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ADJOINT SYMMETRY CONSTRAINTS OF MULTICOMPONENT AKNS EQUATIONS***

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Abstract

A soliton hierarchy of multicomponent AKNS equations is generated from an arbitrary order matrix spectral problem, along with its bi-Hamiltonian formulation. Adjoint symmetry constraints are presented to manipulate binary nonlinearization for the associated arbitrary order matrix spectral problem. The resulting spatial and temporal constrained flows are shown to provide integrable decompositions of the multicomponent AKNS equations.

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§1. Introduction

Integrable systems and soliton theory are receiving more and more respect and recognition in the mathematical and physical communities, both in China and abroad. A three-digit classification 37K in 2000 Mathematics Subject Classification (MSC2000) was newly designed for this science in the historic year 2000, although MSC2000 does not contain all areas of integrable systems and soliton theory. Binary nonlinearization is one of new areas in that nonlinear science, attracting an increasing interest recently.

In the nonlinearization process of matrix spectral problems of soliton equations, symmetry constraints play an exceptional role[1], which generate integrable decompositions for soliton equations and thus show the integrability by quadratures for the underlying soliton equations. Of particular significance in the area of binary nonlinearization is a kind of non-Lie symmetries engendered from the variational derivative of the spectral parameter[1]. It has not realized until very recently that this variational derivative is an adjoint symmetry of the underlying soliton equations, and thus, actually all we need in carrying out nonlinearization is a kind of adjoint symmetry constraints[2]. Moreover, adjoint symmetry constraints have broader applicability than symmetry constraints, because they can be applied to both Hamiltonian and non-Hamiltonian systems of evolution equations.

However, due to the complexity of higher-order matrix spectral problems, there has not been much investigation on binary nonlinearization for soliton equation associated with higher-order matrix spectral problems. The multi-wave interaction equations, in both $1+1$

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dimensions and 2+1 dimensions, is the first example on binary nonlinearization in the case of higher-order matrix spectral problems (see [3] for details). Further investigation is deserved to make more examples of integrable decompositions for soliton equations associated with higher-order matrix spectral problems.

Actually, many nonlinear evolution equations of physical and mathematical interest are associated with the following mth-order matrix spectral problem,

$$\phi_x = (\lambda U_0 + U_1)\phi,\ U_0 = \mathrm{diag}(\alpha_1, \cdots, \alpha_m), \tag{1.1}$$

where $\alpha_1, \cdots, \alpha_m$ are arbitrary constants and U_1 is an off-diagonal potential matrix. The nondegenerate case of $\alpha_i \neq \alpha_j$, $1 \leq i \neq j \leq m$, corresponds to the multi-wave interaction equations[3,4]. For the degenerate case that U_0 possesses multiple eigenvalues, a remarkable example is the coupled nonlinear Schrödinger equation associated with $U_0 = \mathrm{diag}(1, -1, -1)$ (see [5]), which is also one of the models in optical fibers[6]. Some of these systems were shown to possess various integrable properties, for example, Hamiltonian structures[3,7,8], solvability by the inverse scattering transform[9,10] or related Riemann-Hilbert problems[5,11], separated variables[12], and the Liouville integrability by constrained flows[1,13].

In this paper, we would like to make an application of adjoint symmetry constraints to the multicomponent AKNS equations, which are associated with an arbitrary order degenerate matrix spectral problem of the type (1.1). Upon choosing a class of Lie point adjoint symmetries, a set of adjoint symmetry constraints is presented for the multicomponent AKNS equations, and the spatial part and the temporal part of the associated spectral problems and adjoint spectral problems are transformed into two Liouville integrable Hamiltonian systems. The potential constraints resulting from the adjoint symmetry constraints provide integrable decompositions[14,15] or the Bäcklund transformations[16] for the multicomponent AKNS equations.

The paper is structured as follows. Section 2 is devoted to a recall of a general scheme of adjoint symmetry constraints. Section 3 presents a multicomponent AKNS hierarchy of soliton equations from an arbitrary order matrix spectral problem. Then, Section 4 discusses the problem of adjoint symmetry constraints, and Section 5 proceeds to establish the Liouville integrability of the resulting constrained flows and integrable decompositions of the multicomponent AKNS equations as well. Finally, Section 6 gives rise to some concluding remarks.

§2. General Procedure

Let us recall a general procedure of adjoint symmetry constraints for carrying out binary nonlinearization[2]. Assume that we have two square matrix spectral problems

$$\phi_x = U\phi = U(u, \lambda)\phi, \tag{2.1a}$$

$$\phi_{t_n} = V^{(n)}\phi = V^{(n)}(u, u_x, \cdots; \lambda)\phi, \tag{2.1b}$$

where $n \geq 0$, λ is a spectral parameter, and U and $V^{(n)}$ are two square matrices, called spectral matrices. If the Gateaux derivative U' of U is injective, then under the isospectral condition

$$\lambda_{t_n} = 0, \tag{2.2}$$

zero curvature equations

$$U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0, \tag{2.3}$$

where $[U, V^{(n)}] = UV^{(n)} - V^{(n)}U$, will determine a hierarchy of soliton equations:

$$u_{t_n} = K_n(u), \quad K_n = JG_n = J\frac{\delta \tilde{H}_n}{\delta u}, \quad \tilde{H}_n = \int H_n\, dx, \tag{2.4}$$

which is supposed to have the Hamiltonian structure with the Hamiltonian operator $J(u)$ and the Hamiltonian functionals $\tilde{H}_n(u)$. Obviously, the adjoint spectral problem and the adjoint associated spectral problems for all $n \geq 0$,

$$\psi_x = -U^T(u,\lambda)\psi, \quad \psi_{t_n} = -V^{(n)T}(u,u_x,\cdots;\lambda)\psi, \tag{2.5}$$

where T denotes the transpose of matrices, have the compatability conditions

$$(-U^T)_{t_n} - (-V^{(n)T})_x + [(-U^T),(-V^{(n)T})] = 0, \tag{2.6}$$

which determine the same soliton hierarchy (2.4).

