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# TRAVELING WAVE SOLUTIONS FOR A CLASS OF NONLINEAR DISPERSIVE EQUATIONS***

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**Abstract**

The method of the phase plane is emploied to investigate the solitary and periodic traveling waves for a class of nonlinear dispersive partial differential equations. By using the bifurcation theