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TOEPLITZ OPERATORS AND ALGEBRAS ON DIRICHLET SPACES**

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Abstract

The automorphism group of the Toeplitz C^*- algebra, $\mathcal{J}(C^1)$, generated by Toeplitz operators with C^1-symbols on Dirichlet space $\mathcal{D}$ is discussed; the K_0, K_1-groups and the first cohomology group of $\mathcal{J}(C^1)$ are computed. In addition, the author proves that the spectra of Toeplitz operators with C^1-symbols are always connected, and discusses the algebraic properties of Toeplitz operators. In particular, it is proved that there is no nontrivial selfadjoint Toeplitz operator on $\mathcal{D}$ and $T_\varphi^* = T_{\bar{\varphi}}$ if and only if T_φ is a scalar operator.

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§1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$, $dA(z) = \frac{1}{\pi}dxdy$ be the normalized Lebesque measure on $\mathbb{D}$. The Sobolev space $L^{2,1}$ is defined to be the collection of functions on $\mathbb{D}$ which satisty

$$\int_{\mathbb{D}} \Big(\Big|\frac{\partial u}{\partial z}\Big|^2 + \Big|\frac{\partial u}{\partial \bar{z}}\Big|^2 + |u|^2\Big) dA < \infty, \qquad \forall u \in L^{2,1}.$$

Define a sesquilinear form on $L^{2,1}$ as

$$\langle u, v\rangle_{\frac{1}{2}} = \Big\langle \frac{\partial u}{\partial z}, \frac{\partial v}{\partial z}\Big\rangle_{L^2(dA)} + \Big\langle \frac{\partial u}{\partial \bar{z}}, \frac{\partial v}{\partial \bar{z}}\Big\rangle_{L^2(dA)},$$

where $L^2(dA)$ is usually Lebesque space with respect to dA. Then $\langle \cdot, \cdot\rangle_{\frac{1}{2}}$ induces a seminorm on $L^{2,1}$:

$$\|u\|_{\frac{1}{2}} = \Big[\int_{\mathbb{D}} \Big(\Big|\frac{\partial u}{\partial z}\Big|^2 + \Big|\frac{\partial u}{\partial \bar{z}}\Big|^2\Big) dA/\pi\Big]^{\frac{1}{2}}.$$

Write $N = \{u \in L^{2,1} | \|u\|_{\frac{1}{2}} = 0\}$, then $N = \mathbb{C}$ by the properties of generalized derivatives. Thus $\|\cdot\|_{\frac{1}{2}}$ is a norm on $L^{2,1}/\mathbb{C}$. For convenience, denote still the completion of $L^{2,1}/\mathbb{C}$ with respect to above norm by $L^{2,1}$. The Dirichlet space, $\mathcal{D}$, is the subspace of all analytic functions g in $L^{2,1}$. Without loss of generality, we may assume $\mathcal{D}$ is the space of all analytic functions g with $g(0) = 0$. In other words, $\mathcal{D}$ is the space of all analytic functions g whose derivations g' belong to the Bergman space L^2_a. L^2_a is here the space of all analytic functions which are in $L^2(dA)$. Throughout this paper, we use respectively the symbols $\langle \cdot, \cdot\rangle, \|\cdot\|$ to

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denote the inner product and norm in $L^2(dA)$, and $\langle\cdot,\cdot\rangle_{\frac{1}{2}}$, $\|\cdot\|_{\frac{1}{2}}$ to those in $\mathcal{D}$. Let P be the orthogonal projection from $L^{2,1}$ onto $\mathcal{D}$, P is an integral operator represented by

$$P(u)(w)=\langle u,K\rangle_{\frac{1}{2}}=\int_{\mathbb{D}}\frac{\partial u}{\partial z}\overline{\frac{\partial K(z,w)}{\partial z}}dA(z),$$

where $K=K(z,w)=\log\frac{1}{1-z\bar{w}}$ is the reproducing kernel of $\mathcal{D}$. For $\varphi\in C^1(\bar{\mathbb{D}})$, the Toeplitz operator with symbol φ is defined as

$$T_\varphi f=P(\varphi f)\qquad(\forall f\in\mathcal{D}).$$

In recent years, Toeplitz operators on Dirichlet spaces have been studied by some specialists (cf. [1,2]). In [3], we obtained some interesting properties of these operators which are similar to those of Toeplitz operators on Hardy or Bergman spaces. However, some surprising differences between the Toeplitz operators on Dirichlet space and those on Hardy or Bergman spaces are displayed in [4]. In this paper, we first discuss the automorphism group of the Toeplitz C^*-algebra generated by Toeplitz operators with symbols in $C^1(\bar{\mathbb{D}})$, and compute the first cohomology group of the algebra. In addition, the K_0,K_1-groups of this algebra are also computed. In Section 3, we prove that there is no nontrivial selfadjoint Toeplitz operator with C^1-symbol and no Toeplitz operator which satisfies $T^*_\varphi=T_{\bar{\varphi}}$ only if it is a scalar operator. Finally, we obtain that all Toeplitz operators with C^1-symbols have connected spectra.

§2. The Algebra Generated by Toeplitz Operators with Symbols in $C^1(\bar{\mathbb{D}})$

Write $\mathcal{J}(C^1)$ as the C^*-algebra generated by Toeplitz operators with symbols in $C^1(\bar{\mathbb{D}})$. In [3], we proved that the following short sequence

$$\{0\}\longrightarrow\mathcal{K}\xrightarrow{i}\mathcal{J}(C^1)\xrightarrow{\xi}C(\mathbb{T})\longrightarrow\{0\}\tag{2.1}$$

is exact, and $\xi(T_\varphi)=\varphi|_{\mathbb{T}}(\varphi\in C^1(\bar{\mathbb{D}})$ induces a $*$-isometric isomorphism from $\frac{\mathcal{J}(C^1)}{\mathcal{K}}$ onto $C(\mathbb{T})$ (we still use ξ to denote the isomorphism from $\frac{\mathcal{J}(C^1)}{\mathcal{K}}$ onto $C(\mathbb{T})$). Using this result, we obtained following index formula:

Lemma 2.1[3] *Suppose $\varphi\in C^1(\bar{\mathbb{D}})$. If T_φ is Fredholm on $\mathcal{D}$, then*

$$\operatorname{Ind}T_\varphi=-\operatorname{wind}\varphi|_{\mathbb{T}}.$$

It should be pointed out that the conclusion can not be shown by usual homotopy relation between symbols in $C(\mathbb{T})$. In fact, there are functions in $C(\mathbb{T})$ which have no C^1-extensions onto $\bar{\mathbb{D}}$. For example:

Example 2.1. There exists $\varphi\in C(\mathbb{T})$ such that for any $\widetilde{\varphi}\in C^1(\bar{\mathbb{D}})$, $\widetilde{\varphi}|_{\mathbb{T}}\neq\varphi$.

Proposition 2.1. *Suppose $\varphi\in GC(\mathbb{T})$, the set of invertible elements in $C(\mathbb{T})$. Then for any $T\in\xi^{-1}(\varphi)$, we have index formula* $\operatorname{Ind}T=-\operatorname{wind}\varphi$.

Proof. Noting two Toeplitz operators with C^1-symbols are essentially commutative, we know that there is a Toeplitz operator sequence $\{T_{\varphi_n}\}$ with $\varphi_n\in C^1(\bar{\mathbb{D}})$ such that $\|[T_{\varphi_n}]-\xi^{-1}(\varphi)\|\to 0$, thus

$$\|\varphi_n|_{\mathbb{T}}-\varphi\|_\infty=\|\xi([T_{\varphi_n}])-\xi(\xi^{-1}(\varphi))\|_\infty=\|[T_{\varphi_n}]-\xi^{-1}(\varphi)\|\to 0.$$

It is obvious that φ_n is invertible in $C(\mathbb{T})$ for enough large n. By the stability of indexes, we see easily that there is an N such that $\operatorname{wind}\varphi_n|_{\mathbb{T}}=\operatorname{wind}\varphi$ for each $n\geq N$. Hence for every $T\in\xi^{-1}(\varphi)$,

$$\lim_n\operatorname{Ind}T_{\varphi_n}=-\lim_n\operatorname{wind}\varphi_n|_{\mathbb{T}}=-\operatorname{wind}\varphi$$

by Lemma 2.1.

Theorem 2.1. *Suppose $\alpha \in \mathrm{Aut}\,(\mathcal{J}(C^1))$, the automorphism group of $\mathcal{J}(C^1)$. Then there is a $\sigma \in \mathrm{homeo}\,(\mathbb{T})$ which has winding number 1 such that*

$$\xi\hat{\alpha}\xi^{-1} = C_\sigma, \tag{2.2}$$

where C_σ is the composition operator on $C(\mathbb{T})$ defined as $C_\sigma\varphi = \varphi \circ \sigma$, $\hat{\alpha}$ is the automorpgism on $\frac{\mathcal{J}(C^1)}{\mathcal{K}}$ induced by α, that is, $\hat{\alpha}([T]) = [\alpha(T)]$ for any $T \in \mathcal{J}(C^1)$.

Conversely, if $\sigma \in \mathrm{homeo}(\mathbb{T})$ with $\mathrm{wind}\,\sigma = 1$, then there is an $\alpha \in \mathrm{Aut}\,(\mathcal{J}(C^1))$ such that the equation (2.2) holds.

Proof. Let $\alpha \in \mathrm{Aut}\,(\mathcal{J}(C^1))$. Since $\mathcal{K} \subset \mathcal{J}(C^1)$, there is a unitary operator U such that $\alpha(T) = U^*TU$ for each $T \in \mathcal{J}(C^1)$. Note $\mathrm{Aut}\,(\frac{\mathcal{J}(C^1)}{\mathcal{K}}) \cong \mathrm{Aut}\,(C(\mathbb{T}))$, so there is a $\sigma \in \mathrm{homeo}\,(\mathbb{T})$ such that $\xi(\hat{\alpha}[T]) = \xi([T]) \circ \sigma$, thus $\hat{\alpha}([T]) = \xi^{-1}(\xi([T]) \circ \sigma)$, further $\xi\hat{\alpha}\xi^{-1}(\varphi) = \varphi \circ \sigma = C_\sigma\varphi \quad (\forall \varphi \in C(\mathbb{T}))$.

To prove that $\mathrm{wind}\,\sigma = 1$, let $\varphi_0(z) = z$. Then $\xi\hat{\alpha}\xi^{-1}(\varphi_0) = C_\sigma\varphi_0 = \sigma$. It is clear that $T_z \in \xi^{-1}(\varphi_0)$, which shows that $U^*T_zU - T \in \mathcal{K}$ for any $T \in \xi^{-1}(\sigma)$. Hence $\mathrm{Ind}\,T = \mathrm{Ind}\,T_z = -1$. By Prposition 2.1, we have $\mathrm{wind}\,\sigma = 1$ since $\xi(T) = \sigma$.

Conversely, assume $\sigma \in \mathrm{homeo}\,(\mathbb{T})$ with $\mathrm{wind}\,\sigma = 1$. Then the map $\varphi \to \varphi \circ \sigma$ defines an automorphism of $C(\mathbb{T})$, thus $\xi^{-1}(\varphi) \to \xi^{-1}(\varphi \circ \sigma)$ defines an automorphism of $\frac{\mathcal{J}(C^1)}{\mathcal{K}}$. If $\varphi \in GC(\mathbb{T})$, then for each $T \in \xi^{-1}(\varphi)$ and $\widetilde{T} \in \xi^{-1}(\varphi\circ\sigma)$, we have $\mathrm{Ind}\,T = -\mathrm{wind}\,\varphi$, $\mathrm{Ind}\,\widetilde{T} = -\mathrm{wind}\,\varphi \circ \sigma = -\mathrm{wind}\,\varphi$ since $\mathrm{wind}\,\sigma = 1$. In particular, for $T \in \xi^{-1}(\varphi_0), \widetilde{T} \in \xi^{-1}(\sigma)$, $\mathrm{Ind}\,T = \mathrm{Ind}\,\widetilde{T} = -1$, it is not difficult to see that $\sigma_e(T) = \sigma_e(T_z) = \mathbb{T}$, $\sigma_e(\widetilde{T}) = \sigma(\mathbb{T}) = \mathbb{T}$ (in fact, $[\widetilde{T} - \lambda]$ is invertible in $\frac{\mathcal{J}(C^1)}{\mathcal{K}}$ if and only if $\xi([\widetilde{T} - \lambda]) = \xi([\widetilde{T}]) - \xi([\lambda I]) = \sigma - \lambda$ is invertible in $C(\mathbb{T})$). By Brown-Douglas-Fillmore theorem, we know that $T_z \overset{e.u.}{\sim} \widetilde{T}$, that is, there is a unitary U such that $U^*T_zU - \widetilde{T} \in \mathcal{K}$, further $U^*T_z^*U - \widetilde{T}^* \in \mathcal{K}$. Noting $T_z^* - T_{\bar{z}} \in \mathcal{K}$, we have also $U^*T_{\bar{z}}U - \widetilde{T}^* \in \mathcal{K}$. Since ξ is a $*$-isomorphism, $\xi([\widetilde{T}^*]) = \bar{\sigma}$, hence, for any polynormial $p(z, \bar{z})$ of $z, \bar{z}$, we have $U^*T_pU - T \in \mathcal{K}$ for every $T \in \xi^{-1}(p \circ \sigma) = \xi^{-1}(p(\sigma, \bar{\sigma}))$. For any $\varphi \in C(\mathbb{T})$, we can find a polynormial sequence $\{p_k\}$ such that $\|p_k - \varphi\|_\infty \to 0$, thus $\|p_k \circ \sigma - \varphi \circ \sigma\|_\infty \to 0$, consequently,

$$\|\xi^{-1}(p_k) - \xi^{-1}(\varphi)\| \to 0, \quad \|\xi^{-1}(p_k \circ \sigma) - \xi^{-1}(\varphi \circ \sigma)\| \to 0.$$

By above argument, we know that for any $T_k \in \xi^{-1}(p_k)$ and $\widetilde{T}_k \in \xi^{-1}(p_k \circ \sigma)$, $U^*T_kU - \widetilde{T}_k \in \mathcal{K}$, thus $[U^*T_kU - \widetilde{T}_k] = 0$. Set $\alpha(T) = U^*TU$ for each $T \in \mathcal{J}(C^1)$. Then $\alpha \in Aut(\mathcal{J}(C^1))$, and $\hat{\alpha}(\xi^{-1}(p_k)) = \xi^{-1}(p_k \circ \sigma)$, thus $\xi\hat{\alpha}\xi^{-1}(p_k) = p_k \circ \sigma = C_\sigma p_k$. Note

$$\|\xi(\hat{\alpha}\xi^{-1}(\varphi))\|_\infty = \|\hat{\alpha}\xi^{-1}(\varphi)\| \le \|\xi^{-1}(\varphi)\| = \|\varphi\|_\infty,$$

hence $\xi\hat{\alpha}\xi^{-1}(\varphi) = C_\sigma\varphi$. The theorem is thus complete.

Lemma 2.2. *There is no multiplicative linear function on $\mathcal{K}$.*

Proof. It is obvious since $\mathcal{K}$ is generated by commutators of $\mathcal{J}(C^1)$.

Theorem 2.2. *Let $\mathcal{M}$ be the space of multiplicative functions on $\mathcal{J}(C^1)$. Then for any $\varphi \in (\mathcal{J}(C^1))^*$, $\varphi \in \mathcal{M}$ if and only if there is a unique $\zeta \in \mathbb{T}$ such that $\varphi = \varphi_\zeta \circ \xi \circ \pi$, where φ_ζ is the multiplicative linear function which is defined by $\varphi_\zeta(f) = f(\zeta)$ for each $f \in C(\mathbb{T})$ and π is the classical map from $\mathcal{J}(C^1)$ to $\frac{\mathcal{J}(C^1)}{\mathcal{K}}$.*

Proof. It is clear that for each $\zeta \in \mathbb{T}$, $\varphi = \varphi_\zeta \circ \xi \circ \pi$ defines a multiplicative linear function on $\mathcal{J}(C^1)$ which satisfies $\varphi|_{\mathcal{K}} = 0$.

Conversely, if $\varphi \in \mathcal{M}$, then $\varphi|_{\mathcal{K}} = 0$ by Lemma 2.2. Set $\widetilde{\varphi}([T]) = \varphi(T)$ for any $[T] \in$

$\frac{\mathcal{J}(C^1)}{\mathcal{K}}$. Then $\widetilde{\varphi}$ is well-defined and $\widetilde{\varphi} \circ \pi = \varphi$. Since $\frac{\mathcal{J}(C^1)}{\mathcal{K}} \overset{\xi}{\cong} C(\mathbb{T})$, we see that $\widetilde{\varphi} \circ \xi^{-1}$ is a multiplicative linear function on $C(\mathbb{T})$. Thus there is a unique $\zeta \in \mathbb{T}$ such that $\widetilde{\varphi} \circ \xi^{-1} = \varphi_\zeta$. Furthermore, $\varphi_\zeta \circ \xi \circ \pi = \widetilde{\varphi} \circ \xi^{-1} \circ \xi \circ \pi = \widetilde{\varphi} \circ \pi = \varphi$.

Theorem 2.3. $K_0(\mathcal{J}(C^1)) \cong \mathbb{Z}, K_1(\mathcal{J}(C^1)) = 0$.