It is known (for example, see [15] for a detailed deduction) that

$$\frac{\delta\lambda}{\delta u} = E^{-1}\psi^T \frac{\partial U(u,\lambda)}{\partial u}\phi, \quad E = -\int_\Omega \psi^T \frac{\partial U(u,\lambda)}{\partial \lambda}\phi\, dx, \tag{2.7}$$

where E is called the normalized constant, and $\Omega = (0,T_0)$ if u is supposed to be periodic with period T_0 or $\Omega = (-\infty,\infty)$ if u is supposed to belong to the Schwartz space. Therefore, if the Hamiltonian functionals $\widetilde{H}_m$ are assumed to be conserved, then we have the following common adjoint symmetries:

$$G_m, \ \psi^T \frac{\partial U(u,\lambda)}{\partial u}\phi \tag{2.8}$$

for all soliton equations in the hierarchy (2.4), since $\lambda = \lambda(u)$ is a conserved functional due to (2.2). The adjoint symmetries G_m are Lie type. But $\delta\lambda/\delta u$ is not Lie type, since $\phi^{(s)}$ and $\psi^{(s)}$ can not be expressed in terms of x, u and spatial derivatives of u to some finite order.

Let us proceed to introduce N distinct eigenvalues $\lambda_1,\cdots,\lambda_N$, and so we have the spatial part of the spectral problems and adjoint spectral problems

$$\phi_x^{(s)} = U(u,\lambda_s)\phi^{(s)}, \quad \psi_x^{(s)} = -U^T(u,\lambda_s)\psi^{(s)}, \ 1 \leq s \leq N, \tag{2.9}$$

and the temporal part of the spectral problems and adjoint spectral problems

$$\phi_{t_n}^{(s)} = V^{(n)}(u,u_x,\cdots;\lambda_s)\phi^{(s)}, \quad \psi_{t_n}^{(s)} = -V^{(n)T}(u,u_x,\cdots;\lambda_s)\psi^{(s)}, \ 1 \leq s \leq N, \tag{2.10}$$

where the eigenfunction and adjoint eigenfunction corresponding to λ_s are denoted by $\phi^{(s)}$ and $\psi^{(s)}$, $1 \leq s \leq N$. Fix a Lie type adjoint symmetry G_{m_0}, and then we can define the so-called binary adjoint symmetry constraint

$$G_{m_0} = \sum_{s=1}^N E_s\mu_s \frac{\delta\lambda_s}{\delta u} = \sum_{s=1}^N \mu_s \psi^{(s)T}\frac{\partial U(u,\lambda_s)}{\partial u}\phi^{(s)}, \tag{2.11}$$

where μ_s, $1 \leq s \leq N$, are arbitrary nonzero constants, and E_s, $1 \leq s \leq N$, are N normalized constants. The right-hand side of the constraint (2.11) is a general linear combination of N non-Lie type adjoint symmetries $\delta\lambda_s/\delta u$, $1 \leq s \leq N$. But, the left-hand side of the constraint (2.11) is a Lie type adjoint symmetry G_{m_0}. The adjoint symmetry constraints (2.11) contain three kinds of dependent variables: $u, \phi^{(s)}, \psi^{(s)}$. All adjoint symmetry constraints (2.11) can be divided into three categories:

• **Neumann type:** (2.11) does not depend on any spatial derivative of u and it is impossible to solve (2.11) for u.

• **Bargmann type:** (2.11) does not depend on any spatial derivative of u but it is possible to solve (2.11) for u.

• **Ostrogradsky type:** (2.11) depends on spatial derivatives of u.

In a soliton hierarchy, usually the first conserved functional generates the Neumann type constraint, the second conserved functional generates the Bargmann type constraint, and the other conserved functionals generate the Ostragradovsky type constraints.

Let us now focus on the Bargmann type adjoint symmetry constraints, which are a basis of the theory of binary nonlinearization. Upon solving the adjoint symmetry constraint (2.11) for u, we are assumed to have

$$u = \widetilde{u}(\phi^{(1)},\cdots,\phi^{(N)};\psi^{(1)},\cdots,\psi^{(N)}). \tag{2.12}$$

The replacement of u with $\widetilde{u}$ in the Lax systems (2.9) and (2.10) lead to the so-called spatial binary constrained flow:

$$\phi_x^{(s)} = U(\widetilde{u}, \lambda_s)\phi^{(s)}, \quad \psi_x^{(s)} = -U^T(\widetilde{u}, \lambda_s)\psi^{(s)},\ 1 \le s \le N, \tag{2.13}$$

and the so-called temporal binary constrained flow:

$$\phi_{t_n}^{(s)} = V^{(n)}(\widetilde{u}, \widetilde{u}_x, \cdots; \lambda_s)\phi^{(s)}, \quad \psi_{t_n}^{(s)} = -V^{(n)T}(\widetilde{u}, \widetilde{u}_x, \cdots; \lambda_s)\psi^{(s)},\ 1 \le s \le N. \tag{2.14}$$

The spatial constrained flow (2.13) is a system of ordinary differential equations, but in most cases, the temporal constrained flow (2.14) is a system of partial differential equations. Nevertheless, taking advantage of (2.13), the temporal constrained flow (2.14) can be transformed into a system of ordinary differential equations, which is assumed to be denoted by

$$\phi_{t_n}^{(s)} = \widetilde{V}^{(n)}(\widetilde{u}, \lambda_s)\phi^{(s)}, \quad \psi_{t_n}^{(s)} = -\widetilde{V}^{(n)T}(\widetilde{u}, \lambda_s)\psi^{(s)},\ 1 \le s \le N. \tag{2.15}$$

The constrained flows (2.13) and (2.15) still require the nth soliton equation $u_{t_n} = K_n(u)$ as their compatability condition. Therefore, $u = \widetilde{u}$ gives rise to an integrable decomposition of $u_{t_n} = K_n(u)$, if (2.13) and (2.15) are two Liouville integrable Hamiltonian systems.

Note that the constrained flows (2.13) and (2.15) are nonlinear, although the original Lax systems (2.9) and (2.10) are linear with the eigenfunctions and adjoint eigenfunctions. In view of this characteristic and the involvement of the original spectral problems and the adjoint ones, the above whole process of manipulating the Bargmann adjoint symmetry constraints is called binary nonlinearization[1,17].

§3. Multicomponent AKNS Hierarchy

Let m be an arbitrary natural number. We consider the following $(m+1) \times (m+1)$ matrix spectral problem

$$\phi_x = U(u, \lambda)\phi, \quad U(u, \lambda) = \begin{bmatrix} -m\lambda & q \\ r & \lambda I_m \end{bmatrix} = U_0\lambda + U_1, \quad \frac{\partial U_0}{\partial \lambda} = \frac{\partial U_1}{\partial \lambda} = 0, \tag{3.1}$$

where λ is a spectral parameter, I_m is the $m \times m$ unit matrix, and

$$q = (q_1, q_2, \cdots, q_m), \quad r = (r_1, r_2, \cdots, r_m)^T, \tag{3.2a}$$

$$\phi = (\phi_1, \phi_2, \cdots, \phi_{m+1})^T, \quad u = \rho(U_1) = (q, r^T)^T. \tag{3.2b}$$

Because U_0 has multiple eigenvalues, the spectral problem (3.1) is degenerate. A special reduction of $m = 1$ gives rise to the AKNS spectral problem[18], and thus the spectral problem (3.1) is called a multicomponent AKNS spectral problem.

To derive an associated soliton hierarchy, we first solve the adjoint equation

$$W_x = [U, W] \tag{3.3}$$

of the spectral problem (3.1) through the generalized Tu scheme[19]. We assume that a solution W is given by

$$W = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{3.4}$$

where a is a scalar, b^T and c are m-dimensional columns, and d is an $m \times m$ matrix. Then we have

$$[U, W] = \begin{bmatrix} qc - br & -(m+1)\lambda b + qd - aq \\ (m+1)\lambda c + ra - dr & rb - cq \end{bmatrix}.$$

Therefore, the adjoint equation (3.3) is equivalent to

$$a_x = qc - br, \quad b_x = -(m+1)\lambda b + qd - aq, \tag{3.5a}$$

$$c_x = (m+1)\lambda c + ra - dr, \quad d_x = rb - cq. \tag{3.5b}$$

Let us seek a formal solution of the type

$$W = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \sum_{k=0}^{\infty} W_k \lambda^{-k} = \sum_{k=0}^{\infty} \begin{bmatrix} a^{(k)} & b^{(k)} \\ c^{(k)} & d^{(k)} \end{bmatrix} \lambda^{-k} \tag{3.6}$$

with $b^{(k)}, c^{(k)}$ and $d^{(k)}$ being assumed to be

$$b^{(k)} = (b_1^{(k)}, b_2^{(k)}, \cdots, b_m^{(k)}), \quad c^{(k)} = (c_1^{(k)}, c_2^{(k)}, \cdots, c_m^{(k)})^T, \ d^{(k)} = (d_{ij}^{(k)})_{m\times m}. \tag{3.7}$$

Therefore, the condition (3.5) becomes the following recursion relation:

$$b^{(0)} = 0, \quad c^{(0)} = 0, \quad a_x^{(0)} = 0, \quad d_x^{(0)} = 0, \tag{3.8a}$$

$$b^{(k+1)} = \frac{1}{m+1}(-b_x^{(k)} + q d^{(k)} - a^{(k)} q), \quad k \geq 0, \tag{3.8b}$$

$$c^{(k+1)} = \frac{1}{m+1}(c_x^{(k)} - r a^{(k)} + d^{(k)} r), \ k \geq 0, \tag{3.8c}$$

$$a_x^{(k+1)} = q c^{(k+1)} - b^{(k+1)} r, \quad d_x^{(k+1)} = r b^{(k+1)} - c^{(k+1)} q, \quad k \geq 0. \tag{3.8d}$$

We choose the initial values to be

$$a^{(0)} = -m, \quad d^{(0)} = I_m, \tag{3.9}$$

and require that

$$W_k|_{u=0} = 0, \quad k \geq 1. \tag{3.10}$$

The requirement (3.10) implies that we should identify all constants of integration with zero while using (3.8) to determine W, and thus, with $a^{(0)}$ and $d^{(0)}$ being given by (3.9), all matrices $W_k, \ k \geq 1$, will be uniquely determined by (3.8). For example, we can obtain from (3.8) under (3.9) and (3.10) that

$$b_i^{(1)} = q_i, \quad c_i^{(1)} = r_i, \quad a^{(1)} = 0, \quad d_{ij}^{(1)} = 0,$$

$$b_i^{(2)} = -\frac{1}{m+1} q_{i,x}, \quad c_i^{(2)} = \frac{1}{m+1} r_{i,x}, \quad a^{(2)} = \frac{1}{m+1} \sum_{i=1}^{m} q_i r_i, \quad d_{ij}^{(2)} = -\frac{1}{m+1} r_i q_j,$$

$$b_i^{(3)} = \frac{1}{(m+1)^2}(q_{i,xx} - 2q_i \sum_{j=1}^{m} q_j r_j), \quad c_i^{(3)} = \frac{1}{(m+1)^2}(r_{i,xx} - 2r_i \sum_{j=1}^{m} q_j r_j),$$

$$a^{(3)} = \frac{1}{(m+1)^2} \sum_{i=1}^{m} (q_i r_{i,x} - q_{i,x} r_i), \quad d_{ij}^{(3)} = \frac{1}{(m+1)^2}(r_i q_{j,x} - r_{i,x} q_j),$$

where $1 \leq i, j \leq m$. Since from (3.8d) we have

$$a^{(k)} = \partial^{-1}(q c^{(k)} - b^{(k)} r), \quad d^{(k)} = \partial^{-1}(r b^{(k)} - c^{(k)} q), \quad k \geq 1,$$

where $\partial\partial^{-1} = \partial^{-1}\partial = 1$, and $\partial = \frac{\partial}{\partial x}$, we can obtain the following recursion relation for $b^{(k)}$ and $c^{(k)}$:

$$\begin{bmatrix} c^{(k+1)} \\ b^{(k+1)T} \end{bmatrix} = \Psi \begin{bmatrix} c^{(k)} \\ b^{(k)T} \end{bmatrix}, \ k \geq 1, \tag{3.12}$$

where the $2m \times 2m$ matrix operator Ψ is given by

$$\Psi = \frac{1}{m+1} \begin{bmatrix} \Big(\partial - \sum\limits_{k=1}^{m} r_k \partial^{-1} q_k\Big) I_m - r\partial^{-1} q & r\partial^{-1} r^T + (r\partial^{-1} r^T)^T \\ -q^T \partial^{-1} q - (q^T \partial^{-1} q)^T & \Big(-\partial + \sum\limits_{k=1}^{m} q_k \partial^{-1} r_k\Big) I_m + q^T \partial^{-1} r^T \end{bmatrix}. \tag{3.13}$$

As usual, for any integer $n \geq 0$, we take

$$V^{(n)} = (\lambda^n W)_+ = \sum_{j=0}^{n} W_j \lambda^{n-j}, \tag{3.14}$$

and then introduce the time evolution law for the eigenfunction ϕ:

$$\phi_{t_n} = V^{(n)}\phi = V^{(n)}(u, u_x, \cdots, u^{(n-1)}; \lambda)\phi. \tag{3.15}$$

