value zero such that u is Y -periodic and $u \in W^{1,p}(Y, \lambda)$.

We now define the Muckenhoupt A_p class:

Definition 2.1. Let $p > 1$, $K \geq 1$ and let λ be a weight on $\mathbf{R}^n$. Then λ is in the class $A_p(K)$ if

$$\left(\frac{1}{|Q|} \int_Q \lambda \, dz \right) \left(\frac{1}{|Q|} \int_Q \lambda^{-\frac{1}{p-1}} \, dz \right)^{p-1} \leq K$$

for every cube $Q \in \mathbf{R}^n$ with faces parallel to the coordinate planes. We set $A_p = \bigcup_{K \geq 1} A_p(K)$.

Let λ_i be a Y -periodic weight on $\mathbf{R}^n$, i.e. λ_i satisfies (2.1) and is Y -periodic. We define the weights $\bar{\lambda}_h$ and λ_h as

$$\bar{\lambda}_h(x) = \sum_{i=1}^N \chi_{\Omega_i}(x) \lambda_i(hx), \quad \lambda_h(x) = \sum_{i=1}^N \chi_{\Omega_i}(hx) \lambda_i(h^2x). \quad (2.2)$$

Then it follows that $\bar{\lambda}_h$, $\bar{\lambda}_h^{-1/(p-1)}$, λ_h and $\lambda_h^{-1/(p-1)}$ all are in $L^1_{\text{loc}}(\mathbf{R}^n)$. Moreover, we assume that λ_i , $\bar{\lambda}_h$ and λ_h are in $A_p(K)$ for some K .

Let $a : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a function such that

$$a(y, z, \xi) = \sum_{i=1}^N \chi_{\Omega_i}(y) a_i(z, \xi). \quad (2.3)$$

We assume that $a(\cdot, z, \xi)$ is Y -periodic and $a(y, \cdot, \xi)$ is Z -periodic. We also assume that a satisfies certain continuity and monotonicity conditions. To be more specific, assume that there exist constants $c_1^i, c_2^i > 0$ and constants α and β with $0 \leq \alpha \leq \min(1, p-1)$ and $\max(p, 2) \leq \beta < \infty$ such that

$$|a_i(z, \xi_1) - a_i(z, \xi_2)| \leq c_1^i \lambda_i(z) (1 + |\xi_1| + |\xi_2|)^{p-1-\alpha} |\xi_1 - \xi_2|^\alpha, \quad (2.4)$$

$$\langle a_i(z, \xi_1) - a_i(z, \xi_2), \xi_1 - \xi_2 \rangle \geq c_2^i \lambda_i(z) (1 + |\xi_1| + |\xi_2|)^{p-\beta} |\xi_1 - \xi_2|^\beta, \quad (2.5)$$

for a.e. $z \in \mathbf{R}^n$, every $\xi \in \mathbf{R}^n$. Moreover we assume that

$$a_i(z, 0) = 0 \quad (2.6)$$

for a.e. $z \in \mathbf{R}^n$.

As a direct consequence of (2.4), (2.5), and (2.6) the following inequalities hold:

$$|a_i(z, \xi)| \leq c_a^i \lambda_i(z) (1 + |\xi|^{p-1}), \quad (2.7)$$

$$\lambda_i(z) |\xi|^p \leq c_b^i (\lambda_i(z) + \langle a_i(z, \xi), \xi \rangle), \quad (2.8)$$

$$\int_Z |\xi + Dv_i^\xi|^p \lambda_i(z) \, dz \leq c_c^i (1 + |\xi|^p). \quad (2.9)$$

In [8] the following result is proved.

Lemma 2.1. Let $p > 1$ and $K \geq 1$. Then there exist two positive constants $\delta = \delta(n, p, K)$ and $C = C(n, p, K)$ such that

$$\left(\frac{1}{|Q|} \int_Q \lambda^{1+\delta} \, dy \right)^{\frac{1}{1+\delta}} \leq C \frac{1}{|Q|} \int_Q \lambda \, dy, \quad (2.10)$$

$$\left(\frac{1}{|Q|} \int_Q \lambda^{-(1+\delta)/(p-1)} \, dy \right)^{\frac{1}{1+\delta}} \leq C \frac{1}{|Q|} \int_Q \lambda^{-1/(p-1)} \, dy, \quad (2.11)$$

for every cube with faces parallel to the coordinate planes and every $\lambda \in A_p(K)$.

In [9] the following weighted compensated compactness result is proved.

Lemma 2.2. Let $\nu \in A_p$, $K \geq 1$, let (λ_h) be a family in $A_p(K)$ and let Ω be an open bounded set. Let (u_h) be a family of functions satisfying

- (1) $\int_\Omega |Du_h|^p \lambda_h \, dy \leq C_1 < \infty$ for every $h \in \mathbf{N}$,
 - (2) there exists a function $u \in W^{1,p}(\Omega, \nu)$ such that $u_h \rightarrow u$ in $L^1(\Omega)$.
- Moreover, let (a_h) be a family of vector functions in $\mathbf{R}^n$ such that
- (3) $\int_\Omega |a_h|^q \lambda_h^{-1/(p-1)} \, dy \leq C_2 < \infty$ for every $h \in \mathbf{N}$,
 - (4) $\text{div}(a_h) = f \in L^\infty(\Omega)$ on $C_0^1(\Omega)$ for every $h \in \mathbf{N}$,

(5) there exists $a \in [L^q(\Omega, \nu^{-1/(p-1)})]^n$ such that $a_h \rightarrow a$ weakly in $[L^1(\Omega)]^n$.
Then

$$\int_{\Omega} \langle a_h, Du_h \rangle \phi \, dy \rightarrow \int_{\Omega} \langle a, Du \rangle \phi \, dy$$

for every $\phi \in C_0^\infty(\Omega)$.

In [15] the following convergence result for periodic functions is proved.

Lemma 2.3. Let $1 \leq p \leq \infty$ and let $u_h \in L_{\text{loc}}^p(\mathbf{R}^n)$ be Y -periodic for $h \in \mathbf{N}$. Moreover, suppose that $u_h \rightarrow u$ weakly in $L^p(Y)$ (weakly* if $p = \infty$) as $h \rightarrow \infty$. Let w_h be defined by $w_h(x) = u_h(hx)$. Then as $h \rightarrow \infty$ it holds that $w_h \rightarrow \frac{1}{|Y|} \int_Y u(y) \, dy$ weakly in $L^p(\Omega)$ (weakly* if $p = \infty$).

We end this section with a simple extension lemma.

Lemma 2.4. Let λ be a Y -periodic weight on $\mathbf{R}^n$; let $g : Y \rightarrow \mathbf{R}^n$ be a function such that

$$g \in [L^q(Y, \lambda^{-1/(p-1)})]^n, \quad \int_Y \langle g, Dw \rangle \, dy = 0, \quad \forall w \in W_{\text{per}}^{1,p}(Y, \lambda),$$

and let $\tilde{g}$ be the Y -periodic extension to $\mathbf{R}^n$ of g . Then we have

$$\tilde{g} \in [L_{\text{loc}}^q(\mathbf{R}^n, \lambda^{-1/(p-1)})]^n, \quad \int_{\mathbf{R}^n} \langle \tilde{g}, Dv \rangle \, dy = 0, \quad \forall v \in C_0^1(\mathbf{R}^n).$$

In situations where no confusion can occur we will use the same notation for the extended function as for the original one.

§3. The Main Theorem

Let us consider the following Dirichlet problems:

$$\begin{cases} \int_{\Omega} \langle a(hx, h^2x, Du_h), D\phi \rangle \, dx = \int_{\Omega} f\phi \, dx, & \forall \phi \in W_0^{1,p}(\Omega, \lambda_h), \\ u_h \in W_0^{1,p}(\Omega, \lambda_h), \end{cases} \quad (3.1)$$

where $f \in L^\infty(\Omega)$. By standard results in existence theory there exist unique solutions for each h . Below we state the main result of this paper.

Theorem 3.1. Let (u_h) be the solutions of (3.1). Then
 $u_h \rightarrow u$ weakly in $W_0^{1,1}(\Omega)$,

$$a(hx, h^2x, Du_h) \rightarrow b(Du) \text{ weakly in } [L^1(\Omega)]^n,$$

as $h \rightarrow \infty$, where u is the unique solution of

$$\begin{cases} \int_{\Omega} \langle b(Du), D\phi \rangle \, dx = \int_{\Omega} f\phi \, dx \text{ for every } \phi \in W_0^{1,p}(\Omega), \\ u \in W_0^{1,p}(\Omega). \end{cases} \quad (3.2)$$