Proof. By the short exact sequence (2.1), we have following six term exact sequence:

$$\mathbb{Z} = K_0(\mathcal{K}) \xrightarrow{i^*} K_0(\mathcal{J}(C^1)) \xrightarrow{\xi^*} K_0(C(\mathbb{T})) \cong \mathbb{Z},$$

$$\uparrow \qquad\qquad\qquad\qquad \downarrow$$

$$\mathbb{Z} = K_1(C(\mathbb{T})) \xleftarrow{\xi^*} K_1(\mathcal{J}(C^1)) \xleftarrow{i^*} K_1(\mathcal{K}) = \{0\}.$$

Note index map $\chi : K_1(C(\mathbb{T})) \to K_0(\mathcal{K})$ is an isomorphism, hence we have

$$K_1(\mathcal{J}(C^1) = \{0\}$$

by the exactness of above sequence. Further the sequence $0 \to \mathbb{Z} \xrightarrow{\chi} \mathbb{Z} \xrightarrow{i^*} K_0(\mathcal{J}(C^1)) \xrightarrow{\xi^*} \mathbb{Z} \to 0$ is exact, thus the range of i^*, $R(i^*)$, equals $\{0\}$. Consequently, $\operatorname{Ker} \xi^* = \{0\}$, this shows that the sequence $0 \to K_1(\mathcal{J}(C^1)) \xrightarrow{\xi^*} \mathbb{Z} \to 0$ is exact. That is $K_1(\mathcal{J}(C^1)) \cong \mathbb{Z}$. We are done.

Remark 2.1. Theorems 2.1 and 2.2 can be extended to the case of Dirichlet space on unit ball of $\mathbb{C}^n$. We need only to note that if $\sigma \in \operatorname{homeo}(S^n)$ with degree$\sigma = 1$ (S^n is the unit sphere of $\mathbb{C}^n$), $T_z = (T_{z_1}, \cdots, T_{z_n})$ is joint essentially unitary equivalent to $T_\sigma = (T_{\sigma_1}, \cdots, T_{\sigma_n})$, where $\sigma = (\sigma_1, \cdots, \sigma_n)$ (see [5] for detail). Thus the proof of Theorem 2.1 is true for the case of several complex variables if we consider the Toeplitz operator with matrix symbols which have entries in $C^1(\bar{B_n})$, where B_n is the unit ball of $\mathbb{C}^n$.

Recall a linear derivation on an algebra $\mathcal{B}$ is a linear map δ from $\mathcal{B}$ into $\mathcal{B}$ which satisfies that $\delta(fg) = \delta(f)g + f\delta(g)$. Sakai's theorem says that each linear derivation on C^*- algebra is continuous (see [6]). If there is an $f \in \mathcal{B}$ such that $\delta(g) = fg - gf$ for any $g \in \mathcal{B}$, then δ is said to be an inner derivation on $\mathcal{B}$, otherwise, δ is called an outer derivation. Ringrose's deep result indicates that there are only inner derivations on any Von-Neumann algebras. In the case of C^*-algebras, there may be many outer derivations; for instance, each bounded linear operator (on a Hilbert space H) induces a derivation on the compact operator ideal, hence the first cohomology group of $\mathcal{K}$ is $\frac{L(H)}{\mathcal{K}}$. In sequal of this section, we will compute the first cohomology group of $\mathcal{J}(C^1)$.

Proposition 2.2. *Suppose δ is a linear derivation on $\mathcal{J}(C^1)$. Then the range of δ is contained in $\mathcal{K}$.*

Proof. Since the convex combination of unitary elements in $\mathcal{J}(C^1)$ is dense in the unit ball of $\mathcal{J}(C^1)$ (see [7]), we need only to prove that $\delta(U) \in \mathcal{K}$ for any unitary operator U in $\mathcal{J}(C^1)$. Assume $U \in \mathcal{J}(C^1)$ is a unitary operator. Then there is a $\varphi_U \in C(\mathbb{T})$ such that $\xi([U]) = \varphi_U$. By $U^*U = I$, we have

$$|\varphi_U|^2 = \bar{\varphi_U}\varphi_U = \xi([U^*])\xi([U]) = \xi([I]) = 1,$$

which shows that φ_U is a unimodular function. Write $\delta(U) = T_U$. Then there is a $\varphi_{T_U} \in C(\mathbb{T})$ such that $\xi([T_U]) = \varphi_{T_U}$. Noting all elements in $\mathcal{J}(C^1)$ are essentially commutative , we see that $[\delta(U^n)] = n[U^{n-1}\delta(U)]$, hence

$$\xi([\delta(U^n)]) = n\xi([U^{n-1}])\xi([\delta(U)]) = n\varphi_U^{n-1}\varphi_{T_U}.$$

Further $\|\xi([\delta(U^n)])\|_\infty = n\|\varphi_{T_U}\|_\infty$ since $|\varphi_U| = 1$. By $\|U^n\| \le 1$, we have

$$\|\xi([\delta(U^n)])\| = \|[\delta(U^n)]\| \le \|\delta(U^n)\| \le \|\delta\|\|U^n\| = \|\delta\| < \infty \quad \text{for any} \quad n \in \mathbb{N},$$

which follows that $\varphi_{T_U} = 0$, so that $\delta(U) = T_U \in \mathcal{K}$ (since ξ is an isometric isomorphism from $\frac{\mathcal{J}(C^1)}{\mathcal{K}}$ onto $C(\mathbb{T})$). We are done.

Let $B(\mathcal{J}(C^1), \mathcal{J}(C^1))$ be the set of all derivations from $\mathcal{J}(C^1)$ into itself, and $Z(\mathcal{J}(C^1), \mathcal{J}(C^1))$ be the set of all inner derivations on $\mathcal{J}(C^1)$. $H^1(\mathcal{J}(C^1), \mathcal{J}(C^1)) = \frac{B(\mathcal{J}(C^1), \mathcal{J}(C^1))}{Z(\mathcal{J}(C^1), \mathcal{J}(C^1))}$ is said to be the first cohomology group of $\mathcal{J}(C^1)$. (The definition on higher order cohomology groups of Banach algebras can be found in [8].)

Theorem 2.4. $H^1(\mathcal{J}(C^1),\mathcal{J}(C^1)) \cong \frac{\{T_z\}_e'}{\mathcal{J}(C^1)}$.

Proof. For any $f,g \in \mathcal{J}(C^1)$, if $\delta_f = \delta_g$, then for every $h \in \mathcal{J}(C^1)$ we have $\delta_f(h) = \delta_g(h)$, thus $(g-f)h = h(g-f)$, which shows that $g-f \in \{\mathcal{J}(C^1)\}'$, the commutant of $\mathcal{J}(C^1)$. However, it is not difficult to see that $\{\mathcal{J}(C^1)\}' = \mathbb{C}I$. Hence $g-f = \lambda I$ for some $\lambda \in \mathbb{C}$; further $Z(\mathcal{J}(C^1),\mathcal{J}(C^1)) \cong \frac{\mathcal{J}(C^1)}{\mathbb{C}I}$. By Proposition 2.2, if $\delta \in B(\mathcal{J}(C^1),\mathcal{J}(C^1))$, then δ maps $\mathcal{J}(C^1)$ into $\mathcal{K}$. Note δ is induced by a bounded operator T on $\mathcal{D}$ (see [9]), that is $\delta(f) = fT - Tf$ for any $f \in \mathcal{J}(C^1)$, so $T \in \{\mathcal{J}(C^1)\}_e'$, the essential commutant of $\mathcal{J}(C^1)$. An easy checking shows that $\{\mathcal{J}(C^1)\}_e' = \{T_z\}_e'$. In fact, it is obviously that $\{\mathcal{J}(C^1)\}_e' \subset \{T_z\}_e'$. For each $T \in \{T_z\}_e'$, if we prove that T commutes essentially with $T_{\bar z}$, then T commutes essentially with each element in $\mathcal{J}(C^1)$. Noting $T_{\bar z}T_z = T_{|z|^2} = I$ (see [4]), $T_zT_{\bar z} = I + K (K \in \mathcal{K})$, we see that

$$T_{\bar z}(T_zT - TT_z)T_{\bar z} = T_{\bar z}T_zTT_{\bar z} - T_{\bar z}TT_zT_{\bar z} = TT_{\bar z} - T_{\bar z}T + K_1 \ (K_1 \in \mathcal{K}),$$

so $TT_{\bar z} - T_{\bar z}T$ is compact. Furthermore, $T \in \{\mathcal{J}(C^1)\}_e'$. It follows that $\{\mathcal{J}(C^1)\}_e' = \{T_z\}_e'$. Consequently, $B(\mathcal{J}(C^1),\mathcal{J}(C^1)) \cong \frac{\{T_z\}_e'}{\mathbb{C}I}$, hence $H^1(\mathcal{J}(C^1),\mathcal{J}(C^1)) \cong \frac{\{T_z\}_e'}{\mathcal{J}(C^1)}$. The theorem is thus complete.