The compatibility condition of (3.1) and (3.15), i.e., the zero curvature equation $U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0$, leads to a system of evolution equations

$$u_{t_n} = \begin{bmatrix} q^T \\ r \end{bmatrix}_{t_n} = K_n = \begin{bmatrix} -(m+1)b^{(n+1)T} \\ (m+1)c^{(n+1)} \end{bmatrix}. \tag{3.16}$$

Among this soliton hierarchy (3.16), the first nonlinear system reads as

$$q_{i,t_2} = -\frac{1}{m+1}\Big(q_{i,xx} - 2q_i \sum_{j=1}^{m} q_j r_j\Big), \tag{3.17a}$$

$$r_{i,t_2} = \frac{1}{m+1}\Big(r_{i,xx} - 2r_i \sum_{j=1}^{m} q_j r_j\Big), \tag{3.17b}$$

where $1 \leq i \leq m$, which is the multicomponent version of the AKNS nonlinear Schrödinger equations[18]. Hence, the soliton hierarchy (3.16) is called the multicomponent AKNS soliton hierarchy with multiplicity m.

To construct the Hamiltonian structure of the multicomponent AKNS hierarchy (3.16), we apply the trace identity (see [20] for more applications):

$$\frac{\delta}{\delta u}\int \mathrm{tr}\Big(W\frac{\partial U}{\partial \lambda}\Big)dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\Big[\lambda^{\gamma}\mathrm{tr}\Big(W\frac{\partial U}{\partial u}\Big)\Big], \tag{3.18}$$

with γ being a constant to be determined. An application of (3.18) yields

$$\frac{\delta \tilde{H}_{n+1}}{\delta u} = G_n, \quad \tilde{H}_n = -\frac{1}{n}\int \Big(- ma^{(n+1)} + \sum_{i=1}^{m} d_{ii}^{(n+1)}\Big)\, dx, \quad G_{n-1} = \begin{bmatrix} c^{(n)} \\ b^{(n)T} \end{bmatrix}, \ n \geq 0. \tag{3.19}$$

In fact, it is easy to compute that

$$\mathrm{tr}\Big(W\frac{\partial U}{\partial \lambda}\Big) = -ma + \mathrm{tr}(d) = \sum_{k\geq 0}\Big(- ma^{(k)} + \sum_{i=1}^{m} d_{ii}^{(k)}\Big)\lambda^{-k},$$

$$\mathrm{tr}\Big(W\frac{\partial U}{\partial u}\Big) = \begin{bmatrix} c \\ b^T \end{bmatrix} = \sum_{k\geq 0} G_{k-1}\lambda^{-k}.$$

Upon inserting these two expressions into the trace identity (3.18) and considering the case of $k = 2$, we know $\gamma = 0$, and thus we have (3.19). Now it follows from (3.19) that the multicomponent AKNS equations (3.16) have the following bi-Hamiltonian structure

$$u_{t_n} = K_n = JG_n = J\frac{\delta \tilde{H}_{n+1}}{\delta u} = M\frac{\delta \tilde{H}_n}{\delta u}, \tag{3.20}$$

where the Hamiltonian pair of J and $M = J\Psi$ is defined by

$$J = \begin{bmatrix} 0 & -(m+1)I_m \\ (m+1)I_m & 0 \end{bmatrix}, \tag{3.21a}$$

$$M = \begin{bmatrix} q^T\partial^{-1}q + (q^T\partial^{-1}q)^T & \Big(\partial - \sum_{k=1}^{M} q_k\partial^{-1}r_k\Big)I_m - q^T\partial^{-1}r^T \\ \Big(\partial - \sum_{k=1}^{M} r_k\partial^{-1}q_k\Big)I_m - r\partial^{-1}q & r\partial^{-1}r^T + (r\partial^{-1}r^T)^T \end{bmatrix}. \tag{3.21b}$$

§4. Adjoint Symmetry Constraints

Let us proceed to consider the problem of adjoint symmetry constraints for the multi-component AKNS equations $u_{t_2} = K_2$ defined by (3.16) or (3.17). A direct computation tells us that the Gateaux derivative operator of K_2 reads as

$$K_2' = \frac{2}{m+1}\Big(-\frac{1}{2}\partial^2 + qr\Big)\begin{bmatrix} I_m & 0 \\ 0 & -I_m \end{bmatrix} + \frac{2}{m+1}\begin{bmatrix} q^Tr^T & q^Tq \\ -rr^T & -rq \end{bmatrix}, \tag{4.1}$$

and thus its adjoint operator is given by

$$(K_2')^\dagger = \frac{2}{m+1}\Big(-\frac{1}{2}\partial^2 + qr\Big)\begin{bmatrix} I_m & 0 \\ 0 & -I_m \end{bmatrix} + \frac{2}{m+1}\begin{bmatrix} rq & -rr^T \\ q^Tq & -q^Tr^T \end{bmatrix} = (K')^T, \tag{4.2}$$

which is just the transpose of ${K'}_2$. We choose a constant diagonal matrix

$$\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \cdots, \gamma_{m+1}), \quad \gamma_i \neq \gamma_j, \quad 1 \le i \neq j \le m+1. \tag{4.3}$$

Then the commutator of two matrices Γ and U_1,

$$[\Gamma, U_1] = \begin{bmatrix} 0 & (\gamma_1-\gamma_2)q_1 & \cdots & (\gamma_1-\gamma_{m+1})q_m \\ (\gamma_2-\gamma_1)r_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (\gamma_{m+1}-\gamma_1)r_m & 0 & \cdots & 0 \end{bmatrix},$$

will give us a Lie point symmetry for the multicomponent AKNS equations (3.17):

$$\bar{K}_0 := \rho([\Gamma, U_1]) = \begin{bmatrix} (\gamma_1-\gamma_2)q_1 \\ \vdots \\ (\gamma_1-\gamma_{m+1})q_m \\ (\gamma_2-\gamma_1)r_1 \\ \vdots \\ (\gamma_{m+1}-\gamma_1)r_m \end{bmatrix}. \tag{4.4}$$

Taking advantage of the Hamiltonian structure in (3.20), we obtain a Lie point adjoint symmetry for the multicomponent AKNS equations (3.17):

$$\bar{G}_0 := J^{-1}\bar{K}_0 = \begin{bmatrix} 0 & \frac{1}{m+1}I_m \\ -\frac{1}{m+1}I_m & 0 \end{bmatrix}\bar{K}_0 = \frac{1}{m+1}\begin{bmatrix} (\gamma_2-\gamma_1)r_1 \\ \vdots \\ (\gamma_{m+1}-\gamma_1)r_m \\ (\gamma_2-\gamma_1)q_1 \\ \vdots \\ (\gamma_{m+1}-\gamma_1)q_m \end{bmatrix}. \tag{4.5}$$

This also can be shown by directly checking the adjoint linearized system

$$\bar{G}_{0,t_2} = -(K_2')^\dagger\bar{G}_0,$$

while u solves $u_{t_2} = K_2(u)$. The adjoint symmetry $\bar{G}_0$ contains $m+1$ arbitrary distinct constants $\gamma_1, \gamma_2, \cdots, \gamma_{m+1}$, which is very important for guaranteeing that there exist sufficiently many integrals of motion for the Liouville integrability of the corresponding constrained flows.