The operator $b : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is defined as

$$b(\xi) = \frac{1}{|Y|} \int_Y b_Y(y, \xi + Du^\xi(y)) \, dy,$$

where u^ξ is the unique solution of the Y -cell problem

$$\begin{cases} \int_Y \langle b_Y(y, \xi + Du^\xi(y)), D\phi \rangle \, dy = 0 \text{ for every } \phi \in W_{\text{per}}^{1,p}(Y), \\ u^\xi \in W_{\text{per}}^{1,p}(Y). \end{cases}$$

The operator $b_Y : Y \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ is defined as $b_Y(y, \xi) = \sum_{i=1}^N \chi_{\Omega_i}(y) b_i(\xi)$, where

$$b_i(\xi) = \frac{1}{|Z|} \int_Z a_i\left(z, \xi + Dv_i^\xi(z)\right) dz,$$

and v_i^ξ are the unique solutions of the Z -cell problems

$$\begin{cases} \int_Z \left\langle a_i\left(z, \xi + Dv_i^\xi(z)\right), D\phi(z) \right\rangle dz = 0 \text{ for every } \phi \in W_{\text{per}}^{1,p}(Z, \lambda_i), \\ v_i^\xi \in W_{\text{per}}^{1,p}(Z, \lambda_i). \end{cases}$$

Proof. Let us first prove that $\|Du_h\|_{L^p(\Omega, \lambda_h)} \leq C$. By periodicity it follows that

$$\int_{\Omega} \lambda_h(x) dx \leq C, \quad \int_{\Omega} (\lambda_h(x))^{-1/(p-1)} dx \leq C. \quad (3.3)$$

Thus, by (2.8), (3.1), (3.3), Poincaré's and Hölder's inequalities we obtain

$$\begin{aligned} \int_{\Omega} |Du_h|^p \lambda_h(x) dx &\leq c_b \left(\int_{\Omega} \lambda_h(x) dx + \int_{\Omega} f u_h dx \right) \\ &\leq C \left(\int_{\Omega} \lambda_h(x) dx + \|f\|_{L^\infty(\Omega)} \int_{\Omega} |Du_h| dx \right) \leq C \left(1 + \left(\int_{\Omega} |Du_h|^p \lambda_h(x) dx \right)^{1/p} \right). \end{aligned}$$

Hence it is clear that

$$\|Du_h\|_{L^p(\Omega, \lambda_h)} \leq C. \quad (3.4)$$

Let η_h be defined as $\eta_h = a(hx, h^2x, Du_h)$. Then (2.7) and (3.4) implies

$$\|\eta_h\|_{[L^q(\Omega, \lambda_h^{-1/(p-1)})]^n} \leq C. \quad (3.5)$$

Take $\delta > 0$ such that (2.11) holds. Now choose σ_1 such that

$$\frac{1 + \sigma_1}{p - 1 - \sigma_1} = \frac{1 + \delta}{p - 1}.$$

Then $\sigma_1 > 0$ and $p - 1 - \sigma_1 > 0$. Let Q be a cube in $\mathbf{R}^n$ containing Ω . Hölder's inequality and (3.4) then gives

$$\begin{aligned} \int_{\Omega} |Du_h|^{1+\sigma_1} dx &\leq \left(\int_{\Omega} |Du_h|^p \lambda_h dx \right)^{\frac{1+\sigma_1}{p}} \left(\int_{\Omega} \lambda_h^{-\frac{1+\sigma_1}{p-1-\sigma_1}} dx \right)^{\frac{p-1-\sigma_1}{p}} \\ &\leq C \left(\int_{\Omega} \lambda_h^{-\frac{1+\delta}{p-1}} dx \right)^{\frac{p-1}{p+\delta}} \leq C \left(\int_Q \lambda_h^{-\frac{1+\delta}{p-1}} dx \right)^{\frac{p-1}{p+\delta}}. \end{aligned}$$

By applying (2.11) and (3.3) in the inequality above, we obtain

$$\int_{\Omega} |Du_h|^{1+\sigma_1} dx \leq C \left(\int_Q \lambda_h^{1/(1-p)} dx \right)^{(p-1)(1+\delta)/(p+\delta)} \leq C. \quad (3.6)$$

Next, choose $\delta > 0$ such that (2.10) holds and choose σ_2 such that

$$1 + \delta = (1 + \sigma_2) \frac{q-1}{q-1-\sigma_2}.$$

Then $\sigma_2 > 0$ and $q - 1 - \sigma_2 > 0$. By using (2.10) and arguing similarly as for (3.6) we obtain

$$\int_{\Omega} |\eta_h|^{1+\sigma_2} dx \leq C.$$

This means that (u_h) and (η_h) are bounded in $W_0^{1,1+\sigma_1}(\Omega)$ and $[L^{1+\sigma_2}(\Omega)]^n$ respectively. Since these spaces are reflexive, we have that there exist subsequences, still denoted by (u_h) and (η_h) , such that

$$u_h \rightarrow u_* \text{ weakly in } W_0^{1,1+\sigma_1}(\Omega), \quad (3.7)$$

$$\eta_h \rightarrow \eta_* \text{ weakly in } [L^{1+\sigma_2}(\Omega)]^n. \quad (3.8)$$

From (3.7) and (3.8) it follows that

$$u_h \rightarrow u_* \text{ weakly in } W_0^{1,1}(\Omega), \quad (3.9)$$

$$\eta_h \rightarrow \eta_* \text{ weakly in } [L^1(\Omega)]^n. \quad (3.10)$$

From our original problem (3.1) we have

$$\int_{\Omega} \langle \eta_h, D\phi \rangle dx = \int_{\Omega} f\phi dx \text{ for every } \phi \in W_0^{1,p}(\Omega, \lambda_h). \quad (3.11)$$

By using the fact $C_0^1(\Omega) \subset W_0^{1,p}(\Omega, \lambda_h)$ and (3.10) we can pass to the limit in (3.11), thus

$$\int_{\Omega} \langle \eta_*, D\phi \rangle dx = \int_{\Omega} f\phi dx \text{ for every } \phi \in C_0^1(\Omega).$$

Density and the fact that $\eta_* \in [L^q(\Omega)]^n$ (see (3.17)) then gives

$$\int_{\Omega} \langle \eta_*, D\phi \rangle dx = \int_{\Omega} f\phi dx \text{ for every } \phi \in W_0^{1,p}(\Omega). \quad (3.12)$$

Let us now observe that by (3.9), (3.10) and (3.12) the theorem is proved if we show that

$$u_* \in W_0^{1,p}(\Omega), \quad (3.13)$$

$$\eta_* = b(Du_*) \text{ a.e. on } \Omega, \quad (3.14)$$

since the uniqueness of the solution of the homogenized problem (3.2) then implies that $u_* = u$ a.e. on Ω .

We start with the proof of (3.13). We observe that since Ω is a regular bounded open set it is sufficient to show that $Du_* \in [L^p(\Omega)]^n$. Let $\phi \in C_0(\Omega)$. Then Hölder's inequality and (3.4) gives

$$\begin{aligned} \|Du_h \phi\|_{[L^1(\Omega)]^n} &\leq \left(\int_{\Omega} |Du_h|^p \lambda_h(x) dx \right)^{\frac{1}{p}} \left(\int_{\Omega} (\lambda_h(x))^{-\frac{1}{p-1}} |\phi|^q dx \right)^{\frac{1}{q}} \\ &\leq C \left(\int_{\Omega} (\lambda_h(x))^{-\frac{1}{p-1}} |\phi|^q dx \right)^{\frac{1}{q}}. \end{aligned} \quad (3.15)$$

Applying $\liminf$ on both sides of (3.15) and using the weak lower semicontinuity of the norm on the left hand side and periodicity on the right hand side we obtain

$$\int_{\Omega} |Du_*| |\phi| dx \leq C \|\phi\|_{L^q(\Omega)} \text{ for every } \phi \in C_0(\Omega).$$

By density and Landau's theorem we then have that

$$Du_* \in [L^p(\Omega)]^n. \quad (3.16)$$

By using (3.5) and arguments similar to those employed in the proof of (3.16) it can also be deduced that

$$\eta_* \in [L^q(\Omega)]^n. \quad (3.17)$$

It remains to prove (3.14). For this purpose let us define the test function

$$w_h^\xi(x) = (\xi, x) + \frac{1}{h} u_h^\xi(hx),$$

where u_h^ξ is defined as in the auxiliary problem (see Section 4). To be able to apply the compensated compactness result (Lemma 2.2) we have to prove certain facts about w_h^ξ and $a(hx, h^2x, Dw_h^\xi)$. Therefore, by periodicity, (4.5) and the fact that $\lambda_h(x) = \bar{\lambda}_h(hx)$ we get

