Chin. Ann. of Math.
23B:3(2002),361-372.

COMPARISON, SYMMETRY AND MONOTONICITY RESULTS FOR SOME DEGENERATE ELLIPTIC OPERATORS IN CARNOT-CARATHÉODORY SPACES****

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Abstract

This paper studies the properties of solutions of quasilinear equations involving the p-laplacian type operator in general Carnot-Carathéodory spaces. The authors show some comparison results for solutions of the relevant differential inequalities and use them to get some symmetry and monotonicity properties of solutions, in bounded or unbounded domains.

Keywords Carnot-Carathéodory space, Symmetry, Monotonicity, Degenerate elliptic operator

2000 MR Subject Classification 35J60, 35J70

Chinese Library Classification O175.25, O175.29 **Document Code** A

Article ID 0252-9599(2002)03-0361-12

§1. Introduction

Let $X = (X_1, \cdots, X_k)$ be a C^∞ Hörmander field in $\mathbb{R}^n$, namely

$$\mathrm{rankLie}[X_1, \cdots, X_k](x) = n \qquad \text{for any } x \in \mathbb{R}^n. \tag{1.1}$$

Let $\mathcal{H}$ be the space of horizontal curves, i.e. each such curve is a piecewise C^1 function $\gamma : [0, T] \longrightarrow \mathbb{R}^n$ such that, whenever $\gamma'(t)$ exists,

$$\gamma'(t) = \sum_{j=1}^{k} c_j(t) X_j(\gamma(t)) \text{ verifying } \sum_{j=1}^{k} c_j(t)^2 \leq 1. \tag{1.2}$$

Manuscript received October 12, 2001.

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****Project supported by the National Natural Science Foundation of China (No. 10071023).
The third author is also supported in part by MOST and Foundation for University Key Teacher by MEC and Shanghai Priority Academic Discipline.

T is called the horizontal length of γ. For any $x, y \in \mathbb{R}^n$, define

$$d(x,y) = \inf_{\gamma\in\mathcal{H}}\{T;\ \gamma(0)=x, \gamma(T)=y\}. \tag{1.3}$$

It was proved by Chow[4], Nagel, Stein and Wainger[18] that the following basic properties are satisfied:

(H1) $(\mathbb{R}^n, d)$ is a metric space, i.e. $d(x,y) < \infty$ for any $x, y \in \mathbb{R}^n$.

(H2) $\forall\, U$ bounded open set of $\mathbb{R}^n$, there exist C_1 and $R_0 > 0$ such that

$$\text{for any }\ x_0 \in U, R \le R_0, \qquad |B_d(x_0, 2R)| \le C_1|B_d(x_0,R)| \quad \text{(doubling condition)},$$

where $|\cdot|$ denotes Lebesgue's measure on $\mathbb{R}^n$ and B_d denotes the metric ball in $(\mathbb{R}^n, d)$.

Let u be a Lipschitz function. We denote by $|Xu| = \Big[\sum\limits_{1\le j\le k}(X_j u)^2\Big]^{1/2}$ the length of horizontal gradient $Xu = (X_1u, \ldots, X_ku)$, and then we can introduce the corresponding Sobolev spaces. For any U open set of $\mathbb{R}^n$ and $1 \le p < \infty$, define

$$\|u\|_{W^{1,p}} = \Big(\int_U |Xu|^p + |u|^p dx\Big)^{1/p} \tag{1.4}$$

and define $W_0^{1,p}(U)$ (resp. $W^{1,p}(U)$) as the completion of $C_0^\infty(U)$ (resp. that of $\{u \in C^\infty(U), \|u\|_{W^{1,p}} < \infty\}$) under the norm $\|\cdot\|_{W^{1,p}}$.

Clearly, all the definitions (1.2), (1.3) and (1.4) can be generalized for any real valued, locally Lipschitz vector fields

$$X_j = \sum_{l=1}^n a_{jl}\frac{\partial}{\partial x_l} \quad (1 \le j \le k), \tag{1.5}$$

and for any $u \in W^{1,p}(U)$, X_ju is understood in the sense of distribution

$$\langle X_j u, \varphi\rangle = \int_U u X_j^* \varphi dx, \qquad \forall\, \varphi \in C_0^\infty(U), \tag{1.6}$$

where $X_j^* = -\sum\limits_{1\le l\le n}\partial_l(a_{jl}\cdot)$ denotes the formal adjoint of X_j. Throughout the paper we assume that (H1) and (H2) hold always true and we call $(\mathbb{R}^n, d)$ the Carnot-Carathéodory metric space associated to $(X_j)_{1\le j\le k}$.

Many works have been done to understand the density of regular functions, Sobolev embedding properties or isoperimetric inequalities in Carnot-Carathéodory spaces (see [13, 16] and references therein). Assume that

(H) $X = (X_1, \cdots, X_k)$ is a system of vector fields given by (1.5) which satisfies (H1), (H2) and there exist $C_2, R_0 > 0$ and $\alpha \ge 1$ such that for any $x_0 \in U$, $R \le R_0$ and any Lipschitz function u defined on $B_d(x_0, \alpha R)$, we have

$$|\{x \in B_d(x_0,R);\ |u(x) - \bar u_B| > \lambda\}| \le \frac{C_2 R}{\lambda}\int_{B_d(x_0,\alpha R)} |Xu| dx \quad \text{ for all }\ \lambda > 0, \tag{1.7}$$

where $\bar u_B$ stands for the average of u over $B_d(x_0, R)$. Moreover, $(\mathbb{R}^n, d)$ is complete homeomorphic to $(\mathbb{R}^n, |\cdot|)$, i.e. $(\mathbb{R}^n, d)$ defines the usual topology in $\mathbb{R}^n$.

Under such assumptions, the following generalized Sobolev embedding inequalities were proved in [13].

Lemma 1.1. (see [13] and [11]). *Given any bounded open set $U \subset \mathbb{R}^n$, let $Q = \log_2 C_1$ (C_1 is the constant in the doubling condition* (H2)*), called the local homogeneous dimension relative to U. We suppose that $Q \ge 2$. Then there exist R_1, $C > 0$ such that for any metric* *ball* $B = B_d(x_0, R)$ *with* $x_0 \in U$ *and* $R \le R_1$, *one has* $\forall\, 1 \le p < Q$, $1 \le k < Q/(Q-p)$ *and* $u \in W^{1,p}(B)$

$$\Big(\frac{1}{|B|}\int_B |u-\bar u_B|^{kp}dx\Big)^{1/kp} \le CR\Big(\frac{1}{|B|}\int_B |Xu|^p dx\Big)^{1/p}, \tag{1.8}$$

and for any $u \in W_0^{1,p}(B)$

$$\Big(\frac{1}{|B|}\int_B |u|^{kp}dx\Big)^{1/kp} \le C\Big(\frac{1}{|B|}\int_B |Xu|^p dx\Big)^{1/p}. \tag{1.9}$$

If $p \ge Q$, *then for any* $u \in W^{1,p}(B)$ *and* $1 \le q < \infty$, *one has*

$$\Big(\frac{1}{|B|}\int_B |u-\bar u_B|^{q}dx\Big)^{1/q} \le CR\Big(\frac{1}{|B|}\int_B |Xu|^p dx\Big)^{1/p}. \tag{1.10}$$

These inequalities play an important role for analysis in Carnot-Carathéodory spaces. We know that (H) is satisfied for a very large class of spaces and a lot of interesting examples have been given in [13, 16] (see also [11]) and the references therein.

Here, we study the properties of solutions of quasilinear equations involving the p-laplacian type operator in general Carnot-Carathéodory spaces. More precisely, we consider the differential operator L as follows:

$$Lu = \sum_{j=1}^{k} X_j^* A_j(x, Xu), \tag{1.11}$$

where $A(x,\eta) = (A_1(x,\eta), \cdots, A_k(x,\eta))$ satisfies the following assumptions:

(H3) $A \in C^0(\mathbb{R}^n \times \mathbb{R}^k; \mathbb{R}^k) \cap C^1(\mathbb{R}^n \times (\mathbb{R}^k \setminus \{0\}); \mathbb{R}^k)$ and $A(x,0) = 0$ for any $x \in \mathbb{R}^n$.

(H4) For any bounded domain U, there exist suitable constants $\Gamma, \gamma > 0$ such that

$$\sum_{i,j=1}^{k} \Big|\frac{\partial A_i}{\partial \eta_j}(x,\eta)\Big| \le \Gamma\|\eta\|^{p-2} \quad \text{for any } x \in \bar U, \eta \in \mathbb{R}^k \setminus \{0\}, \tag{1.12}$$

$$\sum_{i,j=1}^{k} \frac{\partial A_i}{\partial \eta_j}(x,\eta)\xi_i\xi_j \ge \gamma\|\eta\|^{p-2}\|\xi\|^2 \quad \text{for any } x \in \bar U, \eta \in \mathbb{R}^k \setminus \{0\}, \xi \in \mathbb{R}^k \tag{1.13}$$

with a fixed constant $p \in (1, \infty)$.

In this work, we establish several comparison results, then associating with some generalizations of "moving plane method" developped in [11] and [1] (see also [3, 5] and [14]), we will prove some symmetry or monotonicity properties.

A major difficulty of studing solutions of p-laplacian type equations lies essentially in the degeneracy of the operator L. On one hand, this degeneracy comes from the structure of the operator if $p \ne 2$; on the other hand, it is caused by the degeneracy of vector fields X, that is, $\{X_1(x), \cdots, X_k(x)\}$ does not span $\mathbb{R}^n$ in general.

Our paper is organized as follows. In Section 2, we prove various comparison results, in particular we show a Harnack type inequality. Symmetry and monotonicity properties for bounded domain case are established in Section 3. The last section is devoted to a symmetry result for ground state solution in the whole space $\mathbb{R}^n$. In all this paper, C denotes generic positive constant independent of u, even if its value could be changed from one line to another one.

§2. Comparison Principles and Harnack Inequality

In this section, we prove some comparison results for quasilinear degenerate elliptic operators defined above in Carnot-Carathéodory spaces. In particular, a Harnack type inequality is established. We begin with the following weak comparison principles whose proof is analogous to those of [5] for Euclidean spaces case (with general $p > 1$) and [11] for Carnot-Carathéodory spaces case with $p = 2$.

2.1 Weak Comparison Principles

Theorem 2.1. *Let $\Omega \subset U$ be bounded open sets in $\mathbb{R}^n$ and u, v be in $W^{1,\infty}(\Omega)$ satisfying*

$$Lu + g(x,u) - \Lambda u \le Lv + g(x,v) - \Lambda v \qquad in \quad \Omega$$

with $\Lambda \ge 0$ and $g(x,s) \in C^0(\bar{\Omega} \times \mathbb{R})$ is nondecreasing in s for $|s| \le \max(\|u\|_\infty, \|v\|_\infty)$, for any $x \in \Omega$. Let $\Omega' \subseteq \Omega$ be open and suppose that $u \le v$ on $\partial\Omega'$. Then

(a) *if $\Lambda = 0$, $u \le v$ in $\Omega', \forall\, p > 1$;*

(b) *if $\Lambda > 0$ and $1 < p \le 2$, there exists a constant $\delta_1 > 0$, depending on $p, \Lambda, \gamma, \Gamma, |\Omega|$ and M_Ω, such that if $|\Omega'| \le \delta_1$, then $u \le v$ in Ω';*

(c) *if $\Lambda > 0$, $m_\Omega > 0$ and $p > 2$, there exists a constant $\delta_2 > 0$, depending on p, Λ, γ, Γ, $|\Omega|$ and m_Ω, such that if $|\Omega'| \le \delta_2$, then $u \le v$ in Ω',*

where we define $M_{\mathcal{S}} = \sup\limits_{\mathcal{S}}(|Xu| + |Xv|)$ and $m_{\mathcal{S}} = \inf\limits_{\mathcal{S}}(|Xu| + |Xv|)$ for any subset $\mathcal{S}$ of Ω.

To prove this theorem, we need the following basic estimates for $A(x,\eta)$ satisfying conditions (H3) and (H4), the proof of which is straightforward[5].

