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GLOBAL CLASSICAL SOLUTIONS WITH SMALL INITIAL TOTAL VARIATION FOR QUASILINEAR HYPERBOLIC SYSTEMS

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Abstract

By means of the continuous Glimm functional, a proof is given on the global existence of classical solutions to Cauchy problem for general first order quasilinear hyperbolic systems with small initial total variation.

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§1. Introduction

Consider the following first order quasilinear strictly hyperbolic system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \tag{1.1}$$

where $u = (u_1, \cdots, u_n)^T$ is the unknown vector function of (t, x) and $A(u) = (a_{ij}(u))$ is an $n \times n$ matrix with suitably smooth elements $a_{ij}(u)(i, j = 1, \cdots, n)$.

By the strict hyperbolicity, for any given u on the domain under consideration, $A(u)$ has n distinct real eigenvalues:

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u). \tag{1.2}$$

Let $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))$(resp. $r_i(u) = (r_{i1}(u), \cdots, r_{in}(u))^T$) be a left (resp. right) eigenvector corresponding to $\lambda_i(u)(i = 1, \cdots, n)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \quad (\text{resp. } A(u)r_i(u) = \lambda_i(u)r_i(u)). \tag{1.3}$$

We have

$$\det|l_{ij}(u)| \neq 0 \quad (\text{equivalently, } \det|r_{ij}(u)| \neq 0). \tag{1.4}$$

All $\lambda_i(u), l_{ij}(u)$ and $r_{ij}(u)(i, j = 1, \cdots, n)$ have the same regularity as $a_{ij}(u)(i, j = 1, \cdots, n)$.

Without loss of generality, we may suppose that

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1 \cdots, n), \tag{1.5}$$

$$r_i^T(u)r_i(u) \equiv 1 \quad (i, j = 1 \cdots, n), \tag{1.6}$$

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where δ_{ij} stands for the Kronecker's symbol.

For the following initial data

$$t = 0 : u = \varphi(x), \tag{1.7}$$

where $\varphi(x)$ is a "small" C^1 vector function of x. If $\varphi(x)$ possesses certain decay properties as $|x| \to +\infty$ and System (1.1) is weakly linearly degenerate, Li Ta-tsien, Zhou Yi and Kong De-xing have given in [1, 2] a complete result on the global existence of C^1 solution to Cauchy problem (1.1) and (1.7). Later, by means of the continuous Glimm functional, Li Ta-tsien and Kong De-xing in [3] simplified the proof of the global existence given in [2]. By constructing a counter example, Kong De-xing showed in [4] that the necessary decay property of initial data is essential for guaranteeing the global existence of classical solution to Cauchy problem (1.1) and (1.7). Moreover, in the case that $\varphi(x)$ possesses compact support and small initial total variation, when system (1.1) is strictly hyperbolic and linearly degenerate in the sense of P.D.Lax, A.Bressan also gave in [5] the global existence of classical solution to Cauchy problem (1.1) and (1.7).

The main aim of this paper is to generalize the result of [5] to the case that system (1.1) is weakly linearly degenerate.

First of all, we recall the concept of weak linear degeneracy (see [1, 2]) as follows.

Definition 1.1. *The i-th characteristic $\lambda_i(u)$ is weakly linearly degenerate, if along the i-th characteristic trajectory $u = u^{(i)}(s)$ passing through $u = 0$, defined by*

$$\begin{cases} \frac{du}{ds} = r_i(u), \\ s = 0 : u = 0, \end{cases} \tag{1.8}$$

we have

$$\nabla\lambda_i(u)r_i(u) \equiv 0, \quad \forall |u| \quad \text{small}, \tag{1.9}$$

namely,

$$\lambda_i(u^{(i)}(s)) \equiv \lambda_i(0), \quad \forall |s| \quad \text{small}. \tag{1.10}$$

If all characteristics are weakly linearly degenerate, System (1.1) *is called to be weakly linearly degenerate.*

The main result of this paper is the following

Theorem 1.1. *Suppose that in a neighbourhood of $u = 0$, $A(u) \in c^2$ and System* (1.1) *is strictly hyperbolic and weakly linearly degenerate. Suppose furthermore that the initial data $\varphi(x)$ satisfy the following properties*:

(i) $\varphi(x) \in C^1$;

(ii) $\varphi(x)$ *has compact support: supp*$\varphi(x) \in [\alpha_0, \beta_0]$,*where* $\alpha_0 < \beta_0$;

(iii) *The initial total variation is small enough, namely,*

$$\theta \stackrel{\triangle}{=} \int_{-\infty}^{+\infty} |\varphi'(x)|dx << 1. \tag{1.11}$$

Then there exists $\theta_0 > 0$ so small that for any given $\theta \in [0, \theta_0]$, Cauchy problem (1.1) *and* (1.7) *admits a unique global C^1 solution $u = u(t, x)$ for all $t \in \mathbb{R}$.*

For the sake of completeness, in Section 2 we will briefly recall F. John's formulas on the decomposition of waves with some supplements (see [6,1]), which will play an important role in the sequel. In Section 3, we shall establish a uniform a priori estimate on the C^1 norm of C^1 solution $u = u(t, x)$ to Cauchy problem (1.1) and (1.7), and then prove Theorem 1.1.