$$\int_{\Omega} \left| Dw_h^\xi \right|^p \lambda_h(x) dx \leq C. \quad (3.18)$$

Moreover, by using (2.7) and (3.18) we obtain

$$\int_{\Omega} |a(hx, h^2x, Dw_h^\xi)|^q (\lambda_h(x))^{-1/(p-1)} dx \leq C.$$

By periodicity and Lemma 2.3 we have that

$$\begin{aligned} w_h^\xi(\cdot) &\rightarrow (\xi, \cdot) \text{ in } L^1(\Omega), \\ a(hx, h^2x, Dw_h^\xi) &\rightarrow \frac{1}{|Y|} \int_Y b_Y(y, \xi + Du^\xi) dy = b(\xi) \text{ weakly in } [L^1(\Omega)]^n. \end{aligned}$$

Finally, due to (4.6), we can apply Lemma 2.4 on (4.1) and obtain

$$\operatorname{div} \left(a(hx, h^2x, Dw_h^\xi) \right) = 0 \text{ on } C_0^1(\Omega),$$

and we are now ready to apply the compensated compactness result. Indeed, by the monotonicity of a we have for a fixed ξ that

$$\int_{\Omega} \left\langle \eta_h - a \left(hx, h^2 x, Dw_h^{\xi}(x) \right), Du_h(x) - Dw_h^{\xi}(x) \right\rangle \phi(x) dx \geq 0$$

for every $\phi \in C_0^{\infty}(\Omega)$, $\phi \geq 0$. By the compensated compactness lemma (Lemma 2.2) with the weight $\nu = 1$, we get in the limit

$$\int_{\Omega} \langle \eta_* - b(\xi), Du_*(x) - \xi \rangle \phi(x) dx \geq 0$$

for every $\phi \in C_0^{\infty}(\Omega)$, $\phi \geq 0$. Hence for our fixed $\xi \in \mathbf{R}^n$ we have that

$$\langle \eta_* - b(\xi), Du_*(x) - \xi \rangle \geq 0 \text{ for a.e. } x \in \Omega.$$

By density and the continuity of b (see Lemma 5.2), it follows that

$$\langle \eta_* - b(\xi), Du_*(x) - \xi \rangle \geq 0 \text{ for a.e. } x \in \Omega \text{ and every } \xi \in \mathbf{R}^n.$$

Since b is monotone (5.3) and continuous (5.4), we have that b is maximal monotone and hence (3.14) follows. Finally, let us observe that we have proved the theorem only up to a subsequence, but since the homogenized operator is uniquely defined and the solution of the homogenized problem is unique we can conclude that the theorem holds for the whole sequence.

§4. An Auxiliary Problem

In this section we prove a homogenization result for the auxiliary problem. This result was used in the definition of the special type of test functions defined in the proof of the main result (Theorem 3.1) of this paper.

Fix ξ and consider the following Dirichlet problems:

$$\begin{cases} \int_Y \langle a(y, hy, \xi + Du_h^{\xi}), D\phi \rangle dy = 0, \forall \phi \in W_{\text{per}}^{1,p}(Y, \bar{\lambda}_h), \\ u_h^{\xi} \in W_{\text{per}}^{1,p}(Y, \bar{\lambda}_h). \end{cases} \quad (4.1)$$

By standard results in existence theory there exist unique solutions for each h . Below we state the auxiliary result of this paper.

Theorem 4.1. *Let (u_h^{ξ}) be the solutions of (4.1). We then have that*

$$\begin{aligned} u_h^{\xi} &\rightarrow u^{\xi} \text{ weakly in } W_{\text{per}}^{1,1}(Y), \\ a(y, hy, \xi + Du_h^{\xi}) &\rightarrow b_Y(y, \xi + Du^{\xi}) \text{ weakly in } [L^1(Y)]^n, \end{aligned}$$

as $h \rightarrow \infty$, where u^{ξ} is the unique solution of

$$\begin{cases} \int_Y \langle b_Y(y, \xi + Du^{\xi}), D\phi \rangle dy = 0 \text{ for every } \phi \in W_{\text{per}}^{1,p}(Y), \\ u^{\xi} \in W_{\text{per}}^{1,p}(Y). \end{cases} \quad (4.2)$$

The operator $b_Y : Y \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ is defined as $b_Y(y, \tau) = \sum_{i=1}^N \chi_{\Omega_i}(y) b_i(\tau)$, where

$$b_i(\tau) = \frac{1}{|Z|} \int_Z a_i(z, \tau + Dv_i^{\tau}(z)) dz$$

and v_i^{τ} are the unique solutions of the Z -cell problems

$$\begin{cases} \int_Z \langle a_i(z, \tau + Dv_i^{\tau}(z)), D\phi(z) \rangle dz = 0, \forall \phi \in W_{\text{per}}^{1,p}(Z, \lambda_i), \\ v_i^{\tau} \in W_{\text{per}}^{1,p}(Z, \lambda_i). \end{cases} \quad (4.3)$$

Proof. By (2.8), (4.1) and (2.7) we have that

$$\int_Y |\xi + Du_h^{\xi}|^p \bar{\lambda}_h(y) dy \leq C_b \left(\int_Y \bar{\lambda}_h(y) dy + C_a \int_Y \left(1 + |\xi + Du_h^{\xi}|^{p-1} \right) \bar{\lambda}_h(y) |\xi| dy \right).$$

Moreover, if we use Young's inequality on the right hand side and rearrange the resulting inequality we obtain

$$\begin{aligned} & \left(1 - \frac{C_a C_b 2^{q-1} \eta}{q}\right) \int_Y |\xi + Du_h^\xi|^p \bar{\lambda}_h(y) dy \\ & \leq C_b \left(1 + \frac{C_a 2^{q-1} \eta}{q} + \frac{C_a |\xi|^p \eta^{-(p-1)}}{p}\right) \int_Y \bar{\lambda}_h(y) dy, \end{aligned}$$

where η is a positive real number. By choosing η small enough we get that

$$\int_Y |\xi + Du_h^\xi|^p \bar{\lambda}_h(y) dy \leq C. \quad (4.4)$$

In particular this implies

$$\int_Y |Du_h^\xi|^p \bar{\lambda}_h(y) dy \leq C. \quad (4.5)$$

Let us define $\eta_h^i = a_i(hy, \xi + Du_h^\xi)$. By using (2.7) and (4.4), it follows that

$$\int_{\Omega_i} |\eta_h^i|^q (\lambda_i(hy))^{-1/(p-1)} dy \leq C. \quad (4.6)$$

Moreover, by using (2.11), (4.5), (2.10), (4.6) and arguments similar to those employed in the proof of (3.6), it can be deduced that

$$\int_Y |Du_h^\xi|^{1+\kappa_1} dy \leq C, \quad \int_{\Omega_i} |\eta_h^i|^{1+\kappa_2} dy \leq C.$$

Thus we have that (u_h^ξ) and (η_h^i) are bounded in $W_{\text{per}}^{1,1+\kappa_1}(Y)$ and $[L^{1+\kappa_2}(\Omega_i)]^n$ respectively.

Since these spaces are reflexive, there exist subsequences, still denoted by (u_h^ξ) and (η_h^i) , such that

$$u_h^\xi \rightarrow u_*^\xi \text{ weakly in } W_{\text{per}}^{1,1+\kappa_1}(Y), \quad \eta_h^i \rightarrow \eta_*^i \text{ weakly in } [L^{1+\kappa_2}(\Omega_i)]^n.$$

Hence we can conclude that

$$u_h^\xi \rightarrow u_*^\xi \text{ weakly in } W_{\text{per}}^{1,1}(Y), \quad \eta_h^i \rightarrow \eta_*^i \text{ weakly in } [L^1(\Omega_i)]^n. \quad (4.7)$$

Using similar ideas as in the proof of (3.16), it can be shown that

$$\eta_*^i \in [L^q(Y)]^n. \quad (4.8)$$

From (4.1) and (4.7) it follows that

$$\sum_{i=1}^N \int_{\Omega_i} \langle \eta_*^i, D\phi \rangle dy = 0 \text{ for every } \phi \in C_{\text{per}}^1(Y).$$

Density arguments in conjunction with (4.8) then results in

$$\sum_{i=1}^N \int_{\Omega_i} \langle \eta_*^i, D\phi \rangle dy = 0 \text{ for every } \phi \in W_{\text{per}}^{1,p}(Y).$$

Thus the theorem is proved if we show that

$$u_*^\xi \in W_{\text{per}}^{1,p}(Y), \quad (4.9)$$

$$\eta_*^i = b_i(\xi + Du_*^\xi) \text{ a.e. on } \Omega_i, \quad (4.10)$$

since the uniqueness of the solution of the homogenized problem (4.2) then implies that $u_*^\xi = u^\xi$ a.e. on Y .