Lemma 2.1. *There exist constants C_3, C_4, depending on p, γ and Γ such that for any $\eta, \eta' \in \mathbb{R}^n$ with $|\eta| + |\eta'| > 0$ and any $x \in \Omega$,*

$$|A(x,\eta) - A(x,\eta')| \le C_3(|\eta| + |\eta'|)^{p-2}|\eta - \eta'|, \tag{2.1}$$

$$\langle A(x,\eta) - A(x,\eta'), \eta - \eta'\rangle \ge C_4(|\eta| + |\eta'|)^{p-2}|\eta - \eta'|^2. \tag{2.2}$$

Consequently, we have

$$|A(x,\eta) - A(x,\eta')| \le C_3|\eta - \eta'|^{p-1} \qquad \text{if } 1 < p \le 2, \tag{2.3}$$

$$\langle A(x,\eta) - A(x,\eta'), \eta - \eta'\rangle \ge C_4|\eta - \eta'|^p \qquad \text{if } p \ge 2. \tag{2.4}$$

Proof of Theorem 2.1. Since $u, v \in W^{1,\infty}(\Omega)$ and $u \le v$ on $\partial\Omega'$, we have $(u - v)^+ \in W_0^{1,p}(\Omega')$ for any $p > 1$. Taking $(u - v)^+$ as test function and using the fact that $[g(x,u) - g(x,v)](u - v)^+ \ge 0$ in Ω, we get

$$\int_{\Omega'} \langle A(x,Xu) - A(x,Xv), X(u-v)^+\rangle dx \le \Lambda \int_{\Omega'} |(u-v)^+|^2 dx.$$

According to (2.2), we have

$$C_4 \int_{\Omega' \cap \{u \ge v,\ |Xu| + |Xv| \ne 0\}} (|Xu| + |Xv|)^{p-2}|X(u-v)|^2 dx \le \Lambda \int_{\Omega'} |(u-v)^+|^2 dx.$$

When $\Lambda = 0$, we conclude $X(u - v)^+ = 0$ a.e. in Ω', thus (a) is proved.

Furthermore, using the covering argument as in the proof of Theorem 1 in [11], we can prove that there exist constants $C > 0, \rho > 2$ such that for any open subset $\Omega' \subseteq \Omega$ with $|\Omega'| < C$ and any $u \in W_0^{1,2}(\Omega')$, the following inequality holds:

$$\int_{\Omega'} |u|^2 dx \le C|\Omega'|^{1-\frac{2}{\rho}} \int_{\Omega'} |Xu|^2 dx.$$

Therefore, if $\Lambda > 0$ and $1 < p \le 2$, we get

$$C_4 M_\Omega^{p-2} \int_{\Omega'} |X(u-v)^+|^2 dx \le \Lambda C|\Omega'|^{1-\frac{2}{\rho}} \int_{\Omega'} |X(u-v)^+|^2 dx.$$

So if $|\Omega'|$ is sufficiently small, we have $||X(u-v)^+||_{L^2(\Omega')} = 0$ which implies again $(u-v)^+ = 0$ in Ω' and then (b) is done. Finally, the case (c) is proved analogically with M_Ω substituted by m_Ω.

The following is another version of weak comparison result which will be used later. From now on, we assume that a Sobolev inequality with injection in L^2 holds for U, i.e. there exist constants $\bar{C} > 0$ and $q \in (1, 2)$ depending only on U such that

$$||w||_{L^2(U)} \leq \bar{C}||Xw||_{L^q(U)}, \quad \forall\, w \in W_0^{1,q}(U). \tag{2.5}$$

Theorem 2.2. *Let L, g, U, Ω, u, v be as in Theorem 2.1 and (2.5) holds for U. Then for any $1 < p < 2$, there exist $\delta, M > 0$ depending on $p, \Lambda, \gamma, \Gamma, q, |\Omega|$ and M_Ω such that: if $\Omega' = \Sigma_1 \cup \Sigma_2 \subseteq \Omega$ is an open subset with $|\Sigma_1 \cap \Sigma_2| = 0, |\Sigma_1| < \delta$, and $M_{\Sigma_2} < M$, then $u \leq v$ on $\partial\Omega'$ implies $u \leq v$ in Ω'.*

Proof. Taking $(u-v)^+ \in W_0^{1,p}(\Omega')$ as test function and using the fact that $[g(x,u) - g(x,v)](u-v)^+ \geq 0$ in Ω, we get

$$\int_{\Omega'} \langle A(x, Xu) - A(x, Xv), X(u-v)^+\rangle dx \leq \Lambda \int_{\Omega'} |(u-v)^+|^2 dx.$$

Let $1 < p < 2$ and $\Omega' = \Sigma_1 \cup \Sigma_2$ with $|\Sigma_1 \cap \Sigma_2| = 0$. According to Lemma 2.1,

$$\text{l.h.s.} \ \geq C_4 M_\Omega^{p-2} \int_{\Sigma_1} |X(u-v)^+|^2 dx + C_4 M_{\Sigma_2}^{p-2} \int_{\Sigma_2} |X(u-v)^+|^2 dx.$$

Extending $(u-v)^+$ by 0 outside Ω' and using Sobolev inequality in U, we have

$$\begin{aligned} \text{r.h.s.} \ &\leq \Lambda\bar{C}\Big(\int_{\Omega'} |X(u-v)^+|^q dx\Big)^{\frac{2}{q}} \\ &\leq 2\Lambda\bar{C}\Big(||X(u-v)^+||^2_{L^q(\Sigma_1)} + ||X(u-v)^+||^2_{L^q(\Sigma_2)}\Big) \\ &\leq 2\Lambda\bar{C}\Big(|\Sigma_1|^{\frac{2}{q}-1}\int_{\Sigma_1} |X(u-v)^+|^2 dx + |\Omega'|^{\frac{2}{q}-1}\int_{\Sigma_2} |X(u-v)^+|^2 dx\Big). \end{aligned}$$

Therefore we deduce that if $|\Sigma_1|$ and M_{Σ_2} are small enough, we must have $X(u-v)^+ = 0$ in Ω', which gives the theorem.

Consequently for $p \in (1, 2)$ there exists $M > 0$ such that for any open set $\Omega' \subseteq \Omega$, the inequality $u \leq v$ on $\partial\Omega'$ implies $u \leq v$ in Ω' provided $M_{\Omega'} < M$ (with $\Sigma_1 = \emptyset$). We note that the statement holds true without any assumption on the size of Ω', which is the main difference with respect to Theorem 2.1. In general this is not true even for $p = 2$.

2.2. Harnack Inequality and Strong Comparison Principle

We now state a Harnack type comparison result.

Theorem 2.3. *Let Ω be a bounded open set in $\mathbb{R}^n$ and u, v be in $W^{1,\infty}_{\rm loc}(\Omega)$ satisfying*

$$\begin{cases} Lu + \Lambda u \leq Lv + \Lambda v & \text{in } \Omega, \\ u \leq v & \text{in } \Omega, \end{cases} \tag{2.6}$$

for some constant $\Lambda \in \mathbb{R}$. There exists $r_0 > 0$ such that if $U = \overline{B_d(x_0, 4\delta)} \subset \Omega$ with $\delta \in (0, R_1/4)$ and if $m_U > 0$, then there exists $C > 0$ depending on δ, m_U and M_U such that

$$\|v - u\|_{L^{r_0}(B_d(x_0,2\delta))} \leq C_\delta \inf_{B_d(x_0,\delta)} (v - u).$$

Proof. The proof of this theorem is rather technical and similar to that in [11], so we just write the beginning of the proof and leave other details for readers.