Now the Bargmann adjoint symmetry constraint reads as

$$\bar{G}_0 = \sum_{s=1}^{N} \mu_s \psi^{(s)T} \frac{\partial U(u,\lambda_s)}{\partial u} \phi^{(s)}, \tag{4.6}$$

where for a later use, $\phi^{(s)}$ and $\psi^{(s)}$ are assumed to be

$$\phi^{(s)} = (\phi_{1s}, \phi_{2s}, \cdots, \phi_{m+1,s})^T, \quad \psi^{(s)} = (\psi_{1s}, \psi_{2s}, \cdots, \psi_{m+1,s})^T, \quad 1 \le s \le N. \tag{4.7}$$

Upon introducing the matrix as usual,

$$B = \mathrm{diag}(\mu_1, \mu_2, \cdots, \mu_N), \tag{4.8}$$

and solving the adjoint symmetry constraint (4.6) for q and r, we are led to the potential constraints

$$q_i = \tilde{q}_i := \frac{m+1}{\gamma_{i+1} - \gamma_1} \langle \Phi_1, B\Psi_{i+1} \rangle, \quad r_i = \tilde{r}_i := \frac{m+1}{\gamma_{i+1} - \gamma_1} \langle \Phi_{i+1}, B\Psi_1 \rangle, \tag{4.9}$$

where $1 \le i \le m$, Φ_i and Ψ_i are defined by

$$\Phi_i = (\phi_{i1}, \phi_{i2}, \cdots, \phi_{iN})^T, \quad \Psi_i = (\psi_{i1}, \psi_{i2}, \cdots, \psi_{iN})^T, \ 1 \le i \le m+1, \tag{4.10}$$

and $\langle \cdot, \cdot \rangle$ refers to the standard inner product of the Euclidian space $\mathbb{R}^N$. Now the spatial and temporal constrained flows of the multicomponent AKNS equations (3.17) read as

$$\phi_x^{(s)} = U(\widetilde{u}, \lambda_s)\phi^{(s)}, \quad \psi_x^{(s)} = -U^T(\widetilde{u}, \lambda_s)\psi^{(s)}, \ 1 \le s \le N, \tag{4.11}$$

$$\phi_{t_2}^{(s)} = V^{(2)}(\widetilde{u}, \widetilde{u}_x; \lambda_s)\phi^{(s)}, \quad \psi_{t_2}^{(s)} = -V^{(2)T}(\widetilde{u}, \widetilde{u}_x; \lambda_s)\psi^{(s)}, \ 1 \le s \le N, \tag{4.12}$$

where $\widetilde{u} = (\widetilde{q}_1, \cdots, \widetilde{q}_m, \widetilde{r}_1, \cdots, \widetilde{r}_m)^T$. As did in (2.15), let us denote by $\widetilde{V}^{(2)}(\widetilde{u}, \widetilde{u}_x; \lambda)$ the transformed matrix of $V^{(2)}(\widetilde{u}, \lambda)$ under (4.11), i.e.,

$$\widetilde{V}^{(2)}(\widetilde{u}, \lambda) = V^{(2)}(\widetilde{u}, \widetilde{u}_x; \lambda)|_{\text{spatial constrained flow (4.11)}}. \tag{4.13}$$

Since $\widetilde{V}^{(2)}(\widetilde{u}, \lambda)$ just depends on ϕ_{is} and ψ_{is} but not on any spatial derivative of ϕ_{is} and ψ_{is}, the transformed temporal constrained flow becomes the following system of ordinary differential equations

$$\phi_{t_2}^{(s)} = \widetilde{V}^{(2)}(\widetilde{u}, \lambda_s)\phi^{(s)}, \quad \psi_{t_2}^{(s)} = -\widetilde{V}^{(2)T}(\widetilde{u}, \lambda_s)\psi^{(s)}, \quad 1 \le s \le N. \tag{4.14}$$

The constrained flows (4.11) and (4.14) still require (3.17) as their compatability condition.

§5. Liouville Integrability

We now turn to the Liouville integrability of the constrained flows (4.11) and (4.14). Noting that

$$V^{(2)}(u, u_x; \lambda) = \begin{bmatrix} -m & 0 \\ 0 & I_m \end{bmatrix} \lambda^2 + \begin{bmatrix} 0 & q \\ r & 0 \end{bmatrix} \lambda + \frac{1}{m+1} \begin{bmatrix} qr & -q_x \\ r_x & -rq \end{bmatrix}, \tag{5.1}$$

and denoting a diagonal matrix

$$A = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N), \tag{5.2}$$

the constrained flows (4.11) and (4.14) read as follows:

$$\phi_{1s,x} = -m\lambda_s \phi_{1s} + \sum_{i=1}^{m} \widetilde{q}_i \phi_{i+1,s}, \quad \phi_{j+1,s,x} = \widetilde{r}_j \phi_{1s} + \lambda_s \phi_{i+1,s}, \tag{5.3a}$$

$$\psi_{1s,x} = m\lambda_s \psi_{1s} - \sum_{i=1}^{m} \widetilde{r}_i \psi_{i+1,s}, \quad \psi_{j+1,s,x} = -\widetilde{q}_j \psi_{1s} - \lambda_s \psi_{i+1,s}, \tag{5.3b}$$