§3. Algebraic Properties of Toeplitz Operators

In [3,4], we have seen that there are many differences between Toeplitz operators on $\mathcal{D}$ and those on Hardy or Bergman spaces. A surprising differnce is that a non-zero function may induce a null Toeplitz operator. In fact, we proved following

Theorem 3.1.[4] *Suppose $\varphi \in C^1(\bar{\mathbb{D}})$. Then the following statements are equivalent.*
(1) $T_\varphi = 0$; (2) T_φ *is compact;* (3) $\varphi|_{\mathbb{T}} = 0$.

In this section, we discuss continuously some properties of these operators.

It is well-known that there are a lot of selfadjoint Toeplitz operators on Hardy and Bergman spaces. A natural problem is: Are there also selfadjoint Toeplitz operators on Dirichlet space? The following theorem gives a complete answer for Toeplitz operators with C^1-symbols.

Theorem 3.2. *Suppose $\varphi \in C^1(\bar{\mathbb{D}})$. Then T_φ is selfadjoint if and only if $\varphi|_{\mathbb{T}} = \bar\varphi|_{\mathbb{T}} = c \in \mathbb{R}$. Furthermore. There are no non-trivial selfadjoint Toeplitz operators on $\mathcal{D}$.*

Proof. Suppose T_φ is selfadjoint. Then for any $k,j \ge 1$, we have $\langle T_\varphi z^k, z^j\rangle_{\frac12} = \langle z^k, T_\varphi z^j\rangle_{\frac12}$. Let $p_l = \sum a_{nm}^{(l)} z^n \bar z^m$ be a polynormial which satisfies

$$\|p_l - \varphi\|_\infty \to 0, \quad \Big\|\frac{\partial p_l}{\partial z} - \frac{\partial \varphi}{\partial z}\Big\|_\infty \to 0 \quad \text{as} \quad l \to \infty.$$

It is easy to check that $\|T_\psi\| \le \|\psi\|_\infty + \|\frac{\partial\psi}{\partial z}\|_\infty$ for each $\psi \in C^1(\bar{\mathbb{D}})$, thus $\|T_{p_l} - T_\varphi\| \to 0 \quad (l \to \infty)$. Further

$$\langle T_\varphi z^k, z^j\rangle_{\frac12} = \lim_{l\to\infty}\langle T_{p_l} z^k, z^j\rangle_{\frac12} = \Big(\lim_{l\to\infty}\sum_{n+k=j+m} a_{nm}^{(l)}\Big) j,$$

$$\langle z^k, T_\varphi z^j\rangle_{\frac12} = \lim_{l\to\infty}\langle z^k, T_{p_l} z^j\rangle_{\frac12} = \Big(\lim_{l\to\infty}\sum_{n+j=k+m} \overline{a_{nm}^{(l)}}\Big) k.$$

Write $t = k - j$. Then

$$\Big(\lim_{l\to\infty}\sum_n a_{n,n+t}^{(l)}\Big)(j+ht) = \Big(\lim_{l\to\infty}\sum_n \overline{a_{n,n-t}^{(l)}}\Big)(k+ht) \quad (\forall h \in \mathbb{Z}^+)$$

by $\langle T_\varphi z^{k+ht}, z^{j+ht}\rangle_{\frac{1}{2}} = \langle z^{k+ht}, T_\varphi z^{j+ht}\rangle_{\frac{1}{2}}$. If $\lim\limits_{l\to\infty}\sum\limits_n a^{(l)}_{n,n+t} \neq 0$, then $\lim\limits_{l\to\infty}\sum\limits_n \overline{a^{(l)}_{n,n-t}} \neq 0$. Thus for each $h \in \mathbb{Z}^+$, we have $\frac{j}{j+ht} = \frac{k}{k+ht}$, consequently, $j = k$. This shows that $\lim\limits_{l\to\infty}\sum a^{(l)}_{n,n+t} = 0$ when $t \neq 0$, hence

$$\langle T_\varphi z^k, z^j\rangle_{\frac{1}{2}} = \Big(\lim_{l\to\infty}\sum a^{(l)}_{n,n+t}\Big)j = 0 \quad \text{when} \quad k \neq j.$$

Note $\|\varphi|_{\mathbb{T}} - p_l|_{\mathbb{T}}\|_\infty \to 0$ and

$$\int_0^{2\pi} \varphi|_{\mathbb{T}} e^{it\theta} d\theta = \lim_{l\to\infty}\int_0^{2\pi} p_l|_{\mathbb{T}} e^{it\theta} d\theta = \lim_{l\to\infty}\sum_n a^{(l)}_{n,n+t} = 0 \quad \text{for} \quad t \neq 0.$$

So $\varphi|_{\mathbb{T}}$ =constant; in fact $\varphi|_{\mathbb{T}} = \lim\limits_{l\to\infty}\sum\limits_n a^{(l)}_{nn}$. Similarly, by

$$\langle z^k, T_\varphi z^j\rangle_{\frac{1}{2}} = \Big(\lim_{l\to\infty}\sum_{n+j=k+m}\overline{a^{(l)}_{nm}}\Big)k = 0 \quad \text{for} \quad k \neq j,$$

we see that $\bar\varphi|_{\mathbb{T}} = \lim\limits_{l\to\infty}\sum\limits_n \overline{a^{(l)}_{nn}}$ =constant. On the other hand, for $k = j$, we have

$$\langle T_\varphi z^k, z^j\rangle_{\frac{1}{2}} = \Big(\lim_{l\to\infty}\sum_n a^{(l)}_{nn}\Big)k = \langle z^k, T_\varphi z^k\rangle_{\frac{1}{2}} = \Big(\lim_{l\to\infty}\sum_n \overline{a^{(l)}_{nn}}\Big)k,$$

hence $\lim\limits_{l\to\infty}\sum\limits_n a^{(l)}_{nn} = \lim\limits_{l\to\infty}\sum\limits_n \overline{a^{(l)}_{nn}}$. That is, $\varphi|_{\mathbb{T}} = \bar\varphi|_{\mathbb{T}}$ =constant.

Conversely, if $\varphi|_{\mathbb{T}} = \bar\varphi|_{\mathbb{T}} = c \in \mathbb{R}$, then $T_{\varphi-\bar\varphi} = T_{\varphi-c} = 0$ by Theorem 3.1. Thus $T_\varphi = cI$. The theorem is then complete.

We have known that $T^*_\varphi \neq T_{\bar\varphi}$ in general. The following stronger conclusion indicates that T^*_φ is never $T_{\bar\varphi}$ whenever T_φ is not a scalar operator.

Proposition 3.1. *Suppose $\varphi \in C^1(\bar{\mathbb{D}})$. Then $T^*_\varphi = T_{\bar\varphi}$ if and only if $\varphi|_{\mathbb{T}}$ =constant.*

Proof. For any $f, g \in \mathcal{D}$, we have

$$\langle T^*_\varphi f, g\rangle_{\frac{1}{2}} = \langle f, T_\varphi g\rangle_{\frac{1}{2}} = \langle f, \varphi g\rangle_{\frac{1}{2}} = \langle f', \varphi g'\rangle + \Big\langle f' + \frac{\partial\varphi}{\partial z} g\Big\rangle;$$

$$\langle T_{\bar\varphi} f, g\rangle_{\frac{1}{2}} = \langle \bar\varphi f, g\rangle_{\frac{1}{2}} = \langle \bar\varphi f', g'\rangle + \Big\langle \frac{\partial\bar\varphi}{\partial z} f, g'\Big\rangle.$$

If $\langle T^*_\varphi f, g\rangle_{\frac{1}{2}} = \langle T_{\bar\varphi} f, g\rangle_{\frac{1}{2}}$, then

$$\langle f', \varphi g'\rangle + \Big\langle f' + \frac{\partial\varphi}{\partial z} g\Big\rangle = \langle \bar\varphi f', g'\rangle + \Big\langle \frac{\partial\bar\varphi}{\partial z} f, g'\Big\rangle,$$

thus

$$\Big\langle f', \frac{\partial\varphi}{\partial z} g\Big\rangle = \Big\langle \frac{\partial\bar\varphi}{\partial z} f, g'\Big\rangle = \Big\langle \overline{\frac{\partial\varphi}{\partial\bar z}} f, g'\Big\rangle.$$

In particular, for any $k, j \in \mathbb{Z}^+ - \{0\}$,

$$k\Big\langle z^{k-1}, \frac{\partial\varphi}{\partial z} z^j\Big\rangle = j\Big\langle \frac{\partial\bar\varphi}{\partial z} z^k, z^{j-1}\Big\rangle.$$

Suppose $p_l = \sum\limits_{nm} a^{(l)}_{nm} z^n \bar z^m$, $q_l = \sum\limits_{nm} b^{(l)}_{nm} z^n \bar z^m$ are polynormial sequences such that

$$\|p_l - \varphi\|_\infty \longrightarrow 0, \quad \Big\|\frac{\partial p_l}{\partial z} - \frac{\partial\varphi}{\partial z}\Big\|_\infty \longrightarrow 0,$$

$$\|q_l - \bar\varphi\|_\infty \longrightarrow 0, \quad \Big\|\frac{\partial q_l}{\partial z} - \frac{\partial\bar\varphi}{\partial z}\Big\|_\infty \longrightarrow 0.$$

Thus

$$k\Big\langle z^{k-1},\frac{\partial\varphi}{\partial z}z^j\Big\rangle=\lim_{l\to\infty}\sum_{nm}k\langle z^{k-1},na^{(l)}_{nm}z^{n-1}\bar z^m z^j\rangle=\lim_{l\to\infty}\sum_{nm}nk\overline{a^{(l)}_{nm}}\langle z^{k+m-1},z^{n+j-1}\rangle$$

$$=\lim_{l\to\infty}\Big(\sum_n n\overline{a^{(l)}_{n(n+t)}}\frac{1}{k+n+t}\Big)k\quad (t=j-k),$$

$$j\Big\langle\frac{\partial\bar\varphi}{\partial z}z^k,z^{j-1}\Big\rangle=\lim_{l\to\infty}\sum_{nm}nb^{(l)}_{nm}j\langle z^{n+k-1},z^{j+m-1}\rangle=\lim_{l\to\infty}\sum_n nb^{(l)}_{n(n-t)}j\frac{1}{n+k}\qquad (t=j-k)$$

$$=\lim_{l\to\infty}\sum_n(n+t)b^{(l)}_{(n+t)n}j\frac{1}{n+k+t}.$$

Note

$$\|\varphi|_{\mathbb T}-p_l|_{\mathbb T}\|_{L^2(\mathbb T)}\le\|\varphi-p_l\|_\infty\longrightarrow 0\qquad(\text{as}\ \ l\to\infty),$$
$$\|\bar\varphi|_{\mathbb T}-q_l|_{\mathbb T}\|_{L^2(\mathbb T)}\le\|\bar\varphi-q_l\|_\infty\longrightarrow 0\qquad(\text{as}\ \ l\to\infty),$$

hence

$$\sum_{t=-\infty}^{\infty}\Big|\sum_n(a^{(l)}_{n(n+t)}-\overline{b^{(l)}_{(n+t)n}})\Big|^2\longrightarrow 0\qquad(\text{as}\ \ l\to\infty).$$

Further

$$\begin{aligned}0&=k\Big\langle z^{k-1},\frac{\partial\varphi}{\partial z}z^j\Big\rangle-j\Big\langle\frac{\partial\bar\varphi}{\partial z}z^k,z^{j-1}\Big\rangle\\
&=\lim_{l\to\infty}\Big[\Big(\sum_n n\overline{a^{(l)}_{n(n+t)}}\frac{1}{n+k+t}\Big)k-\Big(\sum_n(n+t)b^{(l)}_{(n+t)n}\frac{1}{n+k+t}\Big)j\Big]\\
&=\lim_{l\to\infty}\Big[\sum_n n\overline{a^{(l)}_{n(n+t)}}\frac{1}{n+k+t}(k-j)\\
&\quad+\sum_n n\overline{a^{(l)}_{n(n+t)}}\frac{1}{n+k+t}j-\sum_n(n+t)b^{(l)}_{(n+t)n}\frac{1}{n+k+t}j\Big]\\
&=\lim_{l\to\infty}\Big[\Big(-\sum_n n\overline{a^{(l)}_{n(n+t)}}\frac{1}{n+k+t}t-\sum_n\overline{a^{(l)}_{n(n+t)}}\frac{1}{n+k+t}tj\Big)\\
&\quad+\sum_n n\overline{a^{(l)}_{n(n+t)}}\frac{1}{n+k+t}j-\sum_n nb^{(l)}_{(n+t)n}\frac{1}{n+k+t}j\\
&\quad+\sum_n\overline{a^{(l)}_{n(n+t)}}\frac{1}{n+k+t}tj-\sum_n b^{(l)}_{(n+t)n}\frac{1}{n+k+t}tj\Big]\\
&=\lim_{l\to\infty}\Big[-\Big(\sum_n(n+j)\overline{a^{(l)}_{n(n+t)}}\frac{1}{n+k+t}\Big)t+\sum_n n(\overline{a^{(l)}_{n(n+t)}}-b^{(l)}_{(n+t)n})\frac{j}{n+k+t}\\
&\quad+\sum_n(\overline{a^{(l)}_{n(n+t)}}-b^{(l)}_{(n+t)n})\frac{1}{n+k+t}tj\Big]\\
&=\lim_{l\to\infty}\Big[-\Big(\sum_n\overline{a^{(l)}_{n(n+t)}}\Big)t+\sum_n(n+t)(\overline{a^{(l)}_{n(n+t)}}-b^{(l)}_{(n+t)n})\frac{j}{n+k+t}\Big]\\
&=\lim_{l\to\infty}\Big[-\Big(\sum_n\overline{a^{(l)}_{n(n+t)}}\Big)t+\sum_n(\overline{a^{(l)}_{n(n+t)}}-b^{(l)}_{(n+t)n})j\Big]\\
&\quad-\lim_{l\to\infty}\sum_n(\overline{a^{(l)}_{n(n+t)}}-b^{(l)}_{(n+t)n})\frac{kj}{n+k+t}.\end{aligned}$$

Noting

$$\lim_{l\to\infty}\sum_n(\overline{a^{(l)}_{n(n+t)}}-b^{(l)}_{(n+t)n})j=0,$$
$$\lim_{l\to\infty}\langle z^{k-1},p_lz^{j-1}\rangle=\langle z^{k-1},\varphi z^{j-1}\rangle=\lim_{l\to\infty}\langle z^{k-1},\overline{q_l}z^{j-1}\rangle,$$

we know easily that

$$\lim_{l\to\infty}\sum_n(\overline{a^{(l)}_{n(n+t)}}-b^{(l)}_{(n+t)n})\frac{1}{n+k+t}=0.$$

Hence $\lim\limits_{l\to\infty}\Big(-\sum\limits_n\overline{a^{(l)}_{n(n+t)}}\Big)=0$ for each $t\neq 0$, which shows that

$$\int_0^{2\pi}p_l|_{\mathbb{T}}e^{it\theta}d\theta\longrightarrow 0\qquad(\forall t\neq 0,\quad l\to\infty).$$

Thus

$$\int_0^{2\pi}\varphi|_{\mathbb{T}}e^{it\theta}d\theta=\lim_{l\to\infty}\int_0^{2\pi}p_l|_{\mathbb{T}}e^{it\theta}d\theta=0\qquad(\forall t\neq 0),$$

further, $\varphi|_{\mathbb{T}}=\lim\limits_{l\to\infty}\sum\limits_n a^{(l)}_{nn}$ =constant.

Conversely, if $\varphi|_{\mathbb{T}}=c$ =constant, then it is clear that $T_\varphi=T_c=cI$ by Theorem 3.1. Thus $T^*_\varphi=\bar{c}I=T_{\bar{c}}=T_{\bar{\varphi}}$.

Proposition 3.2. *Suppose $\varphi,\psi\in C^1(\bar{\mathbb{D}})$. Then $T_\varphi T_\psi=0$ if and only if at least one of $\varphi|_{\mathbb{T}}$ and $\psi|_{\mathbb{T}}$ equals zero.*

Proof. By Theorem 3.1, we need only to prove that if $T_\varphi T_\psi=0$, then either $\varphi|_{\mathbb{T}}$ or $\psi|_{\mathbb{T}}$ equals zero. Assume $\{p_l\}$ and $\{q_l\}$ are polynormial sequences which satisfy

$$\|p_l-\varphi\|_\infty\to 0,\qquad\Big\|\frac{\partial p_l}{\partial z}-\frac{\partial\varphi}{\partial z}\Big\|_\infty\to 0,$$
$$\|q_l-\psi\|_\infty\to 0,\qquad\Big\|\frac{\partial q_l}{\partial z}-\frac{\partial\psi}{\partial z}\Big\|_\infty\to 0,$$

thus $\|T_{p_l}-T_\varphi\|\to 0,\quad \|T_{q_l}-T_\psi\|\to 0$. Suppose

$$p_l=\sum_{nm}a^{(l)}_{nm}z^n\bar{z}^m,\qquad q_l=\sum_{nm}b^{(l)}_{kj}z^k\bar{z}^j.$$

