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# THE SERRE RELATIONS IN RINGEL-HALL ALGEBRAS

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**Abstract**

The author constructs the Casimir element of Hall algebras. By the method of Gabber-Kac theorem (see [4]), it is proved that the Serre relations are the defining relations in composition algebra.

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## §0. Introduction

Let $\mathfrak{g}$ be symmetrizable Kac-Moody algebra (see [7]), and $U_q(\mathfrak{g})$ be the quantized enveloping algebra of $\mathfrak{g}$. There are several ways to realize $U_q(\mathfrak{g})$. A successful model is the Hall algebra associated to a hereditary algebra (see [2, 8, 9]). Ringel in [8] proved that the quantum Serre relation is a zero relation in Ringel-Hall algebra. Therefore, if $\Lambda$ and $\mathfrak{g}$ enjoy a common Cartan datum of finite type, the generic form of $H(\Lambda)$ gives a realization of the positive part of $U_q(\mathfrak{g})$. Green in [2] proved (depending on Lusztig [3] ) that the positive part of $U_q(\mathfrak{g})$ is canonically isomorphic to the generic composition algebra of $\Lambda$ if $\mathfrak{g}$ and $\Lambda$ enjoy a common Cartan datum (of any type).

According to Xiao's work[9], the double Ringel-Hall algebra, more precisely, the (reduced) Drinfeld double of composition algebra of Ringel-Hall algebra gives a realization of $U_q(\mathfrak{g})$. This approach can provide a global method to study the quantum group $U_q(\mathfrak{g})$. In this paper, we first construct the Casimir element of the Hall algebra. Applying the method of Gabber-Kac theorem, we shall prove that the Serre relations are the defining relations of the composition algebra. At this point of view, the Ringel-Green isomorphism theorem still holds for standard composition algebras. This means that the Drinfeld double of composition algebra (where $v$ is not an indeterminate) is naturally isomorphic to the quantum groups in the sense of Drinfeld-Jimbo.

## §1. Preliminaries

For the basic facts about Hopf algebras, their skew Hopf pairing and corresponding Drinfeld double, the readers can refer to [1, 5, 9]. Let $k$ be a finite field, $v = \sqrt{q}$, $q = | k |$, $Q(v)$ be the field of rational functions of $v$. We keep these notations throughout this paper.

Let $(I, (-,-))$ be a symmetrizable Cartan Datum in the sense of Lusztig and $C$ be the corresponding symmetrizable Cartan matrix, where $a_{ij} = \frac{2(i,j)}{(i,i)}$. For basic concepts please refer to [9], for example, the concept of a skew Hopf pairing $(A^+, A^-, \varphi)$ being a member of

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$\mathcal{L}(C)$, a restricted nondegenerate member of $\mathcal{L}(C)$, etc. By [9, Theorem 3.6], any restricted nondegenerate skew-Hopf pairings in $\mathcal{L}(C)$ are canonically isomorphic for the same Cartan Datum $C = (I, (-,-))$.

Let $\Lambda$ be a finite dimensional hereditary $k$-algebra and $\mathcal{P}$ be the set of isomorphism classes of finite dimensional $\Lambda$-modules. We denote by zero both the zero module and its isomorphism class. Let $\mathcal{P}_1 = \mathcal{P} \setminus \{0\}$, and for every $\alpha \in \mathcal{P}$, $V_\alpha$ be a representative in $\alpha$. Given $\alpha \in \mathcal{P}$, $a_\alpha$ is the order of automorphism group of $V_\alpha$, and for $\alpha, \beta, \lambda \in \mathcal{P}$, $g^\lambda_{\alpha\beta}$ is the number of submodules $B$ of $V_\lambda$ such that $B \simeq V_\beta$ and $V_\lambda/B \simeq V_\alpha$.

Given $\Lambda$ -modules $M, N$, let

$$\langle M, N\rangle_R = \dim_k \mathrm{Hom}_\Lambda(M,N) - \dim_k \mathrm{Ext}^1_\Lambda(M,N).$$

Since $\Lambda$ is hereditary, $\langle M, N\rangle_R$ only depends on $\dim M$ and $\dim N$. For $\alpha, \beta \in \mathcal{P}$, we write $\langle \alpha, \beta\rangle = \langle V_\alpha, V_\beta\rangle_R$. The form $\langle -, -\rangle$ is naturally defined on $\mathbb{Z}[I]$, where $I$ is the set of isomorphism classes of simple $\Lambda$-modules. Let $(\alpha, \beta) = \langle \alpha, \beta\rangle_R + \langle \beta, \alpha\rangle_R$. Set $\langle u_\alpha\rangle = v^{-\dim V_\alpha + \langle \alpha,\alpha\rangle} u_\alpha$.

Let $H^+(\Lambda)$ be a free $Q(v)$ - module with basis $\{K_\alpha \langle u^+_\lambda\rangle \mid \alpha \in \mathbb{Z}[I], \lambda \in \mathcal{P}\}$. According to [9, Theorem 4.5], $H^+(\Lambda)$ is of a Hopf algebra structure (see [10] for detail). Obviously, $H^+(\Lambda)$ is an $\mathbb{N}[I]$-graded algebra.

Dually, let $H^-(\Lambda)$ be the free $Q(v)$ -module with the basis $\{K_\alpha \langle u^-_\lambda\rangle \mid \alpha \in \mathbb{Z}[I], \lambda \in \mathcal{P}\}$. It is also of Hopf algebra structure (see [10]).

By [9, Proposition 5.3], there is a skew Hopf pairing: $\varphi : H^+(\Lambda) \times H^-(\Lambda) \longmapsto Q(v)$ defined by

$$\varphi(K_\alpha \langle u^+_\beta\rangle, K_{\alpha'} \langle u^-_{\beta'}\rangle) = v^{-(\alpha,\alpha') - (\beta,\alpha') + (\alpha,\beta') + (\beta,\beta')} a_\beta^{-1} \delta_{\beta\beta'} \tag{1.1}$$

for all $\alpha, \alpha' \in \mathbb{Z}[I]$ and $\beta, \beta' \in \mathcal{P}$. Accordingly, we have the reduced Drinfeld double, which is denoted by $\mathcal{D}(\Lambda)$. Let $C^+(A)$ (resp. $C^-(A)$) be the subalgebra of $H^+(\Lambda)$ (resp. $H^-(A)$) generated by $u^+_i, i \in I$ (resp. $u^-_i$) and $T$. Restricting the Skew Hopf pairing to $\varphi : C^+(\Lambda) \times C^-(\Lambda) \longmapsto Q(v)$, we see that $(C^+(A), C^-(A), \varphi)$ is the member of $\mathcal{L}(C)$. Therefore, we have the reduced Drinfeld double of the skew Hopf pairing $(C^+(\Lambda), C^-(\Lambda), \varphi)$, which we denote by $\mathcal{D}_c(\Lambda)$. Obviously, $\mathcal{D}_c(\Lambda)$ is a Hopf subalgebra of $\mathcal{D}(\Lambda)$ and has the triangular decomposition $\mathcal{D}_c(\Lambda) = C^<(\Lambda) \otimes T \otimes C^>(\Lambda)$, where $C^<(\Lambda)$ and $C^>(\Lambda)$ are the subalgebras generated by $u^-_i$ and $u^+_i$ $(i \in I)$, respectively.

## §2. Casimir Element

Let $B^+$ and $B^-$ be the $\mathbb{Q}(v)$-bases of $C^+(\Lambda)$ and $C^-(\Lambda)$, respectively. Let

$$C^+(\Lambda) = \bigoplus_{\nu \in \mathbb{N}[I]} C^+(\Lambda)_\nu, \quad C^-(\Lambda) = \bigoplus_{\nu \in \mathbb{N}[I]} C^-(\Lambda)_\nu,$$

where $C^\pm(\Lambda)_\nu = C^\pm(\Lambda) \cap H^\pm(\Lambda)_\nu$. Take a basis $\{{}_\nu x^+_{\alpha_1}, {}_\nu x^+_{\alpha_2}, \cdots, {}_\nu x^+_{\alpha_{r(\nu)}}\} = B^+ \cap C^+(\Lambda)_\nu$ of $C^+(\Lambda)_\nu$ and the dual basis $\{{}_\nu x^-_{\alpha_1}, {}_\nu x^-_{\alpha_2}, \cdots, {}_\nu x^-_{\alpha_{r(\nu)}}\} = B^-(\Lambda) \cap C^-(\Lambda)_\nu$ of $C^-(\Lambda)_\nu$ with respect to the skew Hopf pairing $\varphi : C^+(\Lambda) \times C^-(\Lambda) \longmapsto \mathbb{Q}(v)$, where $r(\nu) = \dim_k C^\pm(\Lambda)_\nu$. Note that, for any $\nu \in \mathbb{N}[I]$, the subspace $C^\pm(\Lambda)_\nu$ is finite dimensional. Set

$$\Theta_\nu = \sum_{j=1}^{r(\nu)} {}_\nu x^-_j \otimes {}_\nu x^+_j \in C^-(\Lambda)_\nu \otimes C^+(\Lambda)_\nu.$$

The following result is easy to prove.

**Lemma 2.1.** (a) *For any* ${}_\nu x^+_\alpha \in C^+(\Lambda)_\nu$ *and* ${}_\nu x^-_\alpha \in C^-(\Lambda)_\nu$, *we have*

$$ {}_\nu x^+_\alpha = \sum_{i=1}^{r(\nu)} \varphi({}_\nu x^+_{\alpha_i}, {}_\nu x^-_\alpha)\, {}_\nu x^+_{\alpha_i}, \tag{2.1} $$

$$ {}_\nu x^-_\alpha = \sum_{i=1}^{r(\nu)} \varphi({}_\nu x^+_\alpha, {}_\nu x^-_{\alpha_i})\, {}_\nu x^-_{\alpha_i}. \tag{2.2} $$

(b) *For any* $i \in I$ *and* $\nu \in \mathbb{N}[I]$, *we have*

$$ (u_i^+ \otimes 1)\Theta_\nu + (K_i \otimes u_i^+)\Theta_{\nu-i} = \Theta_\nu(u_i^+ \otimes 1) + \Theta_{\nu-i}(K_{-i} \otimes u_i^+), \tag{2.3} $$

$$ (1 \otimes u_i^-)\Theta_\nu + (u_i^- \otimes K_{-i})\Theta_{\nu-i} = \Theta_\nu(1 \otimes u_i^-) + \Theta_{\nu-i}(u_i^- \otimes K_i), \tag{2.4} $$

*where* $\Theta_{\nu-i} = 0$ *if* $\nu_i = 0$.

Set $\Theta_{\leq p} = \sum\limits_{\substack{\nu \\ \mathrm{tr}\,\nu \leq p}} \Theta_\nu$, where $p$ is a nonnegative integer, $\mathrm{tr}\nu = \sum\limits_{s=1}^{n} \nu_s$ if $\nu = \sum\limits_{s=1}^{n} \nu_s i_s$. Using this Lemma and noting $\sum\limits_{\substack{\nu \\ \mathrm{tr}\,\nu \leq p}} (\Theta_\nu - \Theta_{\nu-i}) = \sum\limits_{\substack{\nu \\ \mathrm{tr}\,\nu \leq p}} \Theta_\nu - \sum\limits_{\substack{\nu \\ \mathrm{tr}\,\nu \leq p}} \Theta_{\nu-i}$, if $\nu_i = 0$, $\Theta_{\nu-i} = 0$, $\Theta_\nu \,|_{\mathrm{tr}\,\nu = p-1} = \Theta_{\nu-i}\,|_{\mathrm{tr}\,\nu = p}$, we get

**Proposition 2.1.**

(a)
$$ \begin{aligned} &(u_i^+ \otimes 1 + K_i \otimes u_i^+)\Theta_{\leq p} - \Theta_{\leq p}(u_i^+ \otimes 1 + K_{-i} \otimes u_i^+) \\ = &\sum_{\substack{\nu \\ \mathrm{tr}\,\nu \leq p}} (K_i \otimes u_i^+)\Theta_\nu - \sum_{\substack{\nu \\ \mathrm{tr}\,\nu \leq p}} \Theta_\nu(K_{-i} \otimes u_i^+), \end{aligned} $$

(b)
$$ \begin{aligned} &(1 \otimes u_i^- + u_i^- \otimes K_{-i})\Theta_{\leq p} - \Theta_{\leq p}(1 \otimes u_i^- + u_i^- \otimes K_i) \\ = &\sum_{\substack{\nu \\ tr\nu \leq p}} (u_i^- \otimes K_{-i})\Theta_\nu - \sum_{\substack{\nu \\ \mathrm{tr}\,\nu \leq p}} \Theta_\nu(u_i^- \otimes K_i). \end{aligned} $$

Applying $m(\sigma \otimes 1)$ to both sides of the identities (a) and (b) in Proposition 2.1, we get

$$ \begin{aligned} &\sum_{\substack{\nu \\ \mathrm{tr}\,\nu \leq p}} \sum_{j=1}^{r(\nu)} [K_{-i} u_i^+ \sigma({}_\nu x_j^-)\, {}_\nu x_j^+ - K_i \sigma({}_\nu x_j^-)\, {}_\nu x_j^+ u_i^+] \\ = &\sum_{\substack{\nu \\ \mathrm{tr}\,\nu = p}} \sum_{j=1}^{r(\nu)} [\sigma({}_\nu x_j^-) K_{-i} u_i^+\, {}_\nu x_j^+ - K_i \sigma({}_\nu x_j^-)\, {}_\nu x_j^+ u_i^+], \\ &\sum_{\substack{\nu \\ \mathrm{tr}\,\nu \leq p}} \sum_{j=1}^{r(\nu)} [u_i^- K_i \sigma({}_\nu x_j^-)\, {}_\nu x_j^+ K_i - \sigma({}_\nu x_j^-)\, {}_\nu x_j^+ u_i^-] \\ = &\sum_{\substack{\nu \\ \mathrm{tr}\,\nu = p}} \sum_{j=1}^{r(\nu)} [\sigma(u_i^-\, {}_\nu x_j^-) K_{-i}\, {}_\nu x_j^+ - \sigma({}_\nu x_j^- u_i^-)\, {}_\nu x_j^+ K_i], \end{aligned} $$

where $m$ is the multiplication, $\sigma$ is the antipode. Setting $\Omega_{\leq p} = \sum\limits_{\substack{\nu \\ \mathrm{tr}\,\nu \leq p}} \sum\limits_{i=1}^{r(\nu)} ({}_\nu x_j^-)\, {}_\nu x_j^+$, we have

$$(1)\qquad K_{-i}u_i^+\Omega_{\le p} - K_i\Omega_{\le p}u_i^+ = \sum_{\substack{\nu \\ tr\nu=p}}\sum_{j=1}^{r(\nu)}[\sigma({}_\nu x_j^-)K_{-i}u_i^+{}_\nu x_j^+ - K_i\sigma({}_\nu x_j^-){}_\nu x_j^+u_i^+],$$

$$(2)\qquad u_i^-K_i\Omega_{\le p}K_i - \Omega_{\le p}u_i^- = \sum_{\substack{\nu \\ tr\nu=p}}\sum_{j=1}^{r(\nu)}[\sigma(u_i^-{}_\nu x_j^-)K_{-i\nu}x_j^+ - \sigma({}_\nu x_j^-u_i^-){}_\nu x_j^+K_i].$$

It is easy to see that $\Omega_{\le p} = \sum\limits_{\substack{\nu \\ \mathrm{tr}\,\nu\le p}}\sum\limits_{i=1}^{r(\nu)}\sigma({}_\nu u_i^-){}_\nu x_i^+$ is independent of basis.

The $\mathcal{D}_c(\Lambda)$-module $M$ is called the highest weight module if $M = \bigoplus\limits_{\lambda\in X} M^\lambda$ where $X$ is the weight lattice of $M$, for any $m \in M^\lambda$, $K_\mu m = v^{(\mu,\lambda)}m$ for all $\mu \in \mathbb{Z}[I]$, and there is $N > 0$ such that for any $s > N$, $i \in I$, we have $(u_i^+)^s m = 0$. Let $\lambda$ be a weight,

$$J_\lambda = \sum_{i\in I}\mathcal{D}_c(\Lambda)u_i^+ + \sum_{\mu\in\mathbb{Z}[I]}\mathcal{D}_c(\Lambda)(K_\mu - v^{(\mu,\lambda)})$$

and $M_\lambda = \mathcal{D}_c(\Lambda)\big/J_\lambda$. The module $M = \bigoplus\limits_{\lambda\in X} M^\lambda$, where $M^\lambda = \{m \in M | K_\mu m = v^{(\mu,\lambda)}m$, for all $\mu \in \mathbb{Z}[I]\}$ is called the integrable module on $\mathcal{D}_c(\Lambda)$, if for any $m \in M$ and $i \in I$, there is $N_0 \ge 1$ such that for any $n > N_o$, we have ${u_i^+}^n m = {u_i^-}^n m = 0$.

Now we are given a highest weight module $M$. For $m \in M$, if $p$ is sufficiently large, $\Omega_{\le p}(m)$ is independent of $p$. Thus, we can denote $\Omega_{\le p}(m)$ by $\Omega(m)$. Clearly, as an operator on $M$, we have

$$\Omega(m) = \sum_{\nu\in\mathbb{N}[I]}\sum_{i=1}^{r(\nu)} S({}_\nu x_i^-){}_\nu x_i^+(m), \quad K_{-i}u_i^+\Omega = K_i\Omega u_i^+,$$

$$u_i^-K_i\Omega K_i = \Omega u_i^-, \quad K_\mu\Omega = \Omega K_\mu.$$

Therefore,

$$\Omega u_i^+(m) = v^{-2(i,\lambda+i)}u_i^+, \quad \Omega u_i^-(m) = v^{2(i,\lambda)}u_i^-\Omega(m).$$

Obviously, $\mathbb{Z}[I]$ is a subgroup of $X$ (as an abelian group). Let $Y$ be a fixed coset of $\mathbb{Z}[I]$ corresponding to $X$, i.e.

$$Y = \{a + x_1i_1 + x_2i_2 + \cdots + x_ni_n \mid x_i \in \mathbb{Z}, \quad a \in X\}.$$

**Lemma 2.2.** *There is a function $G : Y \longmapsto \mathbb{Z}$, such that $G(\lambda) - G(\lambda - i) = 2(i, \lambda)$, for all $\lambda \in Y$ and $i \in I$, and if there are two such functions, their difference is only a scalar.*

**Proof.** First take $G(a) \in \mathbb{Z}$ arbitrarily. Let $\lambda = a + x_1i_1 + x_2i_2 + \cdots + x_ni_n$ $(x_i \in \mathbb{Z})$. Define

$$\begin{aligned} G(\lambda) &= G(a + x_1i_1 + x_2i_2 + \cdots + x_ni_n) \\ &= G(a) + 2\Big(i_1, x_1\lambda - \sum_{s=1}^{x_1-1} s\Big) + 2\Big(i_2, x_2(\lambda - x_1i_1) - i_2\sum_{s=1}^{x_2-1} s\Big) \\ &\quad + \cdots + 2\Big(i_n, x_n(\lambda - x_1i_1 - \cdots - x_{n-1}i_{n-1}) - i_n\sum_{s=1}^{x_n-1} s\Big). \end{aligned}$$

Then $G$ is a function from $Y$ to $\mathbb{Z}$, and for any $i_s \in I$, we have $G(\lambda) - G(\lambda - i_s) = 2(i_s, \lambda)$. The second statement in the lemma is clear.

We set $\Xi :\ M \longmapsto M, m \mapsto v^{G(\lambda)}m$, for all $m \in M^\lambda$ and $\lambda \in Y$. The composition operator $\Omega\Xi :\ M \longmapsto M$ is called Casimir element. Using 2.5, we have following two propositions.

**Proposition 2.2.** *Casimir element commutes with* $\mathcal{D}_c(\Lambda)$.

**Proposition 2.3.** *Let* $M$ *be a quotient of Verma module* $M_{\lambda'}$, *then* $\Omega\Xi : M \longmapsto M$ *is* $v^{G(\lambda')}$ *times of identity morphism.*

## §3. Construction of $Q(v)$-Algebras $\widetilde{\mathcal{C}}^+(\Lambda)$ and $\widetilde{\mathcal{C}}^-(\Lambda)$

Let $\widetilde{\mathcal{C}}_0^+(\Lambda)$ (resp. $\widetilde{\mathcal{C}}_0^-(\Lambda)$) be a $Q(v)$-algebra freely generated by $u_i^+$ $(i \in I)$ (resp. $u_i^-$) and $\widetilde{\mathcal{C}}^+(\Lambda)$ be a $Q(v)$ - algebra generated by $\widetilde{\mathcal{C}}_0^+(\Lambda)$ and $K_\alpha$ $(\alpha \in \mathbb{Z}[I])$ such that the relation $K_\alpha u_i^+ = v^{(i,\alpha)} u_i^+ K_\alpha$ (resp. $K_\alpha u_i^- = v^{(i,\alpha)} u_i^- K_\alpha$) is satisfied.

For any $\nu \in \mathbb{N}[I]$, $\nu = \sum\limits_i \nu_i i$, we denote the $T$-submodule generated by the monomials $u_{i_1}^+ u_{i_2}^+ \cdots u_{i_r}^+$ (resp. $u_{i_1}^- u_{i_2}^- \cdots u_{i_r}^-$) by $\widetilde{\mathcal{C}}^+(\Lambda)_\nu$ (resp. $\widetilde{\mathcal{C}}^-(\Lambda)_\nu$), where the number of occurrence of any $i \in I$ in the sequence $i_1, i_2, \cdots i_r$ is $\nu_i$. $\widetilde{\mathcal{C}}^+(\Lambda)_\nu$ (resp. $\widetilde{\mathcal{C}}^-(\Lambda)_\nu$) is a finite dimensional free $T$-submodule and $\widetilde{\mathcal{C}}^+(\Lambda) = \bigoplus\limits_{\nu \in \mathbb{N}[I]} \widetilde{\mathcal{C}}^+(\Lambda)_\nu$ (resp. $\widetilde{\mathcal{C}}^-(\Lambda) = \bigoplus\limits_{\nu \in \mathbb{N}[I]} \widetilde{\mathcal{C}}^-(\Lambda)_\nu$). Clearly, $\widetilde{\mathcal{C}}^+(\Lambda)_{\nu_1} \widetilde{\mathcal{C}}^+(\Lambda)_{\nu_2} \subset \widetilde{\mathcal{C}}^+(\Lambda)_{\nu_1+\nu_2}$ and $\widetilde{\mathcal{C}}^+(\Lambda)_0 = T$ (resp. $\widetilde{\mathcal{C}}^-(\Lambda)_{\nu_1} \widetilde{\mathcal{C}}^-(\Lambda)_{\nu_2} \subset \widetilde{\mathcal{C}}^-(\Lambda)_{\nu_1+\nu_2}$ and $\widetilde{\mathcal{C}}^-(\Lambda)_0 = T$). We define

$$\Delta(u_i^+) = u_i^+ \otimes 1 + K_i \otimes u_i^+, \quad \Delta(K_\alpha) = K_\alpha \otimes K_\alpha,$$
$$\varepsilon(u_i^+) = 0, \quad \varepsilon(K_\alpha) = 1, \quad \sigma(u_i^+) = -K_{-i} u_i^+, \quad \sigma(K_\alpha) = -K_\alpha,$$

where $i \in I, \alpha \in \mathbb{N}[I]$. Then, $\widetilde{\mathcal{C}}^+(\Lambda)$ is a Hopf algebra. Similarly, we define

$$\Delta(u_i^-) = u_i^+ \otimes 1 + K_i \otimes u_i^-, \quad \Delta(K_\alpha) = K_\alpha \otimes K_\alpha,$$
$$\varepsilon(u_i^-) = 0, \quad \varepsilon(K_\alpha) = 1, \quad \sigma(u_i^-) = -K_{-i} u_i^-, \quad \sigma(K_\alpha) = -K_\alpha,$$

where $i \in I, \alpha \in \mathbb{N}[I]$. Then, $\widetilde{\mathcal{C}}^-(\Lambda)$ is a Hopf algebra.

We define a bilinear form $\widetilde{\varphi} : \widetilde{\mathcal{C}}^+(\Lambda) \times \widetilde{\mathcal{C}}^-(\Lambda) \longmapsto Q(v)$ on $\widetilde{\mathcal{C}}^+(\Lambda) \times \widetilde{\mathcal{C}}^-(\Lambda)$ as follows

$$\widetilde{\varphi}(K_\alpha u_i^+, K_\beta u_j^-) = v^{-(\alpha,\beta)-(i,\beta)+(\alpha,j)} \mid u_i \mid a_i^{-1} \delta_{ij}$$

for all $\alpha, \beta \in \mathbb{Z}[I], i, j \in I$, where $\mid u_i \mid$ is the cardinality of simple module $u_i$, $\delta_{ij}$ is the Kronecker sign. By [9], $(\widetilde{\mathcal{C}}^+(\Lambda), \widetilde{\mathcal{C}}^-(\Lambda), \widetilde{\varphi})$ is a skew Hopf pairing and is a member of $\mathcal{L}(C)$. Therefore, we have the reduced Dinfeld double $\widetilde{\mathcal{D}_{\mathcal{C}}}(\Lambda)$ and triangular decomposition

$$\widetilde{\mathcal{D}}_{\mathcal{C}}(\Lambda) = \widetilde{\mathcal{C}}_0^+(\Lambda) \otimes T \otimes \widetilde{\mathcal{C}}_0^-(\Lambda).$$

Set

$$\mathcal{I}_0^+ = \{x \in \widetilde{\mathcal{C}}_0^+(\Lambda) \mid \widetilde{\varphi}(x, \widetilde{\mathcal{C}}_0^-(\Lambda)) = 0\} = \{x \in \widetilde{\mathcal{C}}_0^+(\Lambda) \mid \widetilde{\varphi}(x, \widetilde{\mathcal{C}}^-(\Lambda)) = 0\}$$

and $\mathcal{I}^+ = T\mathcal{I}_0^+ \simeq T \otimes \mathcal{I}_0^+$. It is easy to see that $\mathcal{I}^+$ is a Hopf ideal of $\widetilde{\mathcal{C}}^+(\Lambda)$. Dually, setting

$$\mathcal{I}_0^- = \{y \in \widetilde{\mathcal{C}}_0^-(\Lambda) \mid \widetilde{\varphi}(\widetilde{\mathcal{C}}_0^+(\Lambda), y) = 0\} = \{y \in \widetilde{\mathcal{C}}_0^-(\Lambda) \mid \widetilde{\varphi}(\widetilde{\mathcal{C}}^-(\Lambda), y) = 0\}$$

and $\mathcal{I}^- = T\mathcal{I}_0^- \simeq T \otimes \mathcal{I}_0^-$, we have that $\mathcal{I}^-$ is a Hopf ideal of $\widetilde{\mathcal{C}}^-(\Lambda)$ (see for example [9]). We set

$$\mathcal{C}_0^+(\Lambda) = \widetilde{\mathcal{C}}_0^+(\Lambda)/\mathcal{I}_0^+, \mathcal{C}^+(\Lambda) = \widetilde{\mathcal{C}}^+(\Lambda)/\mathcal{I}^+, \quad \mathcal{C}_0^-(\Lambda) = \widetilde{\mathcal{C}}_0^-(\Lambda)/\mathcal{I}_0^-, \quad \mathcal{C}^-(\Lambda) = \widetilde{\mathcal{C}}^-(\Lambda)/\mathcal{I}^-.$$

One sees that $\widetilde{\varphi}$ induces a skew Hopf pairing $\varphi : \mathcal{C}^+(\Lambda) \times \mathcal{C}^-(\Lambda) \longmapsto Q(v)$ which is a nondegenerate member of $\mathcal{L}(C)$. Therefore, we have $\mathcal{D}_{\mathcal{C}}(\Lambda) = \widetilde{\mathcal{D}}_{\mathcal{C}}(\Lambda)/\mathcal{I}$, where $\mathcal{I} = \mathcal{I}^+ \otimes \mathcal{C}^-(\Lambda) + \mathcal{C}^+(\Lambda) \otimes \mathcal{I}^-$. From the construction, we also know that $\mathcal{D}_{\mathcal{C}}(\Lambda) = \mathcal{C}_0^+(\Lambda) \otimes T \otimes \mathcal{C}_0^-(\Lambda)$.

We set

$$u_{ij}^{+}=\sum_{t=0}^{n(i,j)}(-1)^t\begin{bmatrix}n(i,j)\\ t\end{bmatrix}_i \upsilon^{d_i(t-1)t}u_i^{+\,n(i,j)-t}u_j^{+}u_i^{+\,t},$$

$$u_{ij}^{-}=\sum_{t=0}^{n(i,j)}(-1)^t\begin{bmatrix}n(i,j)\\ t\end{bmatrix}_i \upsilon^{d_j(t-1)t}u_i^{-\,n(i,j)-t}u_j^{-}u_i^{-\,t},$$

where $n(i,j)=1+\frac{e(i,j)}{d_i}, e(i,j)=\dim_k\mathrm{Ext}(V_i,V_j), d_i=\dim_k\mathrm{End}(V_i)$. By trivial computation, we can get

$$\Delta(u_{ij}^{+})=u_{ij}^{+}\otimes 1+K_i^{n(i,j)}K_j\otimes u_{ij}^{+},$$

$$\Delta(u_{ij}^{-})=1\otimes u_{ij}^{-}+u_{ij}^{-}\otimes K_i^{-n(i,j)}K_j^{-1}.$$

**Lemma 3.1.** *For any $x\in\mathcal{C}^{-}(\Lambda)$ and $y\in\mathcal{C}^{+}(\Lambda)$ ,we have*

(a) $\varphi(u_{ij}^{+},x)=0,$ (b) $\varphi(y,u_{ij}^{-})=0.$

**Proof.** (a) Since $\deg(u_{ij}^{+})=n(i,j)i+j$, if $\deg(x)\neq n(i,j)i+j$, we have $\varphi(u_{ij}^{+},x)=0$, by [9, 2.9.3]. Assume $\deg(x)=n(i,j)i+j, x=u_s^{-}y$, where $s\in\{i,j\}, y\in\mathcal{C}^{-}(\Lambda), \deg(y)=\deg(x)-s$. Obviously, $\deg(y)\neq 0, \deg(y)\neq\deg(x)$.

$$\begin{aligned}\varphi(u_{ij}^{+},x)&=\varphi(u_{ij}^{+},u_s^{-}y)=\varphi(\Delta(u_{ij}^{+}),u_s^{-}\otimes y)\\&=\varphi(u_{ij}^{+}\otimes 1+K_i^{n(i,j)}K_j\otimes u_{ij}^{+},\langle u_s^{-}\rangle\otimes y)\\&=\varphi(u_{ij}^{+}\otimes 1,u_s^{-}\otimes y)+\varphi(K_i^{n(i,j)}K_j\otimes u_{ij}^{+},u_s^{-}\otimes y)\\&=\varphi(u_{ij}^{+}u_s^{-})\varphi(1,y)+\varphi(K_i^{n(i,j)}K_j,u_s^{-})\varphi(u_{ij}^{+},y)=0,\end{aligned}$$

since $\varphi(1,y)=\varphi(K_i^{n(i,j)}K_j,u_s^{-})=0$. Similarly, we can prove (b).

The category $\mathcal{O}$ consisits of $\mathcal{D}_{\mathcal{C}}(\Lambda)$-modules such that each object $M$ has direct sum decomposition $M=\bigoplus_{\lambda\in X}M_\lambda$, and there are finitely many $\lambda_1,\lambda_2,\dots,\lambda_n\in X$ such that for any $\lambda\in X$ , $\lambda\leq\lambda_i$ for some $1\leq i\leq n$, where $X$ is the weight lattice of $\mathcal{D}_{\mathcal{C}}(\Lambda)$-module.

Let $M\in\mathcal{O}$. A vector $x\in M$ is called primitive vector, if there is a submodule $N\subset M$ such that $x\notin N$, but $\mathcal{C}_0^{+}(\Lambda)\cdot x\subset N$.

**Lemma 3.2.** *$\mathcal{D}_{\mathcal{C}}(\Lambda)$-module $M$ in $\mathcal{O}$ is spanned by its primitive vectors* (*even being a $\mathcal{C}_0^{+}(\Lambda)$-module*).

**Proof.** Let $M\in\mathcal{O}$.

We claim that $x$ is not a primitive vector if and only if $x\in\langle\mathcal{C}_0^{+}(\Lambda)\cdot x\rangle$, where $\langle\mathcal{C}_0^{+}(\Lambda)\cdot x\rangle$ is an ideal spanned by $\mathcal{C}_0^{+}(\Lambda)\cdot x$.

In fact, assume $x$ is not a primitive vector. If $x\notin\langle\mathcal{C}_0^{+}(\Lambda)\cdot x\rangle$, we get a contradiction since $\mathcal{C}_0^{+}(\Lambda)\cdot x\subseteq\langle\mathcal{C}_0^{+}(\Lambda)\cdot x)\rangle$. Therefore, $x\in\langle\mathcal{C}_0^{+}(\Lambda)\cdot x\rangle$ and there is a submodule $N$ such that $x\notin N$, but $\mathcal{C}_0^{+}(\Lambda)\cdot x\subset N$ and $x\in\langle\mathcal{C}_0^{+}(\Lambda)\cdot x\rangle$, we also get a contradiction. This means that $x$ is not a primitive vector, and we get the claim.

Let $x\in M$. If $x$ is not a primitive vector, then $x\in\langle\mathcal{C}_0^{+}(\Lambda)\cdot x\rangle$ by the claim. So $x$ is generated by the elements in $\mathcal{C}_0^{+}(\Lambda)\cdot x$. For any $x_1\in\mathcal{C}_0^{+}(\Lambda)\cdot x$, if $x_1$ is not a primitive vector, then $x_1\in\langle\mathcal{C}_0^{+}(\Lambda)\cdot x_1\rangle$. Thus, $x_1$ is generated by the elements in $\mathcal{C}_0^{+}(\Lambda)\cdot x_1$. Repeating the above process finitely many times, we can get $\mathcal{C}_0^{+}(\Lambda)\cdot x_n=0$, where $x_n$ is an element in $\langle\mathcal{C}_0^{+}(\Lambda)\cdot x_{n-1}\rangle$. By definition, $x_n$ is a primitive vector. Therefore, $x_{n-1}$ is generated by primitive vectors, and hence $x$ is generated by primitive vectors.

**Lemma 3.3.** *Let $M\in\mathcal{O}$, $m\in M$ be a primitive vector, and $\lambda$ is its weight. Then, there is a submodule $N\subset M$ such that $m\notin N$, and $\Omega\Xi(m)=\upsilon^{G(\lambda)}m \pmod N$, where $\Omega\Xi$ is the Casimir operator, $G$ is the function in Lemma* 2.2.

**Proof.** Since $m$ is a primitive vector, there is a submodule $N \subset M$ such that $m \notin N$, but $\mathcal{C}_0^+(\Lambda)\cdot m \subset N$. Thus, if $\overline{m} = m + N \in (M/N)_\lambda$, we have $\mathcal{C}_0^+(\Lambda)\cdot\overline{m} = 0$. There is a unique $\mathcal{D}_{\mathcal{C}}(\Lambda)$-module homomorphism $\varphi : M(\lambda) \longmapsto M/N$ such that $\varphi(m_\lambda) = \overline{m}$, where $m_\lambda$ is the highest weight vector of Verma module $M(\lambda)$. Hence $J_{\overline{m}} \simeq M(\lambda)/\mathrm{ker}\varphi$, where $J_{\overline{m}}$ is the submodule spanned by $\overline{m}$. We get

$$\Omega\Xi(m) = v^{G(\lambda)}\cdot m \quad (\mathrm{mod}: N)$$

by the action of $\Omega E$ on $J_{\overline{m}}$.

**Proposition 3.1.** *As an ideal of $\mathcal{C}_0^+(\Lambda)$ (resp. $\mathcal{C}_0^-(\Lambda)$), $\mathcal{I}_0^+$ (resp. $\mathcal{I}_0^-$) is generated by those $\mathcal{I}_{\lambda 0}^+$ (resp. $\mathcal{I}_{\lambda 0}^-$), where $\lambda \in \mathbb{N}[I]\backslash I$, and if $\lambda = \sum\limits_{i\in I} k_i i$, then $(\lambda,\lambda) = \sum\limits_{i\in I} k_i(i,i)$, where $\mathcal{I}_{\lambda 0}^+ = \mathcal{I}_0^+ \cap \mathcal{C}_0^+(\Lambda)_\lambda$ (resp. $\mathcal{I}_{\lambda 0}^- = \mathcal{I}_0^- \cap \mathcal{C}_0^-(\Lambda)_\lambda$).*

**Proof.** We set

$$\overline{M}(\lambda) = \widetilde{\mathcal{D}}_{\mathcal{C}}(\Lambda)\Big/\Big(\sum_{i\in I}\widetilde{\mathcal{D}}_{\mathcal{C}}(\Lambda)u_i^+ + \widetilde{\mathcal{D}}_{\mathcal{C}}(\Lambda)(K_i - v^{(\lambda,i)})\Big),$$

then we have

$$\overline{M}(0) = \widetilde{\mathcal{D}}_{\mathcal{C}}(\Lambda)\Big/\Big(\sum_{i\in I}\widetilde{\mathcal{D}}_{\mathcal{C}}(\Lambda)u_i^+ + \widetilde{\mathcal{D}}_{\mathcal{C}}(\Lambda)(K_i - 1)\Big).$$

Since $u_i^+\cdot\overline{m}_0 = 0$ (here $\overline{m}_0$ is the highest weight vector of $\overline{M}(0)$), we have

$$\begin{aligned}
u_j^+(u_i^-\cdot\overline{m}_0) &= u_i^-(u_j^+\cdot\overline{m}_0) = 0, \quad \text{if } i \neq j,\\
u_i^+(u_i^-\cdot\overline{m}_0) &= \Big(u_i^-u_i^+ - \frac{|\,u_i\,|}{a_i}(K_i - K_i^{-1})\Big)\cdot\overline{m}_0\\
&= u_i^-u_i^+\cdot\overline{m}_0 - \frac{|\,u_i\,|}{a_i}(K_i\overline{m}_0 - K_i^{-1}\overline{m}_0)\\
&= 0 - \frac{|\,u_i\,|}{a_i}(\overline{m}_0 - \overline{m}_0) = 0, \quad \text{if } i = j.
\end{aligned}$$

Therefore, there is a unique $\widetilde{\mathcal{D}}_{\mathcal{C}}(\Lambda)$-module homomorphism $\psi : \overline{M}(-i) \longmapsto \overline{M}(0)$ such that $\overline{m}_{-i} \longmapsto u_i^-\cdot\overline{m}_0$, where $\overline{m}_{-i}$ is the highest weight vector of the Verma module $\overline{M}(-i)$. For any $x_\alpha \in \mathcal{D}_{\mathcal{C}}(\Lambda)$, if $\psi(x_\alpha\overline{m}_{-i}) = x_\alpha\psi(\overline{m}_{-i}) = x_\alpha u_i^-\overline{m}_0 = 0$, then $x_\alpha u_i^- = 0$ and $x_\alpha = 0$. Thus the map $\psi :\ \overline{M}(-i) \longmapsto \overline{M}(0)$ is an injective. We can regard $\overline{M}(-i)$ and hence $\bigoplus\limits_{i\in I}\overline{M}(-i)$ as a submodule of $\overline{M}(0)$. One sees that $\overline{M}(0)/\Big(\bigoplus\limits_{i\in I}\overline{M}(-i)\Big)$ is a simple module. Therefore, $\bigoplus\limits_{i\in I}\overline{M}(-i)$ is a maximal submodule of $\overline{M}(0)$. We denote it by $\overline{M}'(0)$. Now, we have $\widetilde{\mathcal{D}}_{\mathcal{C}}(\Lambda)$-module isomorphisms:

$$\begin{aligned}
\mathcal{D}_{\mathcal{C}}(\Lambda)\bigotimes_{\widetilde{\mathcal{D}}_{\mathcal{C}}(\Lambda)}\overline{M}'(0) &\simeq \mathcal{D}_{\mathcal{C}}(\Lambda)\bigotimes_{\widetilde{\mathcal{D}}_{\mathcal{C}}(\Lambda)}\Big(\bigoplus_{i\in I}\overline{M}(-i)\Big)\\
&\simeq \bigoplus_{i\in I}\mathcal{D}_{\mathcal{C}}(\Lambda)\bigotimes_{\widetilde{\mathcal{D}}_{\mathcal{C}}(\Lambda)}\overline{M}(-i) \simeq \bigoplus_{i\in I}M(-i),
\end{aligned}$$

where $M(-i)$ is a Verma $\mathcal{D}_{\mathcal{C}}(\Lambda)$-module corresponding to $\overline{M}(-i)$. We denote by $\tau$ the composition of these isomorphisms. Let

$$\pi :\ \widetilde{\mathcal{D}}_{\mathcal{C}}(\Lambda) \longmapsto \widetilde{\mathcal{D}}_{\mathcal{C}}(\Lambda)/(\mathcal{I}^+\otimes\widetilde{\mathcal{C}}^-(\Lambda) + \widetilde{\mathcal{C}}^+(\Lambda)\otimes\mathcal{I}^-)$$

be the canonical projection. Since $\mathcal{I}_0^-\cdot\overline{m}_0$ is a submodule of $\overline{M}(0)$, we have $\mathcal{I}_0^-\cdot\overline{m}_0 \subset \overline{M}'(0)$.

We define the action of $\widetilde{\mathcal{D}}_{\mathcal{C}}(\Lambda)$ on $\mathcal{I}_0^-$ as follows

$$\widetilde{\mathcal{C}}^+(\Lambda)\cdot\mathcal{I}_0^-=0,\quad K_\alpha\cdot x=x,\quad \alpha\in\mathbb{Z}[I],x\in\mathcal{I}_0^-,$$

where $\widetilde{\mathcal{C}}^+(\Lambda)\cdot\mathcal{I}_0^-$ is the multiplication in $\widetilde{\mathcal{D}}_{\mathcal{C}}(\Lambda)$. We have a well-defined map

$$\varphi_1:\quad \mathcal{I}_0^-\longmapsto\widetilde{\mathcal{D}}_{\mathcal{C}}(\Lambda)\bigotimes_{\widetilde{\mathcal{D}}_{\mathcal{C}}(\Lambda)}\overline{M}'(0),$$
$$a\longmapsto 1\otimes a(\overline{m}_0),$$

and $\varphi_1$ is a $\widetilde{\mathcal{D}}_{\mathcal{C}}(\Lambda)$-module homomorphism. For any $x\in\widetilde{\mathcal{D}}_{\mathcal{C}}(\Lambda)$ and $a\in\mathcal{I}_0^-$, we have

$$\varphi_1(x\cdot a)=1\bigotimes_{\widetilde{\mathcal{D}}_{\mathcal{C}}(\Lambda)}(x\cdot a)(\overline{m}_0)=\pi(x)\bigotimes_{\widetilde{\mathcal{D}}_{\mathcal{C}}(\Lambda)}a(\overline{m}_0)=x(\varphi_1(a)).$$

Clearly, $\mathcal{I}_0^-\sum\limits_{i\in I}\widetilde{\mathcal{C}}_0^-(\Lambda)u_i^-$ is a $\widetilde{\mathcal{D}}_{\mathcal{C}}(\Lambda)$-submodule of $\mathcal{I}_0^-\ \mathcal{D}_c(\Lambda)$, and $\varphi_1(\mathcal{I}_0^-\sum\limits_{i\in I}\widetilde{\mathcal{C}}_0^-(\Lambda)u_i^-)=0$. Since $\mathcal{I}_0^-\mathcal{I}_0^-\subset\mathcal{I}_0^-\sum\limits_{i\in I}\widetilde{\mathcal{C}}_0^-(\Lambda)u_i^-$, $\mathcal{I}_0^-/(\mathcal{I}_0^-\sum\limits_{i\in I}\widetilde{\mathcal{C}}_0^-(\Lambda)u_i^-)$ is a $\mathcal{C}_0^-(\Lambda)$-module in a natural way. We have $\mathcal{C}(\Lambda)$-module homomorphism

$$\varphi_2\ :\ \mathcal{I}_0^-\Big/\Big(\mathcal{I}_0^-\sum_{i\in I}\widetilde{\mathcal{C}}_0^-(\Lambda)u_i^-\Big)\longmapsto\bigoplus_{i\in I}M(-i).$$

More precisely, $\varphi_2$ is given in the following way: for any $u_i^-\notin\mathcal{I}_0^-$, we write $a\in\mathcal{I}_0^-$ in the form $a=\sum\limits_{i\in I}x_i\cdot u_i^-$, where $x_i\in\sum\limits_{i\in I}\widetilde{\mathcal{C}}_0^-(\Lambda)u_i^-$, then

$$\begin{aligned}\varphi_2\Big(a+\mathcal{I}_0^-\sum_{i\in I}\widetilde{\mathcal{C}}_0^-(\Lambda)u_i^-\Big)&=\sum_{i\in I}\tau\varphi_1(x_i\cdot u_i^-)=\sum_{i\in I}\tau\Big(1\bigotimes_{\widetilde{\mathcal{D}}_{\mathcal{C}}(\Lambda)}(x_i\cdot u_i^-)\cdot\overline{m}_0\Big)\\&=\sum_{i\in I}\tau(\pi(x_i)\otimes\overline{m}_{-i})=\sum_{i\in I}\pi(x_i)\overline{m}_{-i},\end{aligned}$$

where $\overline{m}_{-i}$ is the highest weight vector of $\overline{M}(-i)$. If

$$\varphi_2(\overline{a})=\varphi_2\Big(a+\mathcal{I}_0^-\sum_{i\in I}\widetilde{\mathcal{C}}_0^-(\Lambda)u_i^-\Big)=0,$$

we have $\pi(x_i)=0$ for all $i\in I$, and it follows that $\overline{a}=a+\mathcal{I}_0^-\sum\limits_{i\in I}\widetilde{\mathcal{C}}_0^-(\Lambda)u_i^-=0$. Thus we have an embedding

$$\varphi_2\quad:\mathcal{I}_0^-\Big/\Big(\mathcal{I}_0^-\sum_{i\in I}\widetilde{\mathcal{C}}_0^-(\Lambda)u_i^-\Big)\longmapsto\bigoplus_{i\in I}M(-i).$$

If we define the action of $\mathcal{C}_0^+(\Lambda)$ and $T$ on $\mathcal{I}_0^-/(\mathcal{I}_0^-\sum_{i\in I}\widetilde{\mathcal{C}}_0^-(\Lambda)u_i^-)$ as zero, then

$$\mathcal{I}_0^-\Big/\Big(\mathcal{I}_0^-\sum_{i\in I}\widetilde{\mathcal{C}}_0^-(\Lambda)u_i^-\Big)$$

becomes a $\mathcal{D}_{\mathcal{C}}(\Lambda)$-module. As $\mathcal{D}_{\mathcal{C}}(\Lambda)$-module, both $\mathcal{I}_0^-\Big/\Big(\mathcal{I}_0^-\sum\limits_{i\in I}\widetilde{\mathcal{C}}_0^-(\Lambda)u_i^-\Big)$ and $\bigoplus\limits_{i\in I}M(-i)$ belong to $\mathcal{O}$.

Now, let $-\beta$ be the primitive weight of $\mathcal{I}_0^-\Big/\Big(\mathcal{I}_0^-\sum\limits_{i\in I}\widetilde{\mathcal{C}}_0^-(\Lambda)u_i^-\Big)$. Note that $\beta\notin I$ since for any $(i\in I),u_i^-\notin\mathcal{I}_0^-$. We know that $-\beta$ is a primitive weight of some $M(-i)$ by the injectivity of $\varphi_2$. Thus, $-\beta<-i$, or $i<\beta$. Let $m$ be a primitive vector corresponding to the weight $-\beta$, then there is a submodule $N \subset M(-i)$ such that $m \notin N$ and by Lemma 3.3

$$\Omega\Xi(m) = v^{G(-\beta)}m \ (\mathrm{mod}\ N), \quad \Omega\Xi(m) = v^{G(-i)}m \ (\mathrm{mod}\ N).$$

Thus $v^{G(-i)}m = v^{G(-\beta)}m \ (\mathrm{mod}\, N)$. However, $m \notin N$, so we get $v^{G(-i)} = v^{G(-\beta)}$ and $G(-i) = G(-\beta)$. By Lemma 3.2, we know that $\mathcal{I}_0^-$ is generated by those $\mathcal{I}_{-\beta 0}^-$ with $\beta \notin I, G(-i) = G(-\beta)$ for some $i \in I$.

Let $-\beta = -\sum\limits_{j=1}^{n} k_j i_j$, where $i_j \in I,\ k_j \in \mathbb{N} \cap \{0\}$. Let $i = i_j$ (for some $j$). Since $\beta > i$, we have $k_j \neq 0$. So we can write $-\beta$ in the form

$$\begin{aligned}-\beta &= -i_j - k_1 i_1 - k_2 i_2 - \cdots - k_{j-1} i_{j-1} - (k_j - 1) i_j - k_{j+1} i_{j+1} - \cdots - k_n i_n \\ &= -i - \gamma_1 - \gamma_2 - \cdots - \gamma_m,\end{aligned}$$

where $m = \sum\limits_{i=1}^{n} k_i - 1$, and for each $p \in \{1, 2, \cdots, m\}$, there are $k_p$ of $i_p$ $(p \neq j)$, and $k_j - 1$ of $i_j$ in $\gamma_1, \gamma_2, \ldots, \gamma_m$. By Lemma 2.2, we get

$$\begin{aligned}0 &= G(-i) - G(-\beta) = G(-i_j) - G(-i_j - \gamma_1 - \gamma_2 - \cdots - \gamma_m) \\ &= \sum_{p=1}^{m} (\gamma_p, \gamma_p)\langle \gamma_p, -i_j\rangle - \sum_{1 \le p \neq q \le m} (\gamma_p, \gamma_q) = 2\sum_{p=1}^{m} (\gamma_p, -i_j) - \sum_{1 \le p \neq q \le m} (\gamma_p, \gamma_q) \\ &= -2\sum_{p=1}^{m} (\gamma_p, i_j) + \sum_{p=1}^{m} (\gamma_p, \gamma_p) - \Big(\sum_{p=1}^{m} \gamma_p, \sum_{p=1}^{m} \gamma_p\Big) \\ &= -2(\beta - i_j, i_j) + \sum_{s \neq j} k_s (i_s, i_s) + (k_j - 1)(i_j, i_j) - (\beta - i_j, \beta - i_j) \\ &= -2(\beta, i_j) + 2(i_j, i_j) + \sum_{s \neq j} k_s (i_s, i_s) + (k_j - 1)(i_j, i_j) - (\beta, \beta) + 2(\beta, i_j) - (i_j, i_j) \\ &= \sum_{p=1}^{m} k_p (i_p, i_p) - (\beta, \beta).\end{aligned}$$

Therefore, $(\beta, \beta) = \sum\limits_{p=1}^{m} k_p (i_p, i_p)$.

## §4. Main Result

In this section, we prove the following result.

**Theorem 4.1.** *Let* $\mathcal{D}_{\mathcal{C}}(\Lambda) = \mathcal{C}_0^+(\Lambda) \otimes T \otimes \mathcal{C}_0^-(\Lambda)$, *where* $\widetilde{\mathcal{C}}_0^+(\Lambda)/\mathcal{I}_0^+ = \mathcal{C}_0^+(\Lambda), \widetilde{\mathcal{C}}_0^-(\Lambda)/\mathcal{I}_0^- = \mathcal{C}_0^-(\Lambda)$. *Then the elements*

$$u_{ij}^+ = \sum_{t=0}^{n(i,j)} (-1)^t \begin{bmatrix} n(i,j) \\ t \end{bmatrix}_{\alpha_i} v^{d_i (t-1) t} u_i^{+\, n(i,j) - t} u_j^+ u_i^{+\, t},$$

$$u_{ij}^- = \sum_{t=0}^{n(i,j)} (-1)^t \begin{bmatrix} n(i,j) \\ t \end{bmatrix}_{\alpha_i} v^{d_j (t-1) t} u_i^{-\, n(i,j) - t} u_j^- u_i^{-\, t}$$

*generate the ideals* $\mathcal{I}_0^+$ *and* $\mathcal{I}_0^-$, *respectively.*

**Proof.** Set $\mathcal{D}'_{\mathcal{C}}(\Lambda) = \widetilde{\mathcal{D}}_{\mathcal{C}}(\Lambda)/\mathcal{J}$, where $\mathcal{J}$ is the ideal generated by $u_{ij}^+$ and $u_{ij}^-$ $(i, j \in I,\ i \neq j)$. Then we have the induced $\mathbb{N}[I]$-graded algebra $\mathcal{D}'_{\mathcal{C}}(\Lambda) = \bigoplus\limits_{\alpha \in \mathbb{N}[I]} \mathcal{D}'_{\mathcal{C}}(\Lambda)_\alpha$. Clearly, $\mathcal{J} \subset \mathcal{I}$.

Let ${}_1\mathcal{I}_0^{\pm}$ be the image of $\mathcal{I}_0^{\pm}$ under the canonical projection $\pi\ :\ \widetilde{\mathcal{D}}_{\mathcal{C}}(\Lambda)\longmapsto \mathcal{D}'_{\mathcal{C}}(\Lambda)$. We assume that ${}_1\mathcal{I}_0^{+}\neq 0$ (the proof of ${}_1\mathcal{I}_0^{-}\neq 0$ can be obtained by using $\omega$).

Now, since ${}_1\mathcal{I}_0^{+}=\bigoplus\limits_{\alpha\in\mathbb{N}[I]}{}_1\mathcal{I}_0^{+}\cap\mathcal{D}'_{\mathcal{C}}(\Lambda)_\alpha=\bigoplus\limits_{\alpha\in\mathbb{N}[I]}{}_1\mathcal{I}_{\alpha 0}^{+}\neq 0$, there is $\alpha\in\mathbb{N}[I]$ such that ${}_1\mathcal{I}_{\alpha 0}^{+}\neq 0$. We choose $\alpha=\sum\limits_{t=1}^{n}k_te_t$ such that $\sum\limits_{t=1}^{n}k_t$ is minimal. Obviously, ${}_1\mathcal{I}_{\alpha 0}^{+}$ must occur in any system of homogeneous generators of ${}_1\mathcal{I}_0^{+}$.

Let $T_i$ be the Luzstig symmetry for each $i\in I$ (see for example [6]). One easily sees that for each $i\in I$, we have $T_i\in \mathrm{Aut}\mathcal{D}'_{\mathcal{C}}(\Lambda)$ such that $T_i(\mathcal{D}'_{\mathcal{C}}(\Lambda)_\alpha)=\mathcal{D}'_{\mathcal{C}}(\Lambda)_{s_i(\alpha)}$, where $s_i$ is the fundamental reflection (see for example [10]). Since

$$T_i({}_1\mathcal{I}_{\alpha 0}^{+})=T_i({}_1\mathcal{I}_0^{+}\cap\mathcal{D}'_{\mathcal{C}}(\Lambda)_\alpha)=T_i({}_1\mathcal{I}_0^{+})\cap T_i(\mathcal{D}'_{\mathcal{C}}(\Lambda)_\alpha)=T_i({}_1\mathcal{I}_0^{+})\cap\mathcal{D}'_{\mathcal{C}}(\Lambda)_{s_i(\alpha)},$$

it follows that if $T_i({}_1\mathcal{I}_0^{+})\subset {}_1\mathcal{I}_0^{+}$, then

$$T_i({}_1\mathcal{I}_{\alpha 0}^{+})=T_i({}_1\mathcal{I}_0^{+})\cap\mathcal{D}'_{\mathcal{C}}(\Lambda)_{s_i(\alpha)}\subset {}_1\mathcal{I}_0^{+}\cap\mathcal{D}'_{\mathcal{C}}(\Lambda)_{s_i(\alpha)}={}_1\mathcal{I}_{s_i(\alpha)}^{+}.$$

$T_i$ is an isomorphism and ${}_1\mathcal{I}_{\alpha 0}^{+}\neq 0$, so ${}_1\mathcal{I}_{s_i(\alpha)}^{+}\neq 0$. Hence, to prove ${}_1\mathcal{I}_{s_i(\alpha)}^{+}\neq 0$, it is enough to prove $T_i({}_1\mathcal{I}_0^{+})\subset {}_1\mathcal{I}_0^{+}$.

Since $\widetilde{\varphi}(u_{ij}^{+},\widetilde{\mathcal{C}}^{-}(\Lambda))=\widetilde{\varphi}(\widetilde{\mathcal{C}}^{+}(\Lambda),u_{ij}^{-})=0$, $\widetilde{\varphi}$ naturally induces a pairing

$$\varphi'\ :\ \widetilde{\mathcal{C}}^{+}(\Lambda)/\mathcal{J}^{+}\times\widetilde{\mathcal{C}}^{-}(\Lambda)/\mathcal{J}^{-}\longmapsto\mathbb{Q}(v),$$
$$\varphi'(x+\mathcal{J}^{+},y+\mathcal{J}^{-})=\widetilde{\varphi}(x,y),$$

where $\mathcal{J}^{+}$ and $\mathcal{J}^{-}$ are the ideals generated by $u_{ij}^{+}$ and $u_{ij}^{-}$ respectively. Clearly, we have

$${}_1\mathcal{I}_0^{+}=\{x\in\widetilde{\mathcal{C}}_0^{+}(\Lambda)/\mathcal{J}^{+}\mid\varphi'(x,\widetilde{\mathcal{C}}_0^{-}(\Lambda)/\mathcal{J}^{-})=0\},$$
$${}_1\mathcal{I}_0^{-}=\{y\in\widetilde{\mathcal{C}}_0^{-}(\Lambda)/\mathcal{J}^{-}\mid\varphi'(\widetilde{\mathcal{C}}_0^{+}(\Lambda)/\mathcal{J}^{+},y)=0\}.$$

For any $i\in I$, $u_i^{+}\notin {}_1\mathcal{I}_0^{+}$ and ${}_1\mathcal{I}_0^{+}\cap\mathcal{D}'_{\mathcal{C}}(\Lambda)_i={}_1\mathcal{I}_{i0}^{+}=0$. Hence

$${}_1\mathcal{I}_0^{+}=\bigoplus_{\substack{\alpha\in\mathbb{N}[I]\\ \alpha\neq i}}{}_1\mathcal{I}_{\alpha 0}\subset\bigoplus_{\substack{\alpha\in\mathbb{N}[I]\\ \alpha\neq i}}(\widetilde{\mathcal{C}}_0^{+}(\Lambda)/\mathcal{J}^{+})_\alpha\subset\widetilde{\mathcal{C}}_0^{+}(\Lambda)/\mathcal{J}^{+}.$$

Thus, we have

$$T_i({}_1\mathcal{I}_0^{+})\subset\bigoplus_{\substack{\alpha\in\mathbb{N}[I]\\ \alpha\neq i}}T_i((\widetilde{\mathcal{C}}_0^{+}(\Lambda)/\mathcal{J}^{+})_\alpha)=\bigoplus_{\substack{\alpha\in\mathbb{N}[I]\\ \alpha\neq i}}(\widetilde{\mathcal{C}}_0^{+}(\Lambda)/\mathcal{J}^{+})_{s_i(\alpha)}\subset\widetilde{\mathcal{C}}_0^{+}(\Lambda)/\mathcal{J}^{+},$$

since when $\alpha\in\mathbb{N}[I]$ and $\alpha\neq i$, $s_i(\alpha)\in\mathbb{N}[I]$. Similar to [6, 6.14], we have operators $r_i$ and $r'_i$ on $\widetilde{\mathcal{C}}_0^{+}(\Lambda)/\mathcal{J}^{+}$ satisfying

$$\Delta(x)=u_i^{+}\otimes\sum_{i\in I}r'_i(x)+(\text{rest}),\qquad \Delta(x)=\sum_{i\in I}r_i(x)\otimes u_i^{+}+(\text{rest}),$$

where $x\in(\widetilde{\mathcal{C}}_0^{+}(\Lambda)/\mathcal{J}^{+})_\alpha$. Furthermore,

$$r_i(1)=r'_i(1)=0,\quad r_i(u_j^{+})=r'_i(u_j^{+})=\delta_{ij},$$
$$r_i(xy)=v^{(\alpha,i)}r_i(x)y+xr_i(y),\quad r'_i(xy)=r'(x)y+v^{(i,\beta)}xr'_i(y),$$

where $x\in(\widetilde{\mathcal{C}}_0^{+}(\Lambda)/\mathcal{J}^{+})_\alpha, y\in\widetilde{\mathcal{C}}_0^{+}(\Lambda)/\mathcal{J}^{+})_\beta$. It is easy to prove that

$$\varphi'(x,u_i^{-}y)=\varphi'(u_i^{+},u_i^{-})\varphi'(r'_i(x),y)\quad \varphi'(x,yu_i^{-})=\varphi'(u_i^{+},u_i^{-})\varphi'(r'_i(x),y).$$

Let

$$(\widetilde{\mathcal{C}}_0^{+}(\Lambda)/\mathcal{J}^{+})(i)=\{x\in\widetilde{\mathcal{C}}_0^{+}(\Lambda)/\mathcal{J}^{+}\mid T_i(x)\in\widetilde{\mathcal{C}}_0^{+}(\Lambda)/\mathcal{J}^{+}\},$$
$$(\widetilde{\mathcal{C}}_0^{-}(\Lambda)/\mathcal{J}^{-})(i)=\{x\in\widetilde{\mathcal{C}}_0^{-}(\Lambda)/\mathcal{J}^{-}\mid T_i(x)\in\widetilde{\mathcal{C}}_0^{-}(\Lambda)/\mathcal{J}^{-}\}.$$

Then we have ${}_1\mathcal{I}_0^+ \subset (\widetilde{\mathcal{C}}_0^+(\Lambda)/\mathcal{J}^+)(i)$. Since $u_i^+u_j^- = u_j^-u_i^+(i \neq j)$, it follows that

$$(\widetilde{\mathcal{C}}_0^+(\Lambda)/\mathcal{J}^+) = \sum_{t\geq 0}(\widetilde{\mathcal{C}}_0^+(\Lambda)/\mathcal{J}^+)(i), \quad (\widetilde{\mathcal{C}}_0^-(\Lambda)/\mathcal{J}^-) = \sum_{t\geq 0}(\widetilde{\mathcal{C}}_0^-(\Lambda)/\mathcal{J}^-)(i).$$

We claim that, if $x \in (\widetilde{\mathcal{C}}_0^+(\Lambda)/\mathcal{J}^+)(i)$, then $r_i'(x) = 0$.