$$\phi_{1s,t_2} = \Big(- m\lambda_s^2 + \frac{1}{m+1}\widetilde{q}\,\widetilde{r}\Big)\phi_{1s} + \lambda_s \sum_{i=1}^{m} \widetilde{q}_i\phi_{i+1,s} + \sum_{i=1}^{m} \frac{1}{\gamma_1 - \gamma_{i+1}}$$
$$\times \Big[-(m+1)\langle A\Phi_1, B\Psi_{i+1}\rangle + \sum_{k=1}^{m} \widetilde{q}_k\langle \Phi_{k+1}, B\Psi_{i+1}\rangle - \widetilde{q}_i\langle \Phi_1, B\Psi_1\rangle\Big]\phi_{i+1,s}, \tag{5.4a}$$

$$\phi_{j+1,s,t_2} = \lambda_s^2\phi_{j+1,s} + \lambda_s\widetilde{r}_j\phi_{1s} + \frac{1}{\gamma_{j+1} - \gamma_1}\Big[\widetilde{r}_j\langle \Phi_1, B\Psi_1\rangle + (m+1)\langle A\Phi_{j+1}, B\Psi_1\rangle$$
$$- \sum_{k=1}^{m} \widetilde{r}_k\langle \Phi_{j+1}, B\Psi_{k+1}\rangle\Big]\phi_{1s} - \frac{1}{m+1}\sum_{i=1}^{m} \widetilde{r}_j\widetilde{q}_i\phi_{i+1,s}, \tag{5.4b}$$

$$\psi_{1s,t_2} = \Big(m\lambda_s^2 - \frac{1}{m+1}\widetilde{r}^T\widetilde{q}^T\Big)\psi_{1s} - \lambda_s \sum_{i=1}^{m} \widetilde{r}_i\psi_{i+1,s} - \sum_{i=1}^{m} \frac{1}{\gamma_{i+1} - \gamma_1}$$
$$\times \Big[\widetilde{r}_i\langle \Phi_1, B\Psi_1\rangle + (m+1)\langle A\Phi_{i+1}, B\Psi_1\rangle - \sum_{k=1}^{m} \widetilde{r}_k\langle \Phi_{i+1}, B\Psi_{k+1}\rangle\Big]\psi_{i+1,s}, \tag{5.4c}$$

$$\psi_{j+1,s,t_2} = -\lambda_s^2\psi_{j+1,s} - \lambda_s\widetilde{q}_j\psi_{1s} - \frac{1}{\gamma_1 - \gamma_{j+1}}\Big[-(m+1)\langle A\Phi_1, B\Psi_{j+1}\rangle$$
$$+ \sum_{k=1}^{m} \widetilde{q}_k\langle \Phi_{k+1}, B\Psi_{j+1}\rangle - \widetilde{q}_j\langle \Phi_1, B\Psi_1\rangle\Big]\psi_{1s} + \frac{1}{m+1}\sum_{i=1}^{m} \widetilde{q}_j\widetilde{r}_i\psi_{i+1,s}, \tag{5.4d}$$

respectively, where $1 \le j \le m$, $1 \le s \le N$, and $\widetilde{q}_i$ and $\widetilde{r}_i$ are defined by the potential constraints (4.9).

In order to analyze the Liouville integrability of (4.11) and (4.14), i.e, (5.3) and (5.4), let us define a symplectic structure

$$\omega^2 = \sum_{i=1}^{m+1} Bd\Phi_i \wedge d\Psi_i = \sum_{i=1}^{m+1}\sum_{s=1}^{N} \mu_s d\phi_{is} \wedge d\psi_{is} \tag{5.5}$$

over the Euclidian space $\mathbb{R}^{2(m+1)N}$, and then the corresponding Poisson bracket is given by

$$\{f, g\} = \omega^2(Idg, Idf) = \sum_{i=1}^{m+1} \Big(\Big\langle \frac{\partial f}{\partial \Psi_i}, B^{-1}\frac{\partial g}{\partial \Phi_i}\Big\rangle - \Big\langle \frac{\partial f}{\partial \Phi_i}, B^{-1}\frac{\partial g}{\partial \Psi_i}\Big\rangle\Big)$$
$$= \sum_{i=1}^{m+1}\sum_{s=1}^{N} \mu_s^{-1}\Big(\frac{\partial f}{\partial \psi_{is}}\frac{\partial g}{\partial \phi_{is}} - \frac{\partial f}{\partial \phi_{is}}\frac{\partial g}{\partial \psi_{is}}\Big),\ f, g \in C^\infty(\mathbb{R}^{2(m+1)N}), \tag{5.6a}$$

where the vector field Idf is defined by

$$\omega^2(X, Idf) = df(X),\ X \in T(\mathbb{R}^{2(m+1)N}).$$

A general Hamiltonian system with a Hamiltonian H defined over the symplectic manifold $(\mathbb{R}^{2(m+1)N}, \omega^2)$ is given by

$$\Phi_{i,t} = \{\Phi_i, H\} = -B^{-1}\frac{\partial H}{\partial \Psi_i}, \quad \Psi_{i,t} = \{\Psi_i, H\} = B^{-1}\frac{\partial H}{\partial \Phi_i},\ 1 \le i \le m+1, \tag{5.7}$$

where t is taken as the evolution variable. To present Lax representations of the constrained flows (5.3) and (5.4), we need a square matrix Lax operator

$$L(\lambda) = \Gamma + D(\lambda), \tag{5.8}$$

with Γ being defined by (4.3) and $D(\lambda)$, by

$$D(\lambda) = (D_{ij}(\lambda))_{(m+1)\times(m+1)},\ D_{ij}(\lambda) = \sum_{s=1}^{N} \frac{\mu_s}{\lambda - \lambda_s}\phi_{is}\psi_{js}, \tag{5.9}$$

where $1 \leq i, j \leq m+1$. Now we can state the main result as follows.