Then for any $t\in\mathbb{Z}^+-\{0\}$, we have

$$\begin{aligned}T_{p_l}T_{q_l}z^t&=P\Big[\Big(\sum_{nm}a^{(l)}_{nm}z^n\bar{z}^m\Big)P\Big(\sum_{kj}b^{(l)}_{kj}z^k\bar{z}^jz^t\Big)\Big]\\&=P\Big[\Big(\sum_{nm}a^{(l)}_{nm}z^n\bar{z}^m\Big)\Big(\sum_{k+t\rangle j}b^{(l)}_{kj}z^{k+t-j}\Big)\Big]\\&=\sum_{k+t>j,n+k+t-j>m}a^{(l)}_{nm}b^{(l)}_{kj}z^{k+t+n-m-j}.\end{aligned}$$

Noting

$$\|T_{p_l}T_{q_l}-T_\varphi T_\psi\|\le\|T_{p_l}\|\|T_{q_l}-T_\psi\|+\|T_{p_l}-T_\varphi\|\|T_\psi\|\to 0,$$

we see that $\lim\limits_{l\to\infty}T_{p_l}T_{q_l}z^t=0$ for each t. Hence

$$\lim_{l\to\infty}\sum_{k+t>j,n+k+t-j>m}a^{(l)}_{nm}b^{(l)}_{kj}z^{k+t+n-m-j}=0\qquad\text{in }\mathcal{D}.$$

Let $\widetilde{P}$ denote the orthogonal projection from $L^2(\mathbb{T})$ onto Hardy space $H^2(\mathbb{T})$, and $\widetilde{T}_{\varphi|_\mathbb{T}}$ denote the Toeplitz operator with symbol $\varphi|_\mathbb{T}$ on $H^2(\mathbb{T})$. Then for any $t \in \mathbb{Z}^+$,

$$\begin{aligned}
T_{p_l|_\mathbb{T}} T_{q_l|_\mathbb{T}} \zeta^t &= \widetilde{P}\Big[\Big(\sum_{nm} a_{nm}^{(l)} \zeta^{n-m}\Big)\Big(\sum_{k+t\geq j} b_{kj}^{(l)} \zeta^{k+t-j}\Big)\Big] \\
&= \sum_{k+t\geq j, n+k+t-j\geq m} a_{nm}^{(l)} b_{kj}^{(l)} \zeta^{k+t+n-m-j} \\
&= \sum_{k+t+1\geq j+1, n+k+t+1\geq (j+1)+m} a_{nm}^{(l)} b_{kj}^{(l)} \zeta^{k+n+(t+1)-m-(j+1)} \\
&= \sum_{k+t'\geq j+1, n+k+t'\geq j+1+m} a_{nm}^{(l)} b_{kj}^{(l)} \zeta^{k+n+t'-m-(j+1)} \qquad (t' \in \mathbb{Z}^+ - \{0\}) \\
&= \sum_{k+t'> j, n+k+t'> j+m} a_{nm}^{(l)} b_{kj}^{(l)} \zeta^{k+n+t'-m-(j+1)}.
\end{aligned}$$

Since $\lim\limits_{l\to\infty} \sum\limits_{k+t>j, n+k+t-j>m} a_{nm}^{(l)} b_{kj}^{(l)} z^{k+t+n-m-j} = 0$ in $\mathcal{D}$ $(\forall t \in \mathbb{Z}^+ - \{0\})$, it is obvious that

$$\lim_{l\to\infty} \sum_{k+t'>j, n+k+t'>j+m} a_{nm}^{(l)} b_{kj}^{(l)} \zeta^{k+n+t'-m-(j+1)} = 0$$

in $H^2(\mathbb{T})$ $(\forall t' \in \mathbb{Z}^+ - \{0\})$, further

$$\widetilde{T}_{\varphi|_\mathbb{T}} \widetilde{T}_{\psi|_\mathbb{T}} \zeta^t = \lim_{l\to\infty} \widetilde{T}_{p_l|_\mathbb{T}} \widetilde{T}_{q_l|_\mathbb{T}} \zeta^t = 0 \qquad (\forall t \in \mathbb{Z}^+).$$

Hence $\widetilde{T}_{\varphi|_\mathbb{T}} \widetilde{T}_{\psi|_\mathbb{T}} = 0$ if $T_\varphi T_\psi = 0$. It is well-known that $\widetilde{T}_{\varphi|_\mathbb{T}} \widetilde{T}_{\psi|_\mathbb{T}} = 0$ if and only if $\varphi|_\mathbb{T} = 0$ or $\psi|_\mathbb{T} = 0$. The proposition is thus complete.

Proposition 3.3. *Suppose $\varphi, \psi \in C^1(\bar{\mathbb{D}})$. Then $T_\varphi T_\psi$ is a Toeplitz operator with C^1-symbol if and only if $\bar{\varphi}|_\mathbb{T}$ or $\psi|_\mathbb{T}$ is the boundary value of an analytic function on $\mathbb{D}$.*

Proof. It is not difficult to see that if $T_\varphi T_\psi$ is a Toeplitz operator with C^1-symbol, then $T_\varphi T_\psi = T_{\varphi\psi}$. In fact, since $T_\varphi T_\psi - T_{\varphi\psi}$ is compact, $T_{\varphi\psi - \widetilde{\varphi}}$ is compact if $T_\varphi T_\psi = T_{\widetilde{\varphi}}$ $(\widetilde{\varphi} \in C^1(\mathbb{D}))$. By Theorem 3.1, we have $T_{\varphi\psi} = T_{\widetilde{\varphi}}$. Similar to the proof of Proposition 3.2, we see easily that $T_\varphi T_\psi = T_{\varphi\psi}$ if and only if $\widetilde{T}_{\varphi|_\mathbb{T}} \widetilde{T}_{\psi|_\mathbb{T}} = \widetilde{T}_{(\varphi\psi)|_\mathbb{T}}$ if and only if either $\bar{\varphi}|_\mathbb{T}$ or $\psi|_\mathbb{T}$ is the boundary value of an analytic function, we are done.

§4. Connectivity of Spectra of Toeplitz Operators

Widom theorem tells us that each Toeplitz operator has connected spectrum on Hardy space for one complex variable. However, it is well-known that similar conclusion fails in the case of Bergman space or Hardy space for several complex variables. Is the spectrum of each Toeplitz operator connected on Dirichlet space $\mathcal{D}$? Our goal is to deal with this question for the case of Toeplitz operators with C^1-symbols in this section.

Lemma 4.1. *Suppose $\varphi \in C^1(\bar{\mathbb{T}})$. If T_φ is invertible, then $\widetilde{T}_{\varphi|_\mathbb{T}}$ is an invertible operator on Hardy space $H^2(\mathbb{T})$, where $\widetilde{T}_{\varphi|_\mathbb{T}}$ denotes the usual Toeplitz operator with symbol $\varphi|_\mathbb{T}$ on $H^2(\mathbb{T})$. In particular, $R(\varphi|_\mathbb{T}) = \sigma(\widetilde{T}_{\varphi|_\mathbb{T}}) \subset \sigma(T_\varphi)$.*

Proof. If T_φ is invertible, then $\varphi|_\mathbb{T}$ is invertible in $C(\mathbb{T})$ and $\operatorname{wind} \varphi|_\mathbb{T} = 0$ by index formula (Lemma 2.1). Thus $\widetilde{T}_{\varphi|_\mathbb{T}}$ is Fredholm and $\operatorname{Ind} \widetilde{T}_{\varphi|_\mathbb{T}} = -\operatorname{wind} \varphi|_\mathbb{T} = 0$. Further $\widetilde{T}_{\varphi|_\mathbb{T}}$ is invertible by Corollary 7.25 in [8]. Hence $\sigma(\widetilde{T}_{\varphi|_\mathbb{T}}) \subset \sigma(T_\varphi)$.

Lemma 4.2. *Suppose $\varphi \in C^1(\bar{\mathbb{D}})$ and $\varphi|_{\mathbb{T}}$ is invertible in $C(\mathbb{T})$. Then at least one of* $\operatorname{Ker} T_\varphi$ *and* $\operatorname{Ker} T_\varphi^*$ *equals* $\{0\}$.