We can always suppose that $\Lambda \geq 0$, (if not, we can take $\Lambda = 0$ since $u \leq v$). We replace also v by $v + \tau$ ($\tau > 0$) and the result is proved by taking $\tau \to 0$. Now $v - u \geq \tau > 0$.

Denoting by B_r the metric ball $B_d(x_0,r)$, we choose a cut-off function ξ with support in $B_{4\delta}$ and take $\varphi=(v-u)^\beta\xi^2$ with $\beta<0$ as test function. Therefore

$$\begin{aligned}&|\beta|\int_{B_{4\delta}}\xi^2(v-u)^{\beta-1}\langle A(x,Xu)-A(x,Xv),Xu-Xv\rangle dx\\&+\int_{B_{4\delta}}2\xi(v-u)^\beta\langle A(x,Xu)-A(x,Xv),X\xi\rangle dx\\&\le\Lambda\int_{B_{4\delta}}(v-u)^{\beta+1}\xi^2dx.\end{aligned}$$

Applying Lemma 2.1 for $p\le 2$, we have

$$\begin{aligned}&|\beta|M_U^{p-2}\int_{B_{4\delta}}\xi^2(v-u)^{\beta-1}|Xu-Xv|^2dx\\&\le Cm_U^{p-2}\int_{B_{4\delta}}\xi|X\xi|(v-u)^\beta|X(v-u)|dx+C\Lambda\int_{B_{4\delta}}(v-u)^{\beta+1}\xi^2dx.\end{aligned}$$

Using Cauchy-Schwarz inequality, we get

$$\int_{B_{4\delta}}\xi^2(v-u)^{\beta-1}|X(u-v)|^2dx\le C\Big(1+\frac{1}{|\beta|^2}\Big)\int_{B_{4\delta}}(\xi^2+|X\xi|^2)(v-u)^{\beta+1}dx,$$

where C depends on the constants m_U and M_U. In the $p>2$ case, it suffices to interchange the roles of m_U and M_U for getting the same type inequality. The remainer of the proof is just the standard Moser's iterative technique as in [11] or [5].

We discuss now several useful consequences. Assume that u, v are two C^1 functions verifying $Lu+\Lambda u\le Lv+\Lambda v$ in Ω with $\Lambda\le 0$ and $u\le v$ on $\partial\Omega$. Define $Z=\{x\in\Omega:|Xu|+|Xv|=0\}$. Then if there exists $x_0\in\Omega\setminus Z$ such that $u(x_0)=v(x_0)$, we have $u\equiv v$ in the connected component of $\Omega\setminus Z$ containing x_0. Moreover, in the following three situations, we have $u<v$ in Ω (not only in $\Omega\setminus Z$!) unless $u\equiv v$ in Ω,

(1) when Ω is connected and Z is discrete;

(2) when Z is compact and $\Omega\setminus Z$ is connected;

(3) when $v=0$, Z is compact and $|Z|=0$.

Indeed, denoting $S=\{x\in\Omega:\ u(x)=v(x)\}$, we see that if $u\not\equiv v$, $S\subset Z$ since $\Omega\setminus Z$ is connected in cases (1), (2); in case (3), $u<0$ in $\Omega\setminus Z$ since each connected component is open, so S has nonzero measure. In case (1), taking any $x_0\in S$ and r small enough such that $U=B_d(x_0,r)$ verifies $\overline{U}\cap S=\{x_0\}$. In case (2) or (3), we choose a neighborhood U of Z such that $Z\subset U\subset\overline{U}\subset\Omega$. In each case, we have $u<v$ in ∂U and $Lu\le Lv$ in U. Let $\epsilon>0$ satisfy $v-u\ge\epsilon$ on ∂U. Applying Theorem 2.1 for v and $u+\epsilon$, we get $v\ge u+\epsilon$ in Z, thus $u<v$ in Ω.

§3. Symmetry and Monotonicity Results for Bounded Domain

In this section, we will apply the above comparison results to obtain some symmetry and monotonicity properties. For this purpose, we consider $(\mathbb{R}^n,d)$ a Carnot-Carathéodory space generated by a system X satisfying (H), let $\Omega\subset U$ be the bounded domains in $\mathbb{R}^n$ verifying (2.5) and L be the p-laplacian type operator verifying (H3) and (H4).

3.1. Symmetry Results

We study the degenerate differential operator L in the form of (1.11) and consider the associated isometry group G of L. More precisely, $g\in G$ if and only if $g:\mathbb{R}^n\to\mathbb{R}^n$ is a C^1 diffeomorphism such that $g^*L=L$. Now we assume the following conditions:

(i) Isometries: There is a family of isometries $I_t \in G$ which is C^1 in t, such that $\forall\, t \in (0,2)$, there exists a family of hypersurfaces $U_t \subset \mathbb{R}^n$ such that $I_t(x) = x \Leftrightarrow x \in U_t$, i.e. U_t is the invariant hypersurface under the action of I_t.

(ii) Domain decomposition: There exist pairwise disjoint sets V_t, such that

(a) $V_t \subset U_t$, for all $t \in (0,2)$.

(b) For all $t_1, t_2 \in [0,2]$, $\bigcup\limits_{t_1<t<t_2} V_t$ is an open subset of $\mathbb{R}^n$ and $\Omega = \bigcup\limits_{0<t<2} V_t$.

(iii) Inclusion in increasing t: Let $Q_{t_1} = \bigcup\limits_{0<t<t_1} V_t$ and $Q^{t_1} = \bigcup\limits_{t_1<t<2} V_t$, then

(a) $I_t(Q_t) \subset Q^t$, for all $t \in (0,1)$.

(b) For all $t \in (0,1)$ and for every connected component Σ of Q_t, there exists a point $x \in \partial\Sigma \cap \partial\Omega : I_t(x) \in Q^t$.

Moreover, we say that Ω is symmetric if we also have

(iv) Inclusion in decreasing t:

(a) $I_t(Q^t) \subset Q_t$, for all $t \in (1,2)$ and $Q_1 = I_1(Q^1)$.

(b) For all $t \in (1,2)$ and for every connected component Σ of Q^t, there exists a point $x \in \partial\Sigma \cap \partial\Omega : I_t(x) \in Q_t$.

For $0 < t \le 1$ and $x \in Q_t$ we define $x^t = I_t(x)$ and $u_t(x) = u \circ I_t(x) = u(x^t)$.