Theorem 5.1. *Under the symplectic structure* (5.5), *the spatial constrained flow* (4.11) [*i.e.,* (5.3)] *and the temporal constrained flow* (4.14) [*i.e.,* (5.4)] *of the multicomponent AKNS equations* (3.17) *are Hamiltonian systems with the evolution variables x and t_2, and the Hamiltonians*

$$H^x = m\langle A\Phi_1, B\Psi_1\rangle - \sum_{i=1}^{m}\langle A\Phi_{i+1}, B\Psi_{i+1}\rangle + \sum_{j=1}^{m}\frac{m+1}{\gamma_1-\gamma_{j+1}}\langle \Phi_1, B\Psi_{j+1}\rangle\langle\Phi_{j+1}, B\Psi_1\rangle, \tag{5.10}$$

$$\begin{aligned} H^{t_2} = & -m\langle A^2\Phi_1, B\Psi_1\rangle + \sum_{i=1}^{m}\langle A^2\Phi_{i+1}, B\Psi_{i+1}\rangle + \sum_{i=1}^{m}\widetilde{q}_i\langle A\Phi_{i+1}, B\Psi_1\rangle \\ & + \sum_{j=1}^{m}\widetilde{r}_j\langle A\Phi_1, B\Psi_{j+1}\rangle + \frac{1}{m+1}\sum_{i=1}^{m}\widetilde{q}_i\widetilde{r}_i\langle\Phi_1, B\Psi_1\rangle \\ & - \frac{1}{m+1}\sum_{i,j=1}^{m}\widetilde{q}_i\widetilde{r}_j\langle\Phi_{i+1}, B\Psi_{j+1}\rangle, \end{aligned} \tag{5.11}$$

respectively, where $\widetilde{q}_i$ and $\widetilde{r}_i$ are given by the potential constraints (4.9). *Moreover, the constrained flows* (4.11) *and* (4.14) *admit the Lax representations*

$$(L(\lambda))_x = [U(\widetilde{u}, \lambda), L(\lambda)], \ (L(\lambda))_{t_2} = [\widetilde{V}^{(2)}(\widetilde{u}, \lambda), L(\lambda)], \tag{5.12}$$

respectively, where $L(\lambda)$ is given by (5.8) *and* (5.9), *and $U(\widetilde{u}, \lambda)$ and $\widetilde{V}^{(2)}(\widetilde{u}, \lambda)$ are two constrained spectral matrices generated from U and $V^{(2)}$ under* (4.11).

Proof. It follows immediately from a direct but long calculation that the spatial constrained flow (4.11) and the temporal constrained flow (4.14) possess the Hamiltonian structures with H^x and H^{t_2} defined by (5.10) and (5.11).

Let us then check the Lax representations. Taking advantage of the spatial constrained flow (4.11), we can make the following computation:

$$\begin{aligned} (L(\lambda))_x &= \sum_{s=1}^{N}\frac{\mu_s}{\lambda-\lambda_s}(\phi_x^{(s)}\psi^{(s)T} + \phi^{(s)}\psi_x^{(s)T}) \\ &= \sum_{s=1}^{N}\frac{\mu_s}{\lambda-\lambda_s}(U(\widetilde{u},\lambda_s)\phi^{(s)}\psi^{(s)T} - \phi^{(s)}\psi^{(s)T}U(\widetilde{u},\lambda_s)) \\ &= \sum_{s=1}^{N}\frac{\mu_s}{\lambda-\lambda_s}[U(\widetilde{u},\lambda_s), \phi^{(s)}\psi^{(s)T}] \\ &= \sum_{s=1}^{N}\frac{\mu_s}{\lambda-\lambda_s}[U(\widetilde{u},\lambda) + (U(\widetilde{u},\lambda_s) - U(\widetilde{u},\lambda)), \phi^{(s)}\psi^{(s)T}] \\ &= \sum_{s=1}^{N}\frac{\mu_s}{\lambda-\lambda_s}[U(\widetilde{u},\lambda), \phi^{(s)}\psi^{(s)T}] + \sum_{s=1}^{N}\frac{\mu_s}{\lambda-\lambda_s}[(U(\widetilde{u},\lambda_s) - U(\widetilde{u},\lambda)), \phi^{(s)}\psi^{(s)T}] \\ &= [U(\widetilde{u},\lambda), L(\lambda) - \Gamma] - [U_0, \sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}] \\ &= [U(\widetilde{u},\lambda), L(\lambda)] + [\Gamma, U(\widetilde{u},\lambda)] - [U_0, \sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}] \\ &= [U(\widetilde{u},\lambda), L(\lambda)] + [\Gamma, U_1(\widetilde{u})] - [U_0, \sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}]. \end{aligned}$$

In the last step of the above computation, we have used $[\Gamma, U_0] = 0$. Now it follows that $(L(\lambda))_x = [U(\widetilde{u}, \lambda), L(\lambda)]$ if and only if $[\Gamma, U_1(\widetilde{u})] = \Big[U_0, \sum\limits_{s=1}^{N} \mu_s \phi^{(s)} \psi^{(s)T}\Big]$. It is easy to see that the latter equality is equivalent to the potential constraints shown in (4.9). Therefore, the spatial constrained flow (4.11) admits the Lax representation defined as in (5.12).

It is also direct to prove the other Lax representation $(L(\lambda))_{t_2} = [\widetilde{V}^{(2)}(\widetilde{u}, \lambda), L(\lambda)]$ for the temporal constrained flow (4.14), and so we do not go to the detail. Therefore, the proof of the theorem is finished.

It is known (see [3] for a detailed analysis) that there is a functionally independent and involutive system of polynomial functions $\{F_{is} \,|\, 1 \le i \le m+1,\ 1 \le s \le N\}$, generated from the Lax operator $L(\lambda)$ defined by (5.8). These polynomial functions are defined as follows[3]:

$$\mathcal{F}_i(\lambda) = \sum_{l=0}^{\infty} F_{il} \lambda^{-l}, \quad 1 \le i \le m+1, \tag{5.13a}$$

$$\det(\nu I_{m+1} - L(\lambda)) = \nu^{m+1} - \mathcal{F}_1(\lambda)\nu^m + \cdots + (-1)^{m+1} \mathcal{F}_{m+1}(\lambda),\ \nu = \text{const.} \tag{5.13b}$$

Now it follows from Theorem 5.1 that we have the following result on the Liouville integrability of the constrained flows (4.11) and (4.14).

Theorem 5.2. *The spatial constrained flow* (4.11) *and the temporal constrained flow* (4.14) *of the multicomponent AKNS equations* (3.17) *are Liouville integrable. They possess an involutive system of functionally independent integrals of motion* $\{F_{is} \,|\, 1 \le i \le m+1,\ 1 \le s \le N\}$, *defined by* (5.13).

Theorem 5.2 also implies that the potential constraints (4.9) present an integrable decomposition and thus show the integrability by quadratures for the multicomponent AKNS equations (3.17). Furthermore, the resulting solutions from (4.9) are involutive solutions to the multicomponent AKNS equations (3.17), because we can verify that $\{H^x, H^{t_2}\} = 0$. Let us summarize these results as follows.

Theorem 5.3. *The potential constraints* (4.9) *present an integrable decomposition from the multicomponent AKNS equations* (3.17) *to the spatial and temporal constrained flows* (4.11) *and* (4.14). *Moreover, the resulting solutions from* (4.9) *are involutive solutions to the multicomponent AKNS equations* (3.17).