Let us start with (4.9). We observe that since Y is a regular bounded open set it is sufficient to show that $Du_*^\xi \in [L^p(Y)]^n$ which is obtained using the same ideas as in the proof of (3.16).

It remains to prove (4.10). Therefore let us define the test function $w_h^{\tau,i}$ by

$$w_h^{\tau,i}(y) = (\tau, y) + \frac{1}{h} v_i^\tau(hy), \quad (4.11)$$

where $v_i^\tau \in W_{\text{per}}^{1,p}(Z, \lambda_i)$ is defined as in (4.3). To be able to apply the compensated compactness result (Lemma 2.2) we have to prove certain facts about $w_h^{\tau,i}$ and $a_i(hy, Dw_h^{\tau,i})$. Indeed,

$$\int_{\Omega_i} |Dw_h^{\tau,i}|^p \lambda_i(hy) dy \leq C \quad (4.12)$$

follows from (4.11). Moreover, we also have that

$$\int_{\Omega_i} |a_i(hy, Dw_h^{\tau,i})|^q (\lambda_i(hy))^{-1/(p-1)} dy \leq C$$

by (2.7) and (4.12). By periodicity we obtain

$$\begin{aligned} w_h^{\tau,i}(\cdot) &\rightarrow (\tau, \cdot) \text{ strongly in } L^1(\Omega_i), \\ a_i(hy, Dw_h^{\tau,i}) &\rightarrow \frac{1}{|Z|} \int_Z a_i(z, \tau + Dv_i^\tau(z)) dz = b_i(\tau) \text{ weakly in } [L^1(\Omega_i)]^n. \end{aligned}$$

Application of Lemma 2.4 on (4.3) gives $\text{div}(a_i(hy, Dw_h^{\tau,i})) = 0$ on $C_0^1(\Omega_i)$. By the monotonicity of a_i we have for a fixed τ that

$$\int_{\Omega_i} \langle \eta_*^i - a_i(hy, Dw_h^{\tau,i}(y)), \xi + Du_h^\xi(y) - Dw_h^{\tau,i}(y) \rangle \phi(y) dy \geq 0$$

for every $\phi \in C_0^\infty(\Omega_i)$, $\phi \geq 0$. By the compensated compactness lemma (Lemma 2.2) with $\nu = 1$, we then get in the limit

$$\int_{\Omega_i} \langle \eta_*^i - b_i(\tau), \xi + Du_*^\xi(y) - \tau \rangle \phi(y) dy \geq 0$$

for every $\phi \in C_0^\infty(\Omega_i)$, $\phi \geq 0$. Hence for our fixed $\tau \in \mathbf{R}^n$ we have that

$$\langle \eta_*^i - b_i(\tau), \xi + Du_*^\xi(y) - \tau \rangle \geq 0 \text{ for a.e. } y \in \Omega_i.$$

By density and the continuity of b_i (5.2), it follows that

$$\langle a_* - b_i(\tau), \xi + Du_*^\xi(y) - \tau \rangle \geq 0 \text{ for a.e. } y \in \Omega_i \text{ and every } \tau \in \mathbf{R}^n.$$

Since b_i is monotone (5.1) and continuous (5.2), we have that b_i is maximal monotone and hence (4.10) follows.

§5. Properties of the Homogenized Operators b_i and b

In this section we list some properties of the homogenized operators b_i and b . In particular these properties imply the existence and uniqueness of the solution of the homogenized problem (in the auxiliary and main problem respectively).

Lemma 5.1. *Let b_i be the homogenized operator defined in Theorem 4.1. Then*

(a) *$b_i(\cdot)$ is strictly monotone. In particular, we have that*

$$\langle b_i(\tau_1) - b_i(\tau_2), \tau_1 - \tau_2 \rangle \geq \tilde{c}_2 (1 + |\tau_1| + |\tau_2|)^{p-\beta} |\tau_1 - \tau_2|^\beta \quad (5.1)$$

for every $\tau_1, \tau_2 \in \mathbf{R}^n$.