Theorem 3.1. *Let $1 < p < 2$. Let Ω satisfy conditions* (i), (ii), (iii) *and* (2.5)*, f be a locally Lipschitz function and $u \in C^1(\bar{\Omega})$ be a weak solution of*

$$\begin{cases} Lu = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Assume that for any $\lambda < 1$ and any connected component C_λ of Q_λ, $C_\lambda \setminus Z_\lambda$ is connected where $Z_\lambda = \{x \in Q_\lambda, Xu(x) = Xu_\lambda(x) = 0\}$. Then

$$u(x) \le u(x^\lambda), \quad \text{if} \quad x \in Q_\lambda, \quad 0 < \lambda < 1. \tag{3.2}$$

If $I_1(Q_1) \subset Q^1$, we still have the inequality $u(x^1) \ge u(x)$, $\forall\, x \in Q_1$. Moreover, if Ω is symmetric and the analogous condition holds for any $C^\lambda \setminus Z^\lambda$ with $\lambda > 1$, then $u(x) = u(x^1)$ in Q_1.

Proof. Since u is bounded, we can find $\Lambda > 0$ such that $g_\pm(x) = \Lambda x \pm f(x)$ are nondecreasing in $[0, \|u\|_\infty]$. For $\lambda < 1$, functions u, u_λ satisfy the equation $Lz - \Lambda z + g_-(z) = 0$ in Q_λ and $u \le u_\lambda$ on $\partial Q_\lambda \cap \partial\Omega$, $u = u_\lambda$ on $\partial Q_\lambda \cap V_\lambda$. By Theorem 2.1, since $M_\Omega < \infty$, there exists a δ such that if $|Q_\lambda| \le \delta$, then

$$u \le u_\lambda \quad \text{in } Q_\lambda. \tag{3.3}$$

Thus (3.3) holds for $\lambda > 0$ small enough.

Now we look at $\lambda_0 = \sup\{\lambda > 0, \text{ s.t. } u \le u_\mu, \forall\, \mu \in [0,\lambda]\}$, we shall prove that $\lambda_0 = 1$. If it is false, then $\lambda_0 < 1$ and $u \le u_{\lambda_0}$ in Q_{λ_0} by continuity. We have also $Lu + \Lambda u = g_+(u) \le g_+(u_{\lambda_0}) = Lu_{\lambda_0} + \Lambda u_{\lambda_0}$ in Q_{λ_0}. For any connected component C_{λ_0} of Q_{λ_0}, Theorem 2.3 implies that $u < u_{\lambda_0}$ in $C_{\lambda_0} \setminus Z_{\lambda_0}$, unless $u \equiv u_{\lambda_0}$ in $C_{\lambda_0} \setminus Z_{\lambda_0}$, since the set $C_{\lambda_0} \setminus Z_{\lambda_0}$ is connected. In the second case, the C^1 continuity of $u - u_{\lambda_0}$ will deduce that $u \equiv u_{\lambda_0}$ in whole C_{λ_0}. However, by the assumption on Ω, we get $x \in \partial\Omega \cap C_{\lambda_0}$ such that $x_{\lambda_0} \in \Omega$, thus $0 = u(x) < u(x_{\lambda_0})$, $u \not\equiv u_{\lambda_0}$ in C_{λ_0}. Thus $u < u_{\lambda_0}$ in $C_{\lambda_0} \setminus Z_{\lambda_0}$ for each connected component C_{λ_0}, thus $u < u_{\lambda_0}$ in $Q_{\lambda_0} \setminus Z_{\lambda_0}$.

Let $S = \{x \in Q_{\lambda_0}, u(x) = u_{\lambda_0}(x)\} \subset Z_{\lambda_0}$. Since $Xu = Xu_{\lambda_0} = 0$ in S, there exists an open set O verifying $S \subset O \subset Q_{\lambda_0}$ and $M_{O,\lambda_0} < M/2$, where $M_{O,\lambda} = \sup\limits_O(|Xu| + |Xu_\lambda|)$

and δ, M are constants dertermined by Theorem 2.2. Take now a compact set $K \subset Q_{\lambda_0}$ with $|Q_{\lambda_0} \setminus K| < \delta/2$. Then in the compact subset $K \setminus O$, there exists $\epsilon > 0$ such that $u_{\lambda_0} - u \geq \epsilon$. Applying again the C^1-continuity of u, we get $\eta > 0$ such that $\lambda_0 + \eta < 1$, $u_\lambda - u > 0$ in $K \setminus O$ and $M_{O,\lambda} < M$ for any $\lambda \in [\lambda_0, \lambda_0 + \eta]$. We can also assume that $|Q_{\lambda_0+\eta} \setminus K| < \delta$.

Moreover, for such λ, if $\Sigma_\lambda = (Q_\lambda \setminus (K \setminus O))$, $\partial\Sigma_\lambda$ is included in $\partial Q_\lambda \cup \partial(K \setminus O)$, we obtain then $u \leq u_\lambda$ in $\partial\Sigma_\lambda$. Since $\Sigma_\lambda = \Sigma_1 \cup \Sigma_2$ with disjointed sets $\Sigma_1 = Q_\lambda \setminus K$ and $\Sigma_2 = K \cap O$, using Theorem 2.2 on Σ_λ, we get $u \leq u_\lambda$ in Σ_λ. Recall that $u \leq u_\lambda$ in $K \setminus O$, so $u \leq u_\lambda$ in Q_λ for any $\lambda \leq \lambda_0 + \eta$; this contradicts the definition of λ_0. Hence, (3.2) is proved. Other results are direct consequences of (3.2).

In the same spirit, we can state

Theorem 3.2. *Let Ω satisfy the conditions* (i), (ii), (iii), *f be a locally Lipschitz function and $u \in C^1(\bar{\Omega})$ be a weak solution to* (3.1). *Assume that conditions* (H) *to* (H4) *are satisfied with $1 < p < \infty$. If $Z = \{x \in \Omega, Xu(x) = 0\}$ is a subset of V_1, then $u(x) \leq u(x^\lambda)$, for any $x \in Q_\lambda$, $\lambda \in (0,1)$. Moreover, if Ω is symmetric, then $u(x) = u(x^1)$ in Q_1.*

3.2. Monotonicity Result

Now we state a monotonicity theorem of the same type as Berestycki and Nirenberg's results in [3]. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain such that $|\partial\Omega| = 0$ and Ω can be decomposed as the union of a family of hypersurfaces as follows: there exists a hypersurface without boundary $V \subset \mathbb{R}^n$ such that $\overline{\Omega} = \bigcup_{t\in[0,1]} A_t(V_t)$ where $V_t \subset V$ are compact hypersurfaces with boundary for $0 < t < 1$, C^1 depending on t, and A_t $(t \in \mathbb{R})$ is a C^1 one parameter subgroup of the associated "isometry" group G of L, which is transversal to V at $t = 0$. Furthermore, we assume the following conditions.