§2. Decomposition of Waves

Suppose that $A(u) \in C^k$, where k is an integer ≥ 1. By Lemma 2.5 in [1], there exists a C^{k+1} local diffeomorphism $u = u(\tilde{u})(u(0) = 0)$, such that in $\tilde{u}$-space, for each $i = 1, \cdots, n$, the i-th characteristic trajectory passing through $\tilde{u} = 0$ coincides with the $\tilde{u}$-axis at least for $|\tilde{u}|$ small, namely,

$$\tilde{r}_i(\tilde{u}_i e_i) \| e_i, \quad \forall |u_i| \text{ small} \quad (i = 1, \cdots, n), \tag{2.1}$$

where

$$e_i = (0, \cdots, 0, \overset{(i)}{1}, 0, \cdots, 0)^T. \tag{2.2}$$

Such a diffeomorphism is called the normalized transformation and the corresponding unknown variables $\tilde{u} = (\tilde{u}_1, \cdots, \tilde{u}_n)$ are called the normalized variables or normalized coordinates.

Let

$$v_i = l_i(u) u \quad (i = 1, \cdots, n), \tag{2.3}$$

$$w_i = l_i(u) u_x \quad (i = 1, \cdots, n), \tag{2.4}$$

where $l_i(u)$ denotes the i-th left eigenvector.

By (1.5), it is easy to see that

$$u = \sum_{k=1}^{n} v_k r_k(u), \tag{2.5}$$

$$u_x = \sum_{k=1}^{n} w_k r_k(u). \tag{2.6}$$

Let

$$\frac{d}{d_i t} = \frac{\partial}{\partial t} + \lambda_i(u) \frac{\partial}{\partial x} \tag{2.7}$$

be the directional derivative along the i-th characteristic. We have (see [6,1])

$$\frac{d v_i}{d_i t} = \sum_{j,k=1}^{n} \beta_{ijk}(u) v_j w_k \quad (i = 1, \cdots, n), \tag{2.8}$$

where

$$\beta_{ijk}(u) = (\lambda_k(u) - \lambda_i(u)) l_i(u) \nabla r_j(u) r_k(u). \tag{2.9}$$

Hence, we have

$$\beta_{iji}(u) \equiv 0, \quad \forall j, \tag{2.10}$$

and in normalized coordinates

$$\beta_{ijj}(u_j e_j) \equiv 0, \quad \forall |u_j| \text{ small}, \quad \forall i, j. \tag{2.11}$$

It follows from (2.8) that

$$\frac{\partial v_i}{\partial t} + \frac{\partial(\lambda_i(u) v_i)}{\partial x} = \sum_{j,k=1}^{n} B_{ijk}(u) v_j w_k \quad (i = 1, \cdots, n), \tag{2.12}$$

where

$$B_{ijk}(u) = \beta_{ijk}(u) + \nabla \lambda_i(u) r_k(u) \delta_{ij}. \tag{2.13}$$

By (2.11), in normalized coordinates

$$B_{ijj}(u_j e_j) \equiv 0, \quad \forall |u_j| \quad \text{small}, \quad \forall j \neq i, \tag{2.14}$$

and, when the i-th characteristic $\lambda_i(u)$ is weakly linearly degenerate,

$$B_{iii}(u_i e_i) \equiv 0, \quad \forall |u_i| \quad \text{small}, \quad \forall i. \tag{2.15}$$

Moreover

$$B_{iji}(u) \equiv 0, \quad \forall j \neq i; \tag{2.16}$$

while

$$B_{iii}(u) = \nabla\lambda_i(u)r_i(u), \tag{2.17}$$

and only in the case that $\lambda_i(u)$ is linearly degenerate in the sense of P.D.Lax, we have

$$B_{iii}(u) \equiv 0, \quad \forall i. \tag{2.18}$$

On the other hand, we have (see [6,1])

$$\frac{dw_i}{d_i t} = \sum_{j,k=1}^{n} \gamma_{ijk}(u)w_j w_k \quad (i = 1, \cdots, n), \tag{2.19}$$

where

$$\gamma_{ijk}(u) = \frac{1}{2}[(\lambda_j(u) - \lambda_k(u))l_i(u)\nabla r_k(u)r_j(u) - \nabla\lambda_k(u)r_j(u)\delta_{ik} + (j|k)], \tag{2.20}$$

in which $(j|k)$ stands for all terms obtained by changing j and k in the previous terms. We have

$$\gamma_{ijj}(u) \equiv 0, \quad \forall j \neq i, \tag{2.21}$$

$$\gamma_{iii}(u) = -\nabla\lambda_i(u)r_i(u), \quad \forall i. \tag{2.22}$$

When the i-th characteristic $\lambda_i(u)$ is linearly degenerate in the sense of P.D.Lax, we have

$$\gamma_{iii}(u) \equiv 0, \quad \forall i; \tag{2.23}$$

while, when $\lambda_i(u)$ is weakly linearly degenerate, in normalized coordinates we have

$$\gamma_{iii}(u_i e_i) \equiv 0, \quad \forall |u| \quad \text{small}, \quad \forall i. \tag{2.24}$$

It follows from (2.19) that

$$\frac{\partial w_i}{\partial t} + \frac{\partial(\lambda_i(u)w_i)}{\partial x} = \sum_{j,k=1}^{n} \Gamma_{ijk}(u)w_j w_k \quad (i = 1, \cdots, n), \tag{2.25}$$

where

$$\begin{aligned}\Gamma_{ijk}(u) &= \gamma_{ijk}(u) + \frac{1}{2}[\nabla\lambda_j(u)r_k(u)\delta_{ij} + (j|k)] \\ &= \frac{1}{2}(\lambda_j(u) - \lambda_k(u))l_i(u)[\nabla r_k(u)r_j(u) - \nabla r_j(u)r_k(u)].\end{aligned} \tag{2.26}$$

We have

$$\Gamma_{ijj}(u) \equiv 0, \quad \forall i, j. \tag{2.27}$$

In order to prove Theorem 1.1, we need Lemma 2.1 in [3] as follows.

Lemma 2.1 *Suppose that $u = u(t, x)$ is a C^1 solution to System (1.1), τ_1 and τ_2 are two C^1 arcs which are never tangent to the i-th characteristic direction, and D is the domain* *bounded by τ_1, τ_2 and two i-th characteristic curves L_i^- and L_i^+. Then we have*

$$\int_{\tau_1} |v_i(dx - \lambda_i(u)dt)| \le \int_{\tau_2} |v_i(dx - \lambda_i(u)dt)| + \iint_D \sum_{j,k=1}^{n} |B_{ijk}(u) v_j w_k| dt dx, \tag{2.28}$$

$$\int_{\tau_1} |w_i(dx - \lambda_i(u)dt)| \le \int_{\tau_2} |w_i(dx - \lambda_i(u)dt)| + \iint_D \sum_{j,k=1, j\neq k}^{n} |\Gamma_{ijk}(u) w_j w_k| dt dx, \tag{2.29}$$

where $v_i, w_i, B_{ijk}(u)$ and $\Gamma_{ijk}(u)$ are defined by (2.3), (2.4), (2.13) *and* (2.26) *respectively.*

§3. Global Existence of C^1 Solution—Proof of Theorem 1.1

Without loss of generality, we will prove our result in normalized coordinates. We may also assume that

$$0 < \lambda_1(0) < \lambda_2(0) < \cdots < \lambda_n(0). \tag{3.1}$$

It is easy to know that there exist positive constants δ and δ_0 so small that

$$\lambda_{i+1}(u) - \lambda_i(u) \ge 4\delta_0, \quad \forall |u| \le \delta \quad (i = 1, \cdots, n-1), \tag{3.2}$$

$$|\lambda_i(u) - \lambda_i(v)| \le \frac{\delta_0}{2}, \quad \forall |u|, |v| \le \delta \quad (i = 1, \cdots, n). \tag{3.3}$$

Suppose that $u = u(t, x)$ is the C^1 solution to Cauchy problem (1.1) and (1.7) on the domain $D(T) = \{(t, x) | 0 \le t \le T, |x| < \infty\}$. Let

$$L(u(t)) = \sum_{i=1}^{n} L_i(u(t)) = \sum_{i=1}^{n} \int_{\mathbb{R}} |w_i(t, x)| dx, \tag{3.4}$$

$$Q(u(t)) = \sum_{i<j} Q_{ij}(u(t)) = \sum_{i<j} \iint_{x>y} |w_i(t, x)||w_j(t, y)| dx dy. \tag{3.5}$$

By Lemma 3.3 in [3], we have

Lemma 3.1. *Suppose that System* (1.1) *is strictly hyperbolic in a neighbourhood of $u = 0$ and* (1.5), (1.6) *hold. Suppose furthermore that $u = u(t, x)$ is the C^1 solution to Cauchy problem* (1.1) *and* (1.7) *on the domain $D(T) = \{(t, x) | 0 \le t \le T, |x| < \infty\}$, and the initial data $\varphi(x)$ satisfy the assumptions given in Theorem* 1.1. *Let*

$$\gamma \overset{\triangle}{=} L(\varphi). \tag{3.6}$$

Then there exists $\gamma_0 > 0$ so small that for any given $\gamma \in [0, \gamma_0]$, there exist two positive constants κ_1 and κ_2 independent of γ and T, such that the following uniform a priori estimates hold:

$$\|u(t, \cdot)\|_{C^0} = \sup_{x\in\mathbb{R}} |u(t, x)| \le \kappa_1 \gamma, \quad \forall t \in [0, T], \tag{3.7}$$

$$L(u(t)) \le \gamma + \kappa_2 \gamma^2, \quad \forall t \in [0, T]. \tag{3.8}$$

Remark 3.1. By (3.4) and (3.6), there exists $\theta_0 > 0$ so small that for any given $\theta \in [0, \theta_0]$, we have

$$\gamma \le C_0 \theta, \tag{3.9}$$

where θ is defined by (1.11) and C_0 is a positive constant independent of φ.

Remark 3.2. By (3.8), (3.9), there exists $\theta_0 > 0$ so small that for any given $\theta \in [0, \theta_0]$, we have

$$L(u(t)) \le C\theta, \quad \forall t \in [0, T], \tag{3.10}$$

where C is a positive constant independent of φ.

Remark 3.3. By (3.7) and (3.9), there exists $\theta_0 > 0$ so small that for any given $\theta \in [0, \theta_0]$, on any existence domain $D(T)$ of C^1 solution $u = u(t, x)$, we have

$$|u(t, x)| \le \delta. \tag{3.11}$$

This is the uniform a priori estimate on the C^0 norm of C^1 solution $u = u(t, x)$.

Noting (3.11), by (3.1), it is easy to see that, when $\delta > 0$ is small enough, on the existence domain $D(T)$ of C^1 solution $u = u(t, x)$, we have

$$0 < \lambda_1(u) < \cdots < \lambda_n(u). \tag{3.12}$$

For any fixed $T > 0$, let

$$D_+^T = \{(t, x) | 0 \le t \le T, x \ge (\lambda_n(0) + \delta_0)t\}, \tag{3.13}$$

$$D_-^T = \{(t, x) | 0 \le t \le T, x \le (\lambda_1(0) - \delta_0)t\}, \tag{3.14}$$

$$D^T = \{(t, x) | 0 \le t \le T, (\lambda_1(0) - \delta_0)t \le x \le (\lambda_n(0) + \delta_0)t\}, \tag{3.15}$$

and for $i = 1, \cdots, n$,

$$\begin{gathered} D_i^T = \{(t, x) | 0 \le t \le T, \\ -[\delta_0 + \eta(\lambda_i(0) - \lambda_1(0))]t \le x - \lambda_i(0)t \le [\delta_0 + \eta(\lambda_n(0) - \lambda_i(0))]t\}, \end{gathered} \tag{3.16}$$

where $\eta > 0$ is suitably small (see Fig. 1).

Since $\eta > 0$ is small, by (3.2) we have

$$D_i^T \cap D_j^T = \emptyset, \quad \forall i \ne j, \tag{3.17}$$

$$\bigcup_{i=1}^n D_i^T \subset D^T. \tag{3.18}$$

Fig. 1

For any fixed constant $\mu > 0$, let

$$V(D_\pm^T) = \max_{i=1,\cdots,n} \|(1+t)^{2+\mu} v_i(t,x)\|_{L^\infty(D_\pm^T)}, \tag{3.19}$$

$$W(D_\pm^T) = \max_{i=1,\cdots,n} \|(1+t)^{2+\mu} w_i(t,x)\|_{L^\infty(D_\pm^T)}, \tag{3.20}$$

$$V_\infty^c(T) = \max_{i=1,\cdots,n} \sup_{(t,x)\in D^T/D_i^T} (1+t)^{2+\mu}|v_i(t,x)|, \tag{3.21}$$

$$W_\infty^c(T) = \max_{i=1,\cdots,n} \sup_{(t,x)\in D^T/D_i^T} (1+t)^{2+\mu}|w_i(t,x)|, \tag{3.22}$$

$$U_\infty^c(T) = \max_{i=1,\cdots,n} \sup_{(t,x)\in D^T/D_i^T} (1+t)^{2+\mu}|u_i(t,x)|, \tag{3.23}$$

$$\widetilde{V}_1(T) = \max_{i=1,\cdots,n} \max_{j\neq i} \sup_{\tilde{c}_j} \int_{\tilde{c}_j} |v_i(t,x)|dx, \tag{3.24}$$

where $\tilde{c}_j$ stands for any given j-th characteristic in D_i^T,

$$\widetilde{\widetilde{W}}_1(T) = \max_{i=1,\cdots,n} \max_{j\neq i} \sup_{\tilde{\tilde{c}}_j} \int_{\tilde{\tilde{c}}_j} |w_i(t,x)|dx, \tag{3.25}$$

where $\tilde{\tilde{c}}_j$ stands for any given j-th characteristic in D^T,

$$V_\infty(T) = \max_{i=1,\cdots,n} \sup_{0\le t\le T, x\in\mathbb{R}} |v_i(t,x)|, \tag{3.26}$$

$$U_\infty(T) = \max_{i=1,\cdots,n} \sup_{0\le t\le T, x\in\mathbb{R}} |u_i(t,x)|, \tag{3.27}$$

$$W_\infty(T) = \max_{i=1,\cdots,n} \sup_{0\le t\le T, x\in\mathbb{R}} |w_i(t,x)|. \tag{3.28}$$

Remark 3.4. By (3.7) and (3.9), and noting (2.3), there exists a constant $C > 0$ such that

$$U_\infty(T), V_\infty(T) \le C\theta. \tag{3.29}$$

Lemma 3.2. *Suppose $A(u) \in C^2$ in a neighbourhood of $u = 0$, and $\varphi(x)$ satisfies the assumptions given in Theorem 1.1. Then there exists $\theta_0 > 0$ so small that for any given $\theta \in [0, \theta_0]$, on any existence domain $D(T)$ of C^1 solution $u = u(t,x)$ to Cauchy problem (1.1) and (1.7), there exist two positive constants κ_3 and κ_4 independent of T and θ such that*

$$V(D_\pm^T) \le \kappa_3\theta, \tag{3.30}$$

$$W(D_\pm^T) \le \kappa_4. \tag{3.31}$$

Remark 3.5. Since $w_i(0,x) = l_i(\varphi)\varphi'(x)(i = 1,\cdots,n)$ are not small, we should modify the corresponding proof in [3].

Proof of Lemma 3.2. First of all, on any existence domain $D(T)$ of C^1 solution $u = u(t,x)$, suppose that there exists a positive constant M such that

$$\|w(t,\cdot)\|_{C^0} \le M, \quad \forall t \in [0,T]. \tag{3.32}$$

At the end of the proof of Lemma 3.3 we shall explain that this hypothesis is reasonable.

We now prove (3.30), (3.31) for D_+^T. The proof of (3.30), (3.31) for D_-^T is similar. Let

$$\overline{W}_1(T) = \max_{i=1,\cdots,n} \max_{j\neq i} \sup_{\bar{c}_j} \int_{\bar{c}_j} |w_i(t,x)|dx, \tag{3.33}$$

where $\bar{c}_j : x = x_j(t)$ stands for any given j-th characteristic in D_+^T.

Noting that the initial data possess compact support, it is sufficient to estimate the integral on the right-hand side of (3.33) in a finite time interval $0 \le t \le T_0$. By Lemma 2.1, noting (3.2) and (3.11), and using (3.10) and (3.32), we have (see Fig. 2, where both P_1A_1 and P_2A_2 are i-th characteristics)

$$\begin{aligned}
\int_{\bar{c}_j} |w_i(t,x)|dt &= \int_{t_1}^{t_2} |w_i(t,x_j(t))|dt \\
&= \int_{t_1}^{t_2} \frac{|\lambda_j(u(t,x_j(t))) - \lambda_i(u(t,x_j(t)))|}{|\lambda_j(u(t,x_j(t))) - \lambda_i(u(t,x_j(t)))|} |w_i(t,x_j(t))|dt \\
&\le \frac{1}{4\delta_0} \int_{t_1}^{t_2} |w_i(t,x_j(t))||\lambda_j(u(t,x_j(t))) - \lambda_i(u(t,x_j(t)))|dt \\
&\le \frac{1}{4\delta_0} \int_{x_1}^{x_2} |w_i(0,x)|dx + \frac{1}{4\delta_0} \iint_{A_1A_2P_2P_1} \sum_{j\ne k} |\Gamma_{ijk}(u)w_jw_k|dtdx \\
&\le C_1\{L(u(0)) + \int_0^{T_0}\int_{-\infty}^{+\infty} \sum_{j\ne k} |\Gamma_{ijk}(u)w_jw_k|dxdt\} \\
&\le C_2(1+MT_0)\theta \le C_3\theta, \qquad (3.34)
\end{aligned}$$

henceforth C_j $(j=1,2,\cdots)$ will denote positive constants independent of θ and T. So we have

$$\overline{W}_1(T) \le C_3\theta. \qquad (3.35)$$

Fig.2

We now estimate $V(D_+^T)$ and $W(D_+^T)$.

For any given point $(t,x) \in D_+^T$ with $0 \le t \le T_0$, we draw the i-th characteristic passing through the point (t,x), which intersects the x-axis at a point $(0,x_i)$. Integrating (2.8) and (2.19) along this i-th characteristic, we get

$$v_i(t,x) = v_i(0,x_i) + \int_0^t \sum_{j,k=1,k\ne i}^{n} \beta_{ijk}(u)v_jw_kd\tau, \qquad (3.36)$$

$$w_i(t,x) = w_i(0,x_i) + \int_0^t \sum_{j,k=1}^{n} \gamma_{ijk}(u)w_jw_kd\tau. \qquad (3.37)$$

Multiplying $(1+t)^{(2+\mu)}$ on both sides of (3.36), and noting $0 \le t \le T_0$ and (3.11), by

(3.29) and (3.35), we get

$$\begin{aligned}
&(1+t)^{(2+\mu)}|v_i(t,x)| \\
&\le (1+T_0)^{(2+\mu)}\{|v_i(0,x_i)| + \int_0^{T_0} \sum_{j,k=1,k\neq i}^{n} |\beta_{ijk}(u)w_jw_k|d\tau\} \\
&\le C_4(1+T_0)^{(2+\mu)}\{V_\infty(T) + V_\infty(T)\overline{W}_1(T)\} \\
&\le C_5\theta.
\end{aligned} \tag{3.38}$$

Then, choosing $\kappa_3 \ge C_5$, we have

$$V(D_+^T) \le \kappa_3\theta. \tag{3.39}$$

Similarly, multiplying $(1+t)^{(2+\mu)}$ on both sides of (3.37), and noting (2.21) and (2.24), we have

$$\begin{aligned}
(1+t)^{(2+\mu)}|w_i(t,x)| \le (1+T_0)^{(2+\mu)}\{|w_i(0,x_i)| &+ \int_0^{T_0} \sum_{j\neq k} |\gamma_{ijk}(u)w_jw_k|d\tau \\
&+ \int_0^{T_0} |\gamma_{iii}(u) - \gamma_{iii}(u_ie_i)||w_i|^2 d\tau\}.
\end{aligned} \tag{3.40}$$

For Hadamard's formula, we have

$$\gamma_{iii}(u) - \gamma_{iii}(u_ie_i) = \int_0^1 \sum_{j\neq i} \frac{\partial r_{iii}}{\partial u_j}(su_1,\cdots,su_{j-1},u_j,su_{j+1},\cdots,su_n)u_jds. \tag{3.41}$$

Noting (3.11) and that $\varphi(x)$ is a C^1 function with compact support, and using (3.29), (3.32) and (3.35), when $\theta > 0$ is suitably small, it comes from (3.40) that

$$\begin{aligned}
(1+t)^{(2+\mu)}|w_i(t,x)| &\le C_6\{1 + M\overline{W}_1(T) + M^2T_0V_\infty(T)\} \\
&\le C_6 + C_7M(1+MT_0)\theta \\
&\le 2C_6.
\end{aligned} \tag{3.42}$$

Hence, choosing $\kappa_4 \ge 2C_6$,we get

$$W(D_+^T) \le \kappa_4. \tag{3.43}$$

Remark 3.6. κ_4 can be chosen to be independent of M, provided that $\theta > 0$ is suitably small.

Lemma 3.3. *Under the assumptions of Theorem* 1.1, *there exists* θ_0 *so small that for any given* $\theta \in [0,\theta_0]$, *on any existence domain* $D(T)$ *of* C^1 *solution to Cauchy problem* (1.1) *and* (1.7), *there exist positive constants* $\kappa_i (i = 5,6,7,8,9)$ *independent of* θ *and* T, *such that*

$$W_\infty^c(T) \le \kappa_5, \quad \widetilde{\widetilde{W}}_1(T) \le \kappa_6\theta, \tag{3.44}$$

$$V_\infty^c(T) \le \kappa_7\theta, \quad \widetilde{V}_1(T) \le \kappa_8\theta, \tag{3.45}$$

$$W_\infty(T) \le \kappa_9. \tag{3.46}$$

Proof First of all, by (2.3) and (3.11), when $\delta > 0$ is suitably small, we have

$$U_\infty^c(T) \le C_7V_\infty^c(T). \tag{3.47}$$

We now estimate $\widetilde{\widetilde{W}}_1(T)$ (see Fig.3).

Fig.3

Let

$$\tilde{\tilde{c}}_j : x = x_j(t) \quad (0 \le t_1 \le t \le t_2 \le T). \tag{3.48}$$

Noting (3.11), for $\delta > 0$ small enough, we have

$$\lambda_1(0) - \delta_0 < \lambda_j(u) < \lambda_n(0) + \delta_0.$$

Then $\tilde{\tilde{c}}_j$ intersects $x = (\lambda_1(0) - \delta_0)t$ and $t = T$ at points $P_1(t_1, x_j(t_1))$ and $P_2(T, x_j(T))$ respectively. Passing through the point P_1 (resp. P_2), we draw the i-th characteristic which intersects the x-axis at a point $A_1(0, y_1)$ (resp. $A_2(0, y_2)$). We have

$$\int_{\tilde{\tilde{c}}_j} |w_i(t,x)|dx = \int_{t_1}^{T} |w_i(t, x_j(t))|dt. \tag{3.49}$$

In order to estimate $\int_{t_1}^{T} |w_i(t, x_j(t))|dt$, similarly to (3.34), using (2.29) on the domain $P_1A_1A_2P_2$ and noting (3.10) and (3.31), we see that

$$\begin{aligned}\int_{t_1}^{T} |w_i(t, x_j(t))|dt &\le \frac{1}{4\delta_0}\int_{t_1}^{T} |w_i(t, x_j(t))||\lambda_j(u(t, x_j(t))) - \lambda_i(u(t, x_j(t)))|dt \\ &\le \frac{1}{4\delta_0}\int_0^{y_2} |w_i(0,x)|dx + \frac{1}{4\delta_0}\iint_{P_1A_1A_2P_2} \sum_{j \ne k} |\Gamma_{ijk}(u)w_jw_k|dtdx \\ &\le C_8\theta + C_9(W(D_+^T) + W_\infty^c(T))\int_0^T (1+\tau)^{-(2+\mu)}L(u(\tau))d\tau \\ &\le C_{10}\theta(1 + W_\infty^c(T)). \end{aligned} \tag{3.50}$$

Thus, we have

$$\widetilde{\widetilde{W}}_1(T) \le C_{11}\theta(1 + W_\infty^c(T)). \tag{3.51}$$

Fig.4

We next estimate $W_\infty^c(T)$.

Passing through any given point $(t,x) \in D^T/D_i^T$, we draw the i-th characteristic which never enters into D_i^T. We may assume that this characteristic intersects the right boundary $x = (\lambda_n(0) + \delta_0)t$ of D^T at a point (t_0, y) (see Fig.4). Integrating (2.19) along this characteristic and noting (2.21) and (2.24) yields

$$w_i(t,x) = w_i(t_0,y) + \int_{t_0}^{t} \sum_{j\neq k} \gamma_{ijk}(u) w_j w_k d\tau + \int_{t_0}^{t} (\gamma_{iii}(u) - \gamma_{iii}(u_i e_i)) w_i^2 d\tau. \tag{3.52}$$

On the i-th characteristic $x = x_i(\tau)(t_0 \le \tau \le t)$ passing through the point (t,x), by (3.2) we have (see [2–3] and Fig.4)

$$t_0 \le \tau \le t \le \frac{1}{\eta} t_0. \tag{3.53}$$

Multiplying $(1+t)^{(2+\mu)}$ on both sides of (3.52), noting (3.53) and (3.11) and using (3.31), (3.51), (3.29) and Hadamard's formula, we get

$$\begin{aligned}(1+t)^{(2+\mu)}|w_i(t,x)| &\le (1+t)^{(2+\mu)}|w_i(t_0,y)| + \int_{t_0}^{t} (1+t)^{(2+\mu)} \sum_{j\neq k} |\gamma_{ijk}(u) w_j w_k| d\tau \\ &\quad + \int_{t_0}^{t} (1+t)^{(2+\mu)} |\gamma_{iii}(u) - \gamma_{iii}(u_i e_i)||w_i^2| d\tau \\ &\le C_{12} + C_{13} \widetilde{\widetilde{W}}_1(T) W_\infty^c(T) + C_{14} U_\infty(T)(W_\infty^c(T))^2 \\ &\le C_{12} + C_{15}\theta W_\infty^c(T) + C_{16}\theta (W_\infty^c(T))^2, \end{aligned} \tag{3.54}$$

hence

$$W_\infty^c(T) \le C_{17}(1 + \theta(W_\infty^c(T))^2). \tag{3.55}$$

Noting that $\varphi(x)$ is a C^1 function with compact support, by (2.4) we have

$$|w_i(0,x)| \le M_0, \quad \forall x \in \mathbb{R}, \tag{3.56}$$

where M_0 is a positive constant. By continuity, there exists $\tau_0 > 0$ such that

$$(1+t)^{(2+\mu)}|w_i(t,x)| \le 2M_0, \quad 0 \le t \le \tau_0, \tag{3.57}$$

then

$$W_\infty^c(t) \le 2M_0, \quad 0 \le t \le \tau_0. \tag{3.58}$$

In order to prove the first inequality in (3.44), it is sufficient to show that for any fixed $T_0 (0 \le T_0 \le T)$ and for any $\theta > 0$ suitably small, we can choose $\kappa_5 \ge M_0 > 0$ in such a way that when

$$W_\infty^c(T_0) \le 2\kappa_5, \tag{3.59}$$

we have

$$W_\infty^c(T_0) \le \kappa_5. \tag{3.60}$$

Substituting (3.59) into (3.55), for $\theta > 0$ suitably small, we have

$$W_\infty^c(T_0) \le C_{17}(1 + 4\theta\kappa^2) \le 2C_{17}. \tag{3.61}$$

Hence, taking $\kappa_5 \ge 2C_{17}$, we get (3.60), then the first inequality in (3.44) holds. Hence by (3.51), there exists $\kappa_6 > 0$ such that the second inequality in (3.44) also holds.

Remark 3.7. κ_5 can be chosen to be independent of M.

We next estimate $\widetilde{V}_1(T)$.

Still denote $\tilde{c}_j$ by $x = x_j(t)$. By (3.3), the whole i-th characteristic passing through $O(0, 0)$ must be included in D_i^T. Let $P_0(t_0, x_j(t_0))$ be the intersection point of this characteristic with $\tilde{\tilde{c}}_j$. Similarly to the estimate of $\widetilde{\widetilde{\widetilde{W}}}_1(T)$, it suffices to estimate $\int_{t_0}^{t_2} |v_i(t, x_j(t))|dt$, where $P_2(t_2, x_j(t_2))$ is the intersection point of $\tilde{c}_j$ with the right boundary of D_i^T. The estimate of $\int_{t_1}^{t_0} |v_i(t, x_j(t))|dt$ is similar. The i-th characteristic passing through the point P_2 intersects the line $x = (\lambda_n(0) + \delta_0)t$ at a point $A_2(\bar{t}_2, y_2)$, where $\bar{t}_2 = \frac{y_2}{\lambda_n(0)+\delta_0}$ (see Fig.5).

Fig.5

Using Lemma 2.1 and noting (2.14), (2.15), similarly we have

$$\begin{aligned}\int_{t_0}^{t_2} |v_i(t, x_j(t))|dt \le{}& \frac{1}{4\delta_0}\int_{t_0}^{t_2} |v_i(t, x_j(t))||\lambda_j(u(t, x_j(t))) - \lambda_i(u(t, x_j(t)))|dt \\ \le{}& \frac{1}{4\delta_0}\int_0^{\bar{t}_2} |v_i(t, (\lambda_n(0) + \delta_0)t)|(\lambda_n(0) + \delta_0 - \lambda_i(u(t, (\lambda_n(0) + \delta_0)t)))dt \\ &+ \frac{1}{4\delta_0}\iint_{P_0OA_2P_2} \sum_{j\ne k} |B_{ijk}(u)v_jw_k|dtdx \\ &+ \frac{1}{4\delta_0}\iint_{P_0OA_2P_2} \sum_{j=1}^{n} |(B_{ijj}(u) - B_{ijj}(u_je_j))v_jw_j|dtdx. \end{aligned}\tag{3.62}$$