(b) $b_i(\cdot)$ is continuous. In particular, we have for $\gamma = \frac{\alpha}{\beta-\alpha}$ that

$$|b_i(\tau_1) - b_i(\tau_2)| \leq \tilde{c}_1 (1 + |\tau_1| + |\tau_2|)^{p-1-\gamma} |\tau_1 - \tau_2|^\gamma \quad (5.2)$$

for every $\tau_1, \tau_2 \in \mathbf{R}^n$.

(c) $b_i(0) = 0$.

Proof. These properties follow by using the same ideas as in [3] and [4].

Lemma 5.2. Let b be the homogenized operator defined in Theorem 3.1. Then

(a) $b(\cdot)$ is strictly monotone. In particular, we have that

$$\langle b(\xi_1) - b(\xi_2), \xi_1 - \xi_2 \rangle \geq \tilde{C}_2 (1 + |\xi_1| + |\xi_2|)^{p-\beta} |\xi_1 - \xi_2|^\beta \quad (5.3)$$

for every $\xi_1, \xi_2 \in \mathbf{R}^n$.

(b) $b(\cdot)$ is continuous. In particular, we have for $\delta = \frac{\gamma}{\beta-\gamma} = \frac{\alpha}{(\beta-\alpha)\beta-\alpha}$ that

$$|b(\xi_1) - b(\xi_2)| \leq \tilde{C}_1 (1 + |\xi_1| + |\xi_2|)^{p-1-\delta} |\xi_1 - \xi_2|^\delta \quad (5.4)$$

for every $\xi_1, \xi_2 \in \mathbf{R}^n$.

(c) $b(0) = 0$.

Proof. These properties follow by using (5.1) and (5.2) in the corresponding theorem given in for example [3] and [4].

REFERENCES

- [1] Bensoussan, A., Lions, J. L. & Papanicolaou, G., Asymptotic analysis for periodic structures [M], North Holland, Amsterdam, 1978.
- [2] Braides, A. & Defranceschi, A., Homogenization of multiple integrals [M], Oxford University Press, New York, 1998.
- [3] Byström, J., Correctors for some nonlinear monotone operators [R], Research report, No. 11, ISSN: 1400-4003, Department of Mathematics, Luleå University of Technology, 1999.
- [4] Byström, J., Correctors for some nonlinear monotone operators [J], *J. Nonlinear Math. Phys.*, **8**:1(2001), 8-30.
- [5] Cherkaev, A. & Kohn, R., Topics in the mathematical modelling of composite materials [M], Birkhäuser, Boston, 1997.
- [6] Chiado Piat, V. & Defranceschi, A., Homogenization of monotone operators [J], *Nonlinear Anal.*, **14**:9(1990), 717-732.
- [7] Cioranescu, D. & Donato, P., An introduction to homogenization [M], Oxford University Press, Oxford, 1999.
- [8] Coifman, R. R. & Fefferman, C., Weighted norm inequalities for maximal function and singular integrals [J], *Studia Math.*, **51**(1974), 241-250.
- [9] De Arcangelis, R. & Serra Cassano, F., On the homogenization of degenerate elliptic equations in divergence form [J], *J. Math. Pures Appl.*, **71**(1992), 119-138.
- [10] Defranceschi, A., An introduction to homogenization and G -convergence [M], Lecture notes, School on homogenization, ICTP, Trieste, 1993.
- [11] Fusco, N. & Moscariello, G., On the homogenization of quasilinear divergence structure operators [J], *Annali Mat. Pura Appl.*, **146**(1987), 1-13.
- [12] Jikov, V., Kozlov, S. & Oleinik, O., Homogenization of differential operators and integral functionals [M], Springer-Verlag, Berlin-Heidelberg-New York, 1994.
- [13] Lions, J. -L., Lukkassen, D., Persson, L. -E. & Wall, P., Reiterated homogenization of monotone operators [J], *C. R. Acad. Sci. Paris Series I*, **330**:8(2000), 675-680.
- [14] Lions, J. -L., Lukkassen, D., Persson, L. -E. & Wall, P., Reiterated homogenization of nonlinear monotone operators [J], *Chin. Ann. Math.*, **22B**:1(2001), 1-12.
- [15] Lukkassen, D. & Wall, P., On weak convergence of locally periodic functions [J], *J. Nonlinear. Math. Phys.*, **9**:1(2002), 42-57 2002.