Foliation conditions:

$$\frac{\partial A_t(x)}{\partial t}\Big|_{t=0} \oplus T_x V = T_x\mathbb{R}^n, \quad \forall\, x \in V, \tag{3.4}$$

$$\exists\, \varepsilon > 0 \text{ such that } \forall\, x \in V,\ \forall\, t \in (0, 1+\varepsilon), \quad A_t(x) \notin V, \tag{3.5}$$

Directional convexity conditions:

$$\begin{cases} \forall\, x \in V, \quad \{t \in [0,1], A_t(x) \in \overline{\Omega}\} \text{ is a closed interval denoted by } [s_1(x), s_2(x)], \\ \forall\, x \in V, \quad \{t \in (0,1), A_t(x) \in \Omega\} \text{ is an open interval denoted by } (\tau_1(x), \tau_2(x)). \end{cases} \tag{3.6}$$

Let u be a $C^1(\bar{\Omega})$ solution of

$$\begin{cases} Lu = f(u) & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Suppose that ϕ is a strictly increasing function of t along the orbits of A_t and u is such that along an orbit, it takes its values between those of ϕ at the points where the orbit crosses the boundary. More precisely, for all boundary points $x_b, x_e \in \partial\Omega$, which are on the same orbit of the group action, i.e. $\exists\, t(x_b, x_e) > 0$ such that $x_e = A_t(x_b)$, then

$$\forall\, \tau \in (0, t(x_b, x_e)), \quad \phi(x_b) < u(A_\tau(x_b)) < \phi(x_e). \tag{3.8}$$

Our result reads as follows.

Theorem 3.3. *Let $1 < p < 2$. Suppose that f is a locally Lipschitz function and the assumptions* (3.4)–(3.6) *are satisfied. Let u be a C^1 solution of* (3.7) *and* (3.8). *Denote $D_t = \Omega \cap A_t(\Omega)$ and $Z_t = \{x \in D_t, |Xu| + |X(u \circ A_t)| = 0\}$ for $t \in (0,1)$. If for any $t \in (0,1)$ and any connected component C_t of D_t, the set $C_t \setminus Z_t$ is connected, then the* *function u is nondecreasing along the orbits of the group action, i.e.*

$$\forall\, x \in \overline{\Omega},\ \forall\, t > 0 \ \text{s.t.}\ A_t(x) \in \overline{\Omega},\ \text{we have}\ u(A_t(x)) \geq u(x). \tag{3.9}$$

Proof. Denote $u^\tau = u \circ A_\tau$ and $\tau_1 = \sup\{\tau > 0 \text{ s.t. } D_\tau \neq \emptyset\}$. First, we see that $Lu^\tau = f(u^\tau)$ for all $\tau \in (0,1)$ and $u < u^\tau$ on ∂D_τ. Then working with $g_\pm(u) = \Lambda u \pm f(u)$, by the same argument as in the proof of Theorem 3.1, we have $u \leq u^\tau$ in D_τ for $\tau < \tau_1$ and close to τ_1, since $|D_\tau| < \delta$. Note $\tau_0 = \inf\{\tau > 0 \text{ s.t. } u \leq u^\mu, \forall\, \mu \in (\tau, \tau_1)\}$, the conclusion of this theorem is then equivalent to saying that $\tau_0 = 0$.

If it is not true, by the continuity of u, $u \leq u^{\tau_0}$ in D_{τ_0}. Since $u < u^{\tau_0}$ on ∂D_{τ_0}, Theorem 2.3 implies that $u < u^{\tau_0}$ in $D_{\tau_0} \setminus Z_{\tau_0}$. On the other hand, applying Theorem 2.2, we see that there exists $M > 0$ such that for any $\Sigma \subset D_\tau$ verifying $M_\Sigma < M$ (with $\Sigma_1 = \emptyset$), then $u \leq u^\tau$ in Σ provided $u \leq u^\tau$ on $\partial\Sigma$.

We choose an open subset Σ of D_{τ_0} such that $S = \{x \in D_{\tau_0}, u(x) = u^{\tau_0}(x)\} \subset \Sigma$ and $M_{\Sigma,\tau_0} \leq M/2$, this is possible because $S \subset Z_{\tau_0}$. By the C^1-continuity of u, we know that $M_{\Sigma,\tau} < M$ for τ close to τ_0. Moreover, $K = \bar{D}_{\tau_0} \setminus \Sigma$ is compact, then there exists $\epsilon > 0$ such that $u \leq u^{\tau_0} - \epsilon$ in K. Using again the continuity of u, $u \leq u^\tau$ on $\bar{D}_\tau \setminus \Sigma$ for τ close to τ_0. Thus $u \leq u^\tau$ on $\partial\Sigma$ for such τ, so $u \leq u^\tau$ in Σ by previous remark, which implies that $u \leq u^\tau$ in D_τ for τ small but close to τ_0. This contradicts the choice of τ_0 and completes the proof.

§4. Symmetry Result for Unbounded Domains

Let $X = \{X_1, \cdots, X_k\}$ be a system of C^∞ vector fields of Hörmander type. First, we remark that (H) is verified (see [9, 18, 17] and [19]). Suppose that the following basic peoperty holds:

(P) $\forall\, x \in \mathbb{R}^n$, $\forall\, R > 0$, $B = B_d(x, R) = \{y \in \mathbb{R}^n, d(y,x) < R\}$ is relatively compact and there exist positive constants $1 < q < 2$ and $C > 0$ depending only on R such that for any $u \in W^{1,p}(B)$, we have

$$\Big(\int_B |u - \bar{u}_B|^2 dx\Big)^{1/2} \leq C\Big(\int_B |Xu|^q dx\Big)^{1/q}. \tag{4.1}$$

Moreover, set $E(x) = \mathrm{Vect}\{X_j(x), 1 \leq j \leq k\}$ and

$$\begin{aligned} G_1 = \{g \in G,\ \text{s.t.}\ g^*(dx) = dx, d(g(x), g(y)) = d(x,y),\ \forall\, x, y \in \mathbb{R}^n,\ g^*(E) = E \\ \text{and}\ \exists\, c_g \geq 1 \ \text{s.t.}\ \forall\, x \in \mathbb{R}^n, Y \in E(x), \|Y\|/c_g \leq \|g^*(Y)\| \leq c_g \|Y\|\}. \end{aligned}$$