Since for any point $(t,x) \in D^T$ we have

$$(\lambda_1(0) - \delta_0)t \le x \le (\lambda_n(0) + \delta_0)t, \tag{3.63}$$

noting (3.10), (3.11), and using Hadamard's formula, (3.30),(3.44),(3.47) and (3.29), it follows from (3.62) that

$$\begin{aligned}\int_{t_0}^{t_2} |v_i(t, x_j(t))|dt \le\ & C_{18} V(D_+^T) + C_{19} V_\infty(T) W_\infty^c(T) \\ & + C_{20} V_\infty^c(T) \int_0^T (1+t)^{-(2+\mu)} L(u(t))dt \\ & + C_{21}(U_\infty^c(T) V_\infty(T) + U_\infty(T) V_\infty^c(T)) \int_0^T (1+t)^{-(2+\mu)} L(u(t))dt \\ \le\ & C_{22}\theta(1 + V_\infty^c(T)). \end{aligned} \tag{3.64}$$

Similarly we can estimate $\int_{t_1}^{t_0} |v_i(t, x_j(t))|dt$. Hence

$$\widetilde{V}_1(T) \le C_{23}\theta(1 + V_\infty^c(T)). \tag{3.65}$$

We next estimate $V_\infty^c(T)$.

Similarly to (3.52), integrating (2.8) along the i-th characteristic expressed in Fig. 4 gives

$$v_i(t,x) = v_i(t_0, y) + \int_{t_0}^t \sum_{k \ne i, j \ne k} \beta_{ijk}(u) v_j w_k d\tau + \int_{t_0}^t \sum_{j=1}^n (\beta_{ijj}(u) - \beta_{ijj}(u_j e_j)) v_j w_j d\tau. \tag{3.66}$$

Hence, similarly to (3.54), using Hadamard's formula and (3.32) and noting (3.30),(3.44), (3.65),(3.29) and (3.47), we have

$$\begin{aligned}(1+t)^{(2+\mu)} |v_i(t,x)| \le\ & C_{24}(1+t_0)^{(2+\mu)} |v_i(t_0, y)| \\ & + C_{25} \int_{t_0}^t (1+\tau)^{(2+\mu)} \Big\{ \sum_{k \ne i, j \ne k} |v_j w_k| d\tau + \sum_{j \ne k} |v_j w_j u_k| \Big\} d\tau \\ \le\ & C_{26} \{ V(D_+^T) + V_\infty^c(T) \widetilde{\widetilde{W}}_1(T) + \widetilde{V}_1(T) W_\infty^c(T) + V_\infty(T) W_\infty^c(T) \\ & + M(U_\infty(T) V_\infty^c(T) + U_\infty^c(T) V_\infty(T)) \} \\ \le\ & C_{27}\theta(1 + V_\infty^c(T)). \end{aligned} \tag{3.67}$$

Hence

$$V_\infty^c(T) \le C_{27}\theta(1 + V_\infty^c(T)). \tag{3.68}$$

Thus, when $\theta_0 > 0$ is suitably small, for any given $\theta \in [0, \theta_0]$ we have

$$V_\infty^c(T) \le 2C_{27}\theta. \tag{3.69}$$

Then by (3.65) it is easy to get (3.45).

We finally estimate $W_\infty(T)$.

Without loss of generality, suppose that the i-th characteristic passing through any given point $(t,x) \in D_i^T$ intersects $x = (\lambda_n(0) + \delta_0)t$ at a point (t_0, y) (see Fig. 6). Integrating (2.19) along this characteristic and noting (2.21) and (2.24), we get

$$\begin{aligned} w_i(t,x) =\ & w_i\Big(\frac{y}{\lambda_n(0) + \delta_0}, y\Big) \\ & + \int_{\frac{y}{\lambda_n(0)+\delta_0}}^t \sum_{j \ne k} \gamma_{ijk}(u) w_j w_k d\tau + \int_{\frac{y}{\lambda_n(0)+\delta_0}}^t (\gamma_{iii}(u) - \gamma_{iii}(u_i e_i)) w_i^2 d\tau. \end{aligned} \tag{3.70}$$

Noting (3.29),(3.31) and (3.32), and using (3.44), (3.45), (3.47) and Hadamard's formula, from (3.70) we have that for $\theta > 0$ suitably small,

$$\begin{aligned} |w_i(t,x)| &\le W(D_+^T) + M\widetilde{\widetilde{\widetilde{W}}}_1(T) + M^2 U_\infty^c(T) + U_\infty(T)(W_\infty^c(T))^2 \\ &\le \kappa_4 + C_{28}\theta \le 2\kappa_4. \end{aligned} \tag{3.71}$$

Fig. 6

On the other hand, for any given point $(t,x) \notin D_i^T (i = 1, \cdots, n)$, $|w_i(t,x)|$ can be controlled by $W_\infty^c(T)$ or $W(D_\pm^T)$. Therefore, it follows from (3.31),(3.44) and (3.71) that

$$W_\infty(T) \le \max\{2\kappa_4, \kappa_5\}. \tag{3.72}$$

Then taking $\kappa_9 \ge \max\{2\kappa_4, \kappa_5\}$, we get (3.46).

Since κ_4, κ_5 and κ_9 can be chosen to be independent of M, we may assume $M \ge 2\kappa_9$, then by (3.46) we have

$$W_\infty(T) \le \kappa_9 \le \frac{M}{2}. \tag{3.73}$$

This shows the validity of hypothesis (3.32).

Thus, by (3.11) and (3.46), and using the existence and uniqueness of local C^1 solution to the Cauchy problem (cf. [7]) to extend the solution successively, we arrive at the conclusion of Theorem 1.1.

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