Noting that $u_j^+u_i^- - u_i^-u_j^+ = \frac{K_i - K_i^{-1}}{a_i}\delta_{ij} \mid u_i \mid$, we have for any $x \in (\widetilde{\mathcal{C}}_0^+(\Lambda)/\mathcal{J}^+)(i)$,

(a) $$xu_i^- - u_i^-x = \frac{r_i(x)K_i - K_i^{-1}r_i'(x)}{a_i} \mid u_i \mid .$$

Let

$$\frac{r_i(x)}{a_i} \mid u_i \mid = \sum_{t\geq 0}(u_i^+)^t y_t, \quad \frac{r_i'(x)}{a_i} \mid u_i \mid = \sum_{t\geq 0}(u_i^+)^t z_t,$$

where $y_t$ and $z_t \in (\widetilde{\mathcal{C}}_0^+(\Lambda)/\mathcal{J}^+)(i)$. Now, (a) can be written as

(a') $$xu_i^- - u_i^-x = \sum_{t\geq 0}(u_i^+)^t y_t K_i - K_i^{-1}\sum_{t\geq 0}(u_i^+)^t z_t.$$

By definition of $(\widetilde{\mathcal{C}}_0^+(\Lambda)/\mathcal{J}^+)(i)$, $T_i(y_t), T_i(z_t) \in (\widetilde{\mathcal{C}}_0^+(\Lambda)/\mathcal{J}^+)(i)$. Applying $T_i$ to both sides of $(a')$, we get

$$\begin{aligned} & - T_i(x)u^+iK_{-i} + u_i^+K_{-i}T_i(x) \\ = & \sum_{t\geq 0}[T_i((u_i^+)^t)T_i(y_t)T_i(K_i) - T_i(K_{-i})T_i((u_i^+)^t)T_i(z_t)] \\ = & \sum_{t\geq 0}(-1)^t v^{-t(i,i)}[K_{ti}(u_i^-)^t T_i(Y_t)K_{-i} - K_iK_{ti}(u_i^-)^t T_i(z_t)] \\ = & \sum_{t\geq 0}(-1)^t v^{-t(i,i)}K_{ti}(u_i^-)^t[T_i(y_t)K_{-i} - v^{t(i,i)}K_iT_i(z_t)]. \end{aligned}$$

The left side of the identity belongs to $K_{-i}\widetilde{\mathcal{C}}_0^+(\Lambda)/\mathcal{J}^+$, so does the right hand side. By the triangular decomposition, we get $T_i(y_t) = 0$ and $T_i(z_t) = 0$ for all $t \leq 0$. Therefore, $r_i'(x) = 0$.

Since $T_i(u_i^+) = \nu^{-(i,i)}K_iu_i^-$, we have $u_i^+ \notin (\widetilde{\mathcal{C}}_0^+(\Lambda)/\mathcal{J}^+)(i)$. For any $\alpha \in \mathbb{N}[I]$, we have

$$T_i((\widetilde{\mathcal{C}}_0^+(\Lambda)/\mathcal{J}^+)_\alpha) = \widetilde{\mathcal{C}}_0^+(\Lambda)/\mathcal{J}^+_{s_i(\alpha)}.$$

Thus, $T_i(T_i((\widetilde{\mathcal{C}}_0^+(\Lambda)/\mathcal{J}^+)(i))) \subseteq \widetilde{\mathcal{C}}_0^+(\Lambda)/\mathcal{J}^+$, and hence

$$T_i((\widetilde{\mathcal{C}}_0^+(\Lambda)/\mathcal{J}^+)(i)) \subseteq (\widetilde{\mathcal{C}}_0^+(\Lambda)/\mathcal{J}^+)(i).$$

So, $T_i((\widetilde{\mathcal{C}}_0^+(\Lambda)/\mathcal{J}^+)(i)) = (\widetilde{\mathcal{C}}_0^+(\Lambda)/\mathcal{J}^+)(i)$.

By [10], $x \in \mathfrak{h}^+(A)\langle i\rangle, y \in \mathfrak{h}^-(A)\langle i\rangle$, we have $\varphi(T_i(x), T_i(y)) = \varphi(x, y)$. Hence for any $\overline{x} \in (\widetilde{\mathcal{C}}_0^+(\Lambda)/\mathcal{J}^+)(i)$ and $\overline{y} \in (\widetilde{\mathcal{C}}_0^-(\Lambda)/\mathcal{J}^-)(i)$, $\varphi'(T_i(\overline{x}), T_i(\overline{y})) = \varphi'(\overline{x}, \overline{y})$. Therefore

$$\begin{aligned} & \varphi'(T_i({}_1\mathcal{I}_0^+), \widetilde{\mathcal{C}}_0^-(\Lambda)/\mathcal{J}^-) = \varphi'(T_i({}_1\mathcal{I}_0^+), \sum_{t\geq}(u_i^-)^t(\widetilde{\mathcal{C}}_0^-(\Lambda)/\mathcal{J}^-)(i)) \\ = & \varphi'(T_i({}_1\mathcal{I}_0^+), (\widetilde{\mathcal{C}}_0^+(\Lambda)/\mathcal{J}^+)(i)) + \sum_{t\geq 1}\varphi'(T_i({}_1\mathcal{I}_0^+), (u_i^-)^t(\widetilde{\mathcal{C}}_0^-(\Lambda)/\mathcal{J}^-)(i)) \\ = & \varphi'(T_i({}_1\mathcal{I}_0^+), T_i((\widetilde{\mathcal{C}}_0^-(\Lambda)/\mathcal{J}^-)(i))) \\ & + \sum_{t\geq 1}\varphi'(u_i^+, u_i^-)\varphi'(r_i'(T_i({}_1\mathcal{I}_0^+), (u_i^-)^{t-1}((\widetilde{\mathcal{C}}_0^-(\Lambda)/\mathcal{J}^-)(i)) \\ & \cdot \varphi'(T_i({}_1\mathcal{I}_0^+), (\widetilde{\mathcal{C}}_0^-(\Lambda)/\mathcal{J}^-)(i)) + 0 = 0. \end{aligned}$$

This implies that $T_i({}_1\mathcal{I}^+) \subset {}_1\mathcal{I}^+$. By the choice of $\alpha$ and fundamental reflection (see for example [10]), we know that

$$\sum_{t=1}^{n} k_t \leq \sum_{\substack{t=1\\ t\neq i}}^{n} k_i + \sum_{j\in\Gamma} d_{ij}k_j - k_i.$$

Thus, $0 \leq \sum\limits_{j\in\Gamma} d_{ji}k_j - 2k_i$. On the other hand,

$$\langle \alpha, e_i\rangle = (k_i - \sum_{j\to i} d_{ji}k_j)\varepsilon_i, \quad \langle e_i, \alpha\rangle = (k_i - \sum_{i\to j} d_{ji}k_j)\varepsilon_i.$$

So

$$(\alpha, e_i) = \langle \alpha, e_i\rangle + \langle e_i, \alpha\rangle = \varepsilon_i\Big(2k_i - \sum_{j\to i} d_{ji}k_j - \sum_{i\to j} d_{ji}k_j\Big),$$

$$\sum_{j\in\Gamma} d_{ji}k_j = \sum_{j\to i} d_{ji}k_j + \sum_{i\to j} d_{ji}k_j.$$

Thus, $(\alpha, e_i))\varepsilon_i^{-1} \leq 0$. So for all $i \in I$, we have $(\alpha, e_i) \leq 0$ (since $\varepsilon_i > 0$). Hence we have $(\alpha,\alpha) = \sum\limits_{i=1}^{n} k_i(\alpha, e_i) \leq 0$. By Proposition 3.1, $(\alpha,\alpha) = \sum\limits_{i=1}^{n} k_i(e_i, e_i) > 0$. This is a contradiction. So, we have ${}_1\mathcal{I}_0^+ = 0$. This means that $\mathcal{I}^+ \subseteq \mathcal{J}^+$. Thus, $\mathcal{I}^+ = \mathcal{J}^+$. Hence, $\mathcal{I}^+$ is generated by $u_{ij}^+$ $(i, j \in I, i \neq j)$. The proof is completed.

By Theorem 4.1, we have $\widetilde{\mathcal{C}}^+(\Lambda)/\mathcal{I}^+ \simeq f^+$ (for $f^+$ see [9]). Therefore, we can rewrite the Green-Ringel isomorphism theorem for Double Ringel-Hall composition algebras (without the condition 'generic').

**Corollary 4.1.** *Keep notations as before, then the map* $\mathcal{D}_c(\Lambda) \longmapsto U_q(\mathfrak{g})$ *defined by* $u_i^+ \longmapsto E_i, u_i^- \longmapsto -v_iF_i, K_i \longmapsto \widetilde{K}_i$ *induces a Hopf algebra isomorphism.*

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## References


[1] Abe, E., Hopf algebras [M], Cambridge Tracts in Math., **74**(1997).

[2] Green, J. A., Hall algebras, hereditary Aagebras and quantum groups [J], *Invent. Math.,* **120**(1995), 361–377.

[3] Lusztig, G., Introduction to quantum groups [M], Progr. Math., **110**(1993).

[4] Gabber & Kac, V. G., On defining relations of certain infinite-dimensional Lie algebras [J], *Bull. Amer. Math. Soc. (N.S.),* **5:**2(1981), 185–189.

[5] Joseph, A., Quantum groups and their primitive ideals [J], *Ergeb. Math. Grenzgeb,* **3**(1995), 29.

[6] Jantzen, J. C., Lectures on quantum groups [M], Amer. Math. Soc, GSM, Vol. 6, 1995.

[7] Kac, V. G., Infinite dimensional Lie algebras [M], Third edition, Cambridge Univ., Press, 1990.

[8] Ringel, C. M., Hall algebras and quantum groups [J], *Invent. Math.,* **101**(1990), 583–592.

[9] Xiao Jie, Drinfeld double and ringel-Green theory of Hall algebras [J], *J. Algebra,* **190**(1997), 100–144.

[10] Xiao Jie & Yang Shilin, BGF-reflection functors and Lusztig's symmetries: A Ringel-Hall algebra approach to quantum groups [J], *J. Algebra,* **241**(2001), 204–246.