Proof. Let $\{p_l\} = \sum\limits_{nm} a_{nm}^{(l)} z^n \bar{z}^m$ be a polynormial sequence such that $\|\varphi - p_l\|_\infty \to 0$ and $\|\frac{\partial \varphi}{\partial z} - \frac{\partial p_l}{\partial z}\|_\infty \to 0$ (as $l \to \infty$). Then $\|T_\varphi - T_{p_l}\| = \|T_\varphi^* - T_{p_l}^*\| \to 0$. If $f \in \operatorname{Ker} T_\varphi$, and $f = \sum\limits_{k>0} b_k z^k$, then $p_l f = \sum\limits_{nm}\sum\limits_{k>0} b_k a_{nm}^{(l)} z^{n+k} \bar{z}^m$. Thus

$$P(p_l f) = \sum_{n+k>m} b_k a_{nm}^{(l)} z^{n+k-m} = \sum_{t>0}\Big(\sum_{n+k\ge t} b_k a_{n(n+k-t)}^{(l)}\Big) z^t,$$

and hence

$$\lim_{l\to\infty} P(p_l f) = \lim_{l\to\infty} \sum_{t>0}\Big(\sum_{n+k\ge t} b_k a_{n(n+k-t)}^{(l)}\Big) z^t = 0 \quad (\text{in} \quad \mathcal{D}).$$

On the other hand, if $g \in KerT_\varphi^*$, and $g = \sum_{k>0} d_k z^k$, then for any $j > 0$ we have

$$\begin{aligned}\langle T_\varphi^* g, z^j\rangle_{\frac12} &= \lim_{l\to\infty}\langle T_{p_l}^* g, z^j\rangle_{\frac12} = \lim_{l\to\infty}\Big\langle g, \sum_{t>0}\Big(\sum_{n+j\ge t} a_{n(n+j-t)}^{(l)}\Big) z^t\Big\rangle_{\frac12} \\ &= \lim_{l\to\infty}\Big\langle \sum_k d_k z^k, \sum_{t>0}\Big(\sum_{n+j\ge t} a_{n(n+j-t)}^{(l)}\Big) z^t\Big\rangle_{\frac12} \\ &= \lim_{l\to\infty}\sum_{t>0}\Big(\sum_{n+j\ge t}\overline{a_{n(n+j-t)}^{(l)}}\Big)\frac{d_t}{t} = 0.\end{aligned}$$

Set

$$A_l = (c_{ij}^{(l)}) = \begin{pmatrix} \sum_{n\ge0} a_{nn}^{(l)} & \sum\limits_{n\ge-1} a_{n(n+1)}^{(l)} & \sum\limits_{n\ge-2} a_{n(n+2)}^{(l)} & \sum\limits_{n\ge-3} a_{n(n+3)}^{(l)} & \cdots \\ \sum\limits_{n\ge1} a_{n(n-1)}^{(l)} & \sum\limits_{n\ge0} a_{nn}^{(l)} & \sum\limits_{n\ge-1} a_{n(n+1)}^{(l)} & \sum\limits_{n\ge-2} a_{n(n+2)}^{(l)} & \cdots \\ \sum\limits_{n\ge2} a_{n(n-2)}^{(l)} & \sum\limits_{n\ge1} a_{n(n-1)}^{(l)} & \sum\limits_{n\ge0} a_{nn}^{(l)} & \sum\limits_{n\ge-1} a_{n(n+1)}^{(l)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Then A_l is a Toeplitz matrix with symbol $p_l|_{\mathbb{T}} = \sum\limits_{j>0} c_{(1+j)1}^{(l)} z^j + \sum\limits_{j\ge0} c_{1(1+j)}^{(l)} \bar{z}^j$. Noting $\varphi|_{\mathbb{T}} \in C(\mathbb{T})$ and $\|\varphi|_{\mathbb{T}} - p_l|_{\mathbb{T}}\|_\infty \to 0$, we see that

$$\|\widetilde{T}_{\varphi|_{\mathbb{T}}} - \widetilde{T}_{p_l|_{\mathbb{T}}}\| = \|\widetilde{T}^*_{\varphi|_{\mathbb{T}}} - \widetilde{T}^*_{p_l|_{\mathbb{T}}}\| \to 0.$$

Let $\widetilde{f} = \sum\limits_{k=0}^{\infty} b_{k+1}\zeta^k$. Then $\widetilde{f} \in H^2(\mathbb{T})$ by $f = \sum\limits_{k=1}^{\infty} b_k z^k \in \mathcal{D}$ and

$$\widetilde{T}_{\varphi|_{\mathbb{T}}}\widetilde{f} = \lim_{l\to\infty} A_l \widetilde{f} = \lim_{l\to\infty}\sum_{t\ge0}\Big(\sum_{n+k\ge t} b_{k+1} a_{n(n+k-t)}^{(l)}\Big)\zeta^t = 0 \quad \text{in} \quad H^2(\mathbb{T})$$

since $\sum\limits_{t>0}\Big(\sum_{n+k\ge t} b_k a_{n(n+k-t)}^{(l)}\Big) z^t \to 0$ in $\mathcal{D}$. Again, let $\widetilde{g} = \sum\limits_{t=0}^{\infty}\frac{d_{t+1}}{t+1}\zeta^t$. Then $\widetilde{g} \in H^2(\mathbb{T})$ since $g = \sum\limits_{t=1}^{\infty} d_t z^t \in \mathcal{D}$. Thus

$$\begin{aligned}\lim_{l\to\infty}\widetilde{T}^*_{p_l|_{\mathbb{T}}}\widetilde{g} &= \lim_{l\to\infty} A_l^*\widetilde{g} = \lim_{l\to\infty}\sum_{i=1}^{\infty}\Big(\sum_{j=1}^{\infty}\overline{c_{ji}^{(l)}}\frac{d_j}{j}\Big)\zeta^{i-1} \\ &= \lim_{l\to\infty}\sum_{i=1}^{\infty}\Big[\sum_{j=1}^{\infty}\Big(\sum_{n+i\ge j}\overline{a_{n(n+i-j)}^{(l)}}\Big)\frac{d_j}{j}\Big]\zeta^{i-1} = 0\end{aligned}$$

since $\lim\limits_{l\to\infty}\sum\limits_{t>0}\sum\limits_{n+j\geq t}\overline{a^{(l)}_{n(n+j-t)}}\frac{d_t}{t}=0$ for each j. Hence $\widetilde{T}^*_{\varphi|_{\mathbb{T}}}\widetilde{g}=0$. It is obvious that $f\neq 0$ if and only if $\widetilde{f}\neq 0$, and $g\neq 0$ if and only if $\widetilde{g}\neq 0$. However, at least one of $\operatorname{Ker}\widetilde{T}_{\varphi|_{\mathbb{T}}}$ and $\operatorname{Ker}\widetilde{T}^*_{\varphi|_{\mathbb{T}}}$ equals $\{0\}$, so that either $\widetilde{f}$ or $\widetilde{g}$ equals $\{0\}$. Further at least one of $\operatorname{Ker}T_\varphi$ and $\operatorname{Ker}T^*_\varphi$ is zero. The lemma is thus complete.

Theorem 4.1. *Suppose $\varphi\in C^1(\bar{\mathbb{D}})$. Then*

$$\sigma(T_\varphi)=\sigma(\widetilde{T}_{\varphi|_{\mathbb{T}}}).$$

In particular, $\sigma(T_\varphi)$ is a connected subset of $\mathbb{C}$.

Proof. By Lemma 4.1, we need only to prove that $\sigma(T_\varphi)\subset\sigma(\widetilde{T}_{\varphi|_{\mathbb{T}}})$. Without loss of generality, assume $0\notin\sigma(\widetilde{T}_{\varphi|_{\mathbb{T}}})$. Then $\varphi|_{\mathbb{T}}$ is invertible in $C(\mathbb{T})$. Thus T_φ is a Fredholm operator on $\mathcal{D}$, and $\operatorname{Ind}T_\varphi=-\operatorname{wind}\varphi|_{\mathbb{T}}$. Note $-\operatorname{wind}\varphi|_{\mathbb{T}}=\operatorname{Ind}\widetilde{T}_{\varphi|_{\mathbb{T}}}=0$, so $\operatorname{Ind}T_\varphi=0$. However, either $\operatorname{Ker}T_\varphi$ or $\operatorname{Ker}T^*_\varphi$ is trivial by Lemma 4.2, we see that $\operatorname{Ker}T_\varphi=\operatorname{Ker}T^*_\varphi=\{0\}$. Thus T_φ is invertible, that is, $0\notin\sigma(T_\varphi)$. Then follows the theorem.

Corollary 4.1. *For any $\varphi\in C^1(\bar{\mathbb{D}})$,*

$$\sigma(T_\varphi)=R(\varphi|_{\mathbb{T}})\cup\{\lambda\in\rho_e(T_\varphi)|\operatorname{Ind}T_{\varphi-\lambda}\neq 0\}.$$

Corollary 4.2. *If $\varphi\in C^1(\bar{\mathbb{D}})$ is a nonconstant real function, then T_φ has no eigenvalues.*

Proof. Since $\sigma(T_\varphi)=\sigma(\widetilde{T}_{\varphi|_{\mathbb{T}}})$, we know that

$$\sigma(T_\varphi)=[\inf\varphi|_{\mathbb{T}},\sup\varphi|_{\mathbb{T}}].$$

For any $\lambda\in\sigma(T_\varphi)$, it is easy to see that

$$\dim\operatorname{Ker}(T_\varphi-\lambda)\leq\dim\operatorname{Ker}(\widetilde{T}_{\varphi|_{\mathbb{T}}}-\lambda),\qquad \dim\operatorname{Ker}(T^*_\varphi-\lambda)\leq\dim\operatorname{Ker}(\widetilde{T}^*_{\varphi|_{\mathbb{T}}}-\lambda)$$

by the proof of Lemma 4.2. Thus $\operatorname{Ker}(T_\varphi-\lambda)=\{0\}$ since $\operatorname{Ker}(\widetilde{T}_{\varphi|_{\mathbb{T}}}-\lambda)=\dim\operatorname{Ker}(\widetilde{T}^*_{\varphi|_{\mathbb{T}}}-\lambda)=\{0\}$ (see [8]); that is, $\lambda\notin\sigma_p(T_\varphi)$. Hence $\sigma_p(T_\varphi)=\emptyset$.