We assume (i), (ii) by replacing $t \in (0,2)$ by $t \in \mathbb{R}$ and $I_t \in G_1$ such that $\mathbb{R}^n = \bigcup\limits_{t\in\mathbb{R}} V_t$ with $V_t = U_t$ connected and $\lim\limits_{t\to\infty} \Big(\inf\limits_{x \in U_t} |x|\Big) = +\infty$. Furthermore, for any $t_1 \in \mathbb{R}$, assume that $Q_{t_1} = \bigcup\limits_{t<t_1} V_t$, $Q^{t_1} = \bigcup\limits_{t>t_1} V_t$ are connected open sets in $\mathbb{R}^n$ and $I_t(Q_t) = Q^t$ for any $t \in \mathbb{R}$. We consider positive solutions of the following equation in $\mathbb{R}^n$, with the ground state condition at infinity, namely

$$\begin{cases} Lu = f(u) & \text{in } \mathbb{R}^n, \\ u > 0 & \text{in } \mathbb{R}^n, \\ u(x) \to 0, & \text{as } |x| \to \infty. \end{cases} \tag{4.2}$$

Theorem 4.1. *Under the above assumptions, let $1 < p < 2$. Assume that* (H), (H3) *and* (H4) *are verified. Let u be a C^1 solution of* (4.2) *where f is locally Lipschitz on* $(0, +\infty)$ *and there exists $s_0 > 0$ such that f is nonincreasing on $(0, s_0)$. Assume that for any $t \in \mathbb{R}$,*

$Q_t \setminus Z_t$ is connected with $Z_t = \{x \in Q_t, |Xu(x)| + |X(u \circ I_t)(x)| = 0\}$. Then there exists $t_0 \in \mathbb{R}$, such that $u \circ I_{t_0} = u$.

Proof. The proof is divided in several steps.

Step 1. Set $\Gamma = \{t \in \mathbb{R}, u \leq u_t = u \circ I_t \text{ in } Q_t\}$. We claim that $\exists\, t_1 \in \mathbb{R}$, such that $(-\infty, t_1] \subset \Gamma$.

Indeed, by our assumption, there exists $t_1 \in \mathbb{R}$ such that we have $0 < u(x) < s_0/2$ in Q_{t_1}. Recall that $I_t \in G_1$, thus u_t satisfies the equation $Lu_t = f(u_t)$ in Q_t for any $t \in \mathbb{R}$. For any $t \leq t_1$, we fix $0 < \epsilon < s_0/2$ and consider the test function $(u - u_t - \epsilon)^+$ in Q_t. Using again the ground state assumption, we can see that $(u - u_t - \epsilon)^+$ has compact support, which implies

$$\int_{Q_t} (f(u) - f(u_t))(u - u_t - \epsilon)^+ dx = \int_{Q_t} (Lu - Lu_t)(u - u_t - \epsilon)^+ dx. \tag{4.3}$$

Remark that if $x \in \operatorname{supp}(u - u_t - \epsilon)^+ \cap Q_t$, then $(f(u) - f(u_t))(u - u_t - \epsilon)^+ \leq 0$, since f is nonincreasing on $(0, s_0)$. On the other hand, we deduce from Lemma 2.1,

$$\int_{Q_t} (Lu - Lu_t)(u - u_t - \epsilon)^+ dx \geq C_4 \int_{Q_t} (|Xu| + |Xu_t|)^{p-2} |X(u - u_t - \epsilon)^+|^2 dx \geq 0.$$

Therefore, $X(u - u_t - \epsilon)^+ = 0$ a.e. in Q_t and $u(x) \leq u_t(x) + \epsilon$ for any $x \in Q_t$. We get our claim by passing $\epsilon \to 0$.

Step 2. Let $\lambda \in \Gamma$. Fix $x \in U_\lambda$. For any $r > 0$, there exist positive numbers m_r and M_r such that $\forall\, y \in B_d(x, r)$, we have $0 < m_r \leq u(y) \leq M_r$. We can choose then $\Lambda > 0$ such that $g(x) = \Lambda x + f(x)$ is nondecreasing in $[m_r, M_r]$, thus $Lu + \Lambda u = g(u) \leq g(u_\lambda) = Lu_\lambda + \Lambda u_\lambda$ in $Q_\lambda \cap B_d(x, r)$. By Theorem 2.3, we deduce that either $u \equiv u_\lambda$ in $Q_\lambda \cap B_d(x, r)$ or $u < u_\lambda$ in one connected component of $(Q_\lambda \setminus Z_\lambda) \cap B_d(x, r)$. As $Q_\lambda \setminus Z_\lambda$ is connected, for any points y_1, $y_2 \in Q_\lambda \setminus Z_\lambda$, we can join them by a loop γ in $(Q_\lambda \setminus Z_\lambda) \cap B_d(x, r)$ for some $r > 0$. Hence, we conclude that either $u \equiv u_\lambda$ in Q_λ or $u < u_\lambda$ in $Q_\lambda \setminus Z_\lambda$. The proof is finished in the first case; otherwise, we claim that

$$\text{there exists } \delta > 0 \text{ such that } [\lambda, \lambda + \delta] \subset \Gamma.$$

Suppose that $u < u_\lambda$ in $Q_\lambda \setminus Z_\lambda$. We choose $R > 0$ such that $u(x) < s_0/2$ in $\mathbb{R}^n \setminus B(0, R)$. We choose also positive constants δ_1, R_1 and R_2 such that for any $t \in [\lambda, \lambda + \delta_1]$, there exists $x_t \in U_t$ satisfying

$$B(0, R) \subset B_d(x_t, R_1) \subset B(0, R_2).$$

Fix $\epsilon \in (0, s_0/2)$, as in Step 1, we get (4.3) and we have $(f(u) - f(u_t))(u - u_t - \epsilon)^+ \leq 0$ in $Q_t \cap (B(0, R))^c$ by the choice of R. Set $m = \underline{\min}_{B(0,R)}\, u$ and $M = \overline{\max}_{B(0,R_2)}\, u$. We see that $u(Q_t \cap B(0, R)) \subset [m, M]$ and $u_t(Q_t \cap B(0, R)) \subset u_t(B_d(x_t, R_1)) = u(B_d(x_t, R_1)) \subset [m, M]$. Recall that f is Lipschitz in $[m, M]$, therefore

$$\begin{aligned}
&\int_{Q_t} (f(u) - f(u_t))(u - u_t - \epsilon)^+ dx \\
&\leq \int_{Q_t \cap B(0,R)} (f(u) - f(u_t))(u - u_t - \epsilon)^+ dx \leq C \int_{Q_t \cap B(0,R)} (u - u_t)^+ (u - u_t - \epsilon)^+ dx.
\end{aligned}$$