§6. Concluding Remarks

We remark that the introduction of the adjoint symmetry $\bar{G}_0$ is very crucial in making integrable decompositions for the multicomponent AKNS equations (3.17). If we choose the original vector function G_0 as the required adjoint symmetry in the adjoint symmetry constraint (2.11), then the resulting constrained flows have the Lax operator which does not provide sufficiently many functionally independent integrals of motion, and thus, no Liouville integrability can be guaranteed.

Our example of the multicomponent AKNS equations also convinced us that binary nonlinearization can result from adjoint symmetry constraints. Actually, if the underlying equations possess a Hamiltonian structure, then adjoint symmetries generate symmetries under the Hamiltonian transformation and thus adjoint symmetry constraints also yield symmetry constraints which are required in binary nonlinearization[1]. However, adjoint symmetry constraints do not require any Hamiltonian structure[2], and hence, they can be applied to non-Hamiltonian equations, for example, the Burgers type equations, for which symmetry constraints do not succeed. Therefore, adjoint symmetry constraints offer a good basis for the theory of binary nonlinearization.

It is worth noting that for the Neumann type adjoint symmetry constraints, the Moser-reduced technique[21] may be used to show the Liouville integrability for the resulting constrained flows. The Ostrogradsky type adjoint symmetry constraints, in which involved

Lie-Bäcklund symmetries have non-degenerate Hamiltonians, can also result in the Liouville integrable constrained flows, by use of the Ostrogradsky coordinates[22]. But the case of degenerate Hamiltonians needs particular consideration while constructing canonical variables[23]. In this case, there is no universal way to determine Hamiltonian structures for the resulting constrained flows.

References

[1] Ma, W. X. & Strampp, W., An explicit symmetry constraint for the Lax pairs and the adjoint Lax pairs of AKNS systems [J], *Phys. Lett. A*, **185**(1994), 277–286.

[2] Ma, W. X. & Zhou, R. G., Adjoint symmetry sonstraints leading to binary nonlinearization [J], *J. Nonlinear Math. Phys.,* **9**(2002), 106–126.

[3] Ma, W. X. & Zhou, Z. X., Binary symmetry constraints of $\mathcal{N}$-wave interaction equations in $1+1$ and $2+1$ dimensions [J], *J. Math. Phys.*, **42**(2001), 4345–4382.

[4] Ablowitz, M. J. & Haberman, R., Resonantly coupled nonlinear evolution equations [J], *J. Math. Phys.*, **16** (1975), 2301–2305.

[5] Manakov, S. V., On the theory of two-dimensional stationary self-focusing of electromagnetic waves [J], *Sov. Phys. JETP*, **38**(1974), 248–253.

[6] Sasa, N. & Satsuma, J., New-type of soliton solutions for a higher-order nonlinear Schrödinger equation [J], *J. Phys. Soc. Jpn.*, **60**(1991), 409–417.

[7] Alberty, J. M., Koikawa, T. & Sasaki, R., Canonical structure of soliton equations, I [J], *Physica D*, **5:**1(1982), 43–65.

[8] Corona-Corona, G., Hamiltonian structure for degenerate AKNS systems [J], *J. Math. Phys.*, **38:**1 (1997), 212–225.

[9] Beals, R. & Coifman, R. R., Scattering and inverse scattering for first order systems [J], *Commun. Pure Appl. Math.,* **37:**1(1984), 39–90.

[10] Zhou, X., Direct and inverse scattering transforms with arbitrary spectral singularities [J], *Commun. Pure Appl. Math.*, **42:**7(1989), 895–938.

[11] Newton, R. G. & Jost, R., The construction of potentials from the S-matrix for systems of differential equations [J], *Nuovo Cimento*, **1:**10(1955), 590–622.

[12] Zeng, Y. B. & Ma, W. X., Separation of variables for soliton equations via their binary constrained flows [J], *J. Math. Phys.*, **40:**12(1999), 6526–6557.

[13] Ma, W. X. & Zhou, R. G., Binary nonlinearization of spectral problems of the perturbation AKNS systems [J], *Chaos, Soliton and Fractals*, **13**(2002), 1451–1463.

[14] Ma, W. X., Fuchssteiner, B. & Oevel, W., A 3×3 matrix spectral problem for AKNS hierarchy and its binary nonlinearization [J], *Physica A*, **233:**1-2(1996), 331–354.

[15] Ma, W. X., Ding, Q., Zhang, W. G. & Lu, B. G., Binary nonlinearization of Lax pairs of Kaup-Newell soliton hierarchy [J], *Il Nuovo Cimento B*, **111**(1996), 1135–1149.

[16] Ma, W. X. & Geng, X. G., Bäcklund transformations of soliton systems from symmetry constraints [A], in Proceedings of the workshop on Bäcklund & Darboux transformations [C], Halifax, Canada, 1999, CRM Proc. Lecture Notes, 29, Amer. Math. Soc., Providence, RI, 2001, 313–323.

[17] Ma, W. X. & Fuchssteiner, B., Binary nonlinearization of Lax pairs [A], in Nonlinear Physics: Theory and Experiment [C], Editors Alfinito, E., Boiti, M., Martina, L. & Pempinelli, F., World Scientific - River Edge, NJ, 1996, 217–224.

[18] Ablowitz, M. J., Kaup, D. J., Newell, A. C. & Segur, H., The inverse scattering transform-Fourier analysis for nonlinear problems [J], *Stud. Appl. Math.*, **53:**4(1974), 249–315.

[19] Ma, W. X., Some aspects of nonlinear integrable systems [D], Ph.D. dissertation, Academia Sinica, Beijing, China, 1990.

[20] Tu, G. Z., The trace identity, a powerful tool for constructing the Hamiltonian structure of integrable systems [J], *J. Math. Phys.*, **30**(1989), 330–338.

[21] Moser, J., Integrable Hamiltonian systems and spectral theory [A], in Proceedings of the 1983 Beijing Symposium on Differential Geometry and Differential Equations [C], Science Press, Beijing, 1986, 157–229.

[22] Deleón, M. & Rodrigues, P. R., Generalized classical mechanics and field theory [M], North-Holland - Amsterdam, 1985.

[23] Li, Y. S. & Ma, W. X., Binary nonlinearization of AKNS spectral problem under higher-order symmetry constraints [J], *Chaos, Solitons and Fractals*, **11**(2000), 697–710.