Corollary 4.3. *Suppose $\varphi\in C^1(\bar{\mathbb{D}})$. Then T_φ is invertible if and only if the following statements are true*:

(i) *$\varphi|_{\mathbb{T}}$ is invertible in $C(\mathbb{T})$;*

(ii) *there exist $\epsilon>0$ and an outer function such that* $\operatorname{Re}(\varphi|_{\mathbb{T}}f)>\epsilon$.

Proof. By Theorem 4.1, we know that T_φ is invertible if and only if $\widetilde{T}_{\varphi|_{\mathbb{T}}}$ is invertible. The theorem is thus complete by Widom-Devinatz theorem (see [8]).

Corollary 4.4. *Suppose $\varphi\in C^1(\bar{\mathbb{D}})$ is a real function. Then T_φ is invertible if and only if $z\in R(T_\varphi)$, the range of T_φ.*

Proof. We need only to prove that T_φ is invertible if $z\in R(T_\varphi)$. In fact, assume $\{p_l\}$ is a polynormial sequence such that $\|p_l-\varphi\|_\infty\to 0$, and $\|\frac{\partial p_l}{\partial z}-\frac{\partial\varphi}{\partial z}\|_\infty\to 0$, thus $\|T_{p_l}-T_\varphi\|\to 0$. Let $p_l=\sum\limits_{nm}^{(l)}z^n\bar{z}^m$, and $f=\sum\limits_{k=1}^{\infty}b_kz^k$ such that $T_\varphi f=z$. We see that $\lim\limits_{l\to\infty}T_{p_l}f=z$. It is obvious that $T_{p_l}f=\sum\limits_{t=1}^{\infty}\sum\limits_{n+k\geq t}a^{(l)}_{n(n+k-t)}b_kz^t$. Hence

$$\lim_{l\to\infty}\sum_{n+k\geq t}a^{(l)}_{n(n+k-t)}b_k=\begin{cases}1, & t=1,\\ 0, & t>1.\end{cases}$$

Now let $\widetilde{f}(\zeta)=\sum\limits_{k=0}^{\infty}b_{k+1}\zeta^k\in H^2(\mathbb{T})$ (it is obvious since $f=\sum\limits_{k=1}^{\infty}b_kz^k\in\mathcal{D}$). Then

$$\begin{aligned}\widetilde{T}_{p_l|_{\mathbb{T}}}\widetilde{f}&=P\Big[\Big(\sum_{nm}a_{nm}^{(l)}\zeta^n\bar{\zeta}^m\Big)\Big(\sum_{k=0}^{\infty}b_{k+1}\zeta^k\Big)\Big]\\&=\sum_{n+k\geq m}a_{nm}^{(l)}b_{k+1}\zeta^{n+k-m}=\sum_{t=0}^{\infty}\sum_{n+k\geq t}a_{n(n+k-t)}^{(l)}b_{k+1}\zeta^t\\&=\sum_{t=0}^{\infty}\sum_{n+k+1\geq t+1}a_{n(n+(k+1)-(t+1))}^{(l)}b_{k+1}\zeta^t=\sum_{t=0}^{\infty}\sum_{n+k'\geq t+1}a_{n(n+k'-(t+1))}^{(l)}b_{k'}\zeta^t.\end{aligned}$$

By previous argument, we have $\lim\limits_{l\to\infty}\widetilde{T}_{p_l|_{\mathbb{T}}}\widetilde{f}=1$, and consequently, $\widetilde{T}_{\varphi|_{\mathbb{T}}}\widetilde{f}=\widetilde{T}_{p_l|_{\mathbb{T}}}\widetilde{f}=1$. This shows that $\widetilde{T}_{\varphi|_{\mathbb{T}}}$ is invertible (see [8]). Hence T_φ is invertible by Theorem 4.1.

Remark 4.1. We know that $T_\varphi^*\neq T_\varphi$ for each nonconstant real function which satisfies $\varphi|_{\mathbb{T}}\neq$constant by Proposition 3.1. Hence for such a φ, $W(T_\varphi)\subsetneqq\mathbb{R}$, the real field. However, the proof of Corollary 4.2 shows that $\sigma(T_\varphi)$ is a closed interval in $\mathbb{R}$. Thus we see again that T_φ is not a convexoid operator. For instance, if $\varphi=(Rez)^2$, then $\sigma(T_\varphi)=[0,1]$. Let $f=iz^3+z^2+z\in\mathcal{D}$. We have that

$$\begin{aligned}\Big\langle T_\varphi\frac{f}{\|f\|_{\mathcal{D}}},\frac{f}{\|f\|_{\mathcal{D}}}\Big\rangle_{\frac12}&=\Big\langle\varphi\frac{f'}{\|f\|_{\mathcal{D}}},\frac{f'}{\|f\|_{\mathcal{D}}}\Big\rangle+\Big\langle\frac{\partial\varphi}{\partial z}\frac{f}{\|f\|_{\mathcal{D}}},\frac{f'}{\|f\|_{\mathcal{D}}}\Big\rangle\\&=\Big\langle\varphi\frac{f'}{\|f\|_{\mathcal{D}}},\frac{f'}{\|f\|_{\mathcal{D}}}\Big\rangle+\frac{1}{\|f\|_{\mathcal{D}}^2}\langle 2\mathrm{Re}(iz^3+z^2+z),(3iz^2+2z+1)\rangle\\&=\Big\langle\varphi\frac{f'}{\|f\|_{\mathcal{D}}},\frac{f'}{\|f\|_{\mathcal{D}}}\Big\rangle+\frac{1}{\|f\|_{\mathcal{D}}^2}\langle(\bar z+z)(iz^3+z^2+z),(3iz^2+2z+1)\rangle\\&=\Big\langle\varphi\frac{f'}{\|f\|_{\mathcal{D}}},\frac{f'}{\|f\|_{\mathcal{D}}}\Big\rangle+\frac{1}{\|f\|_{\mathcal{D}}^2}[-3\langle z^3,z^3\rangle\\&\quad+2\langle z^2,z^2\rangle+\langle z,z\rangle-3i\langle z^2,z^2\rangle].\end{aligned}$$

It is obvious that $\Big\langle T_\varphi\frac{f}{\|f\|_{\mathcal{D}}},\frac{f}{\|f\|_{\mathcal{D}}}\Big\rangle_{\frac12}\notin\mathbb{R}$. Hence $con\sigma(T_\varphi)\subsetneqq\overline{W(T_\varphi)}$.

References

[1] Rochberg, R. & Wu, Z., Toeplitz operators on Dirichlet spaces [J], *Integr. Equat. Oper. Th.,* **15**(1992), 325–342.
[2] Wu, Z., Hankel and Toeplitz operators on Dirichlet spaces [J], *Integr. Equat. Oper. Th.,* **15**(1992), 503–525.
[3] Cao, G. F., Fredholm properties of Toeplitz operators on Dirichlet spaces [J], *Pacific J. Math.,* **188:**2 (1999), 209–224.
[4] Cao, G. F. & Zhong, Ch. Y., Toeplitz operators with C^1-symbols on Dirichlet spaces [J], *Acta Analysis Functionatis Applicata,* **2**(2000), 289–297.
[5] Cao, G. F. & Zhong, Ch. Y., Automorphism groups of Toeplitz algebras on Hardy spaces of certain strong psedoconvex domains [J], *Northeastern Math. J.,* **1**(1998).
[6] Sakai, S., On a problem of Kaplansky [J], *Tohoku Math. J.,* **12**(1960), 31–33.
[7] Kadison, R. V., Notes on the Gelfand-Neumark theorem [J], *Contemporary Math.,* **167**(1994), 21–53.
[8] Helemskii, A. Ya., The homology of Banach and topological algebras [M], Kluwer Academic Publishers, 1989.
[9] Kadison, R. V., Derivations of operator algebras [J], *Ann. of Math.,* **83**(1966), 280–293.
[10] Douglas, R. G., Banach algebra techniques in operator theory [M], Academic Press, New York and London, 1972.