For any $t \in [\lambda, \lambda+\delta_1]$, using (4.3) and Lemma 2.1, we have

$$\int_{Q_t} (|Xu| + |Xu_t|)^{p-2} |X(u-u_t-\epsilon)^+|^2 dx \le C \int_{Q_t \cap B(0,R)} (u-u_t)^+ (u-u_t-\epsilon)^+ dx$$
$$\le C \int_{Q_t \cap B(0,R)} \left((u-u_t)^+\right)^2 dx \le C \int_{Q_t \cap B_d(x_t,R_1)} \left((u-u_t)^+\right)^2 dx.$$

Letting $\epsilon \to 0$, we get

$$\int_{Q_t} (|Xu| + |Xu_t|)^{p-2} |X(u-u_t)^+|^2 dx \le C \int_{Q_t \cap B_d(x_t,R_1)} \left((u-u_t)^+\right)^2 dx.$$

Define now

$$\tilde{\omega}(x) = \begin{cases} (u-u_t)^+(x) & \text{if } x \in Q_t, \\ -(u-u_t)^+(x) & \text{if } x \in Q^t. \end{cases}$$

Obviously, $\tilde{\omega} \in W^{1,2}(B_d(x_t,R_1))$ and

$$\bar{w}_{B_d} = -\!\!\!\!\!\!\int_{B_d(x_t,R_1)} \tilde{\omega} dx = 0,$$

since $I_t(B_d(x_t,R_1) \cap Q_t) = B_d(x_t,R_1) \cap Q^t$, $I_t^*(dx) = dx$ according to the definition of G_1. Then, it follows the hypothesis (P) that there exists $1 < q < 2$ such that

$$\int_{Q_t \cap B_d(x_t,R_1)} \left((u-u_t)^+\right)^2 dx = \frac{1}{2} \int_{B_d(x_t,R_1)} \left(\tilde{\omega} - \bar{w}_{B_d}\right)^2 dx$$
$$\le C \Big(\int_{B_d(x_t,R_1)} |X\tilde{\omega}|^q dx \Big)^{2/q} \le C \Big(\int_{Q_t \cap B_d(x_t,R_1)} |X\tilde{\omega}|^q dx \Big)^{2/q}. \tag{4.4}$$

Hence

$$\int_{Q_t} (|Xu| + |Xu_t|)^{p-2} |X(u-u_t)^+|^2 dx$$
$$\le C \Big(\int_{Q_t \cap B(0,R_2) \cap \mathrm{supp}(u-u_t)^+} |X(u-u_t)^+|^q dx \Big)^{2/q}. \tag{4.5}$$

Moreover, for any disjointed union $\Sigma_1 \cup \Sigma_2 = Q_t \cap B(0,R_2) \cap \mathrm{supp}(u-u_t)^+$, we denote

$$\overline{M} = \sup_{x \in \bar{B}(0,R_2),\, t \in [\lambda, \lambda+\delta_1]} (|Xu(x)| + |Xu_t(x)|) \quad \text{and} \quad M_{\Sigma_2,t} = \sup_{x \in \Sigma_2} \left(|Xu(x)| + |Xu_t(x)|\right).$$

By (4.5) and Hölder inequality, we obtain

$$\overline{M}^{p-2} \int_{\Sigma_1} |X(u-u_t)^+|^2 dx + M_{\Sigma_2,t}^{p-2} \int_{\Sigma_2} |X(u-u_t)^+|^2 dx$$
$$\le \int_{Q_t} (|Xu| + |Xu_t|)^{p-2} |X(u-u_t)^+|^2 dx$$
$$\le C \Big(|\Sigma_1|^{2/q-1} \int_{\Sigma_1} |X(u-u_t)^+|^2 dx + |B(0,R_2)|^{2/q-1} \int_{\Sigma_2} |X(u-u_t)^+|^2 dx \Big). \tag{4.6}$$

Clearly, there exists $M_0, \eta > 0$ such that if $|\Sigma_1| \le \eta$ and $M_{\Sigma_2,t} \le M_0$ with $t \in [\lambda, \lambda+\delta_1]$, (4.6) will deduce that $X(u-u_t)^+ \equiv 0$ in Q_t, i.e. $u \le u_t$ in Q_t.

Coming back to consider u_λ, we can take an open set U such that $Z_\lambda \cap B(0,R_2) \subset U \subset Q_\lambda$ and $M_{U,\lambda} \le M_0/2$. We choose also $K \subset\subset Q_\lambda \cap B(0,R_2)$ verifying $|(Q_\lambda \cap B(0,R_2)) \setminus K| \le \eta/2$. Since $u < u_\lambda$ in $Q_\lambda \setminus Z_\lambda$, we have $\epsilon > 0$ such that $u \le u_\lambda - \epsilon$ in the compact subset $K \setminus U$. As I_t is a C^1 foliation and u is C^1, there exists $\delta \in (0,\delta_1)$ such that for any $t \in [\lambda, \lambda+\delta]$, $u \le u_t$ in $K \setminus U$ and $|(Q_t \cap B(0,R_2)) \setminus K| \le \eta$ and $M_{U,t} \le M_0$. Hence, for such t, letting

$$\Sigma_1 = (\mathrm{supp}(u-u_t)^+ \cap Q_t \cap B(0,R_2)) \setminus K \quad \text{and} \quad \Sigma_2 = \mathrm{supp}(u-u_t)^+ \cap Q_t \cap B(0,R_2) \cap U \cap K$$

be two disjointed sets, we see that $Q_t \cap B(0,R_2) \cap \mathrm{supp}(u-u_t)^+ = \Sigma_1 \cup \Sigma_2$. Since $|\Sigma_1| \le \eta$ and $M_{\Sigma_2,t} \le M_{U,t} \le M_0$, by the above remark, we get $(u-u_t)^+ \equiv 0$ in Q_t for any $t \in [\lambda, \lambda+\delta]$. The claim is proved.

Step 3. By the assumption $\lim\limits_{|x|\to\infty} u(x) = 0$ and $u(x) > 0$ in $\mathbb{R}^n$, clearly there exists $\alpha \in \mathbb{R}$ such that $\Gamma \cap [\alpha, \infty) = \emptyset$. Setting $t_0 = \sup_\Gamma \lambda$, we see that $u \le u_{t_0}$ in Q_{t_0} and the desired result yields from Step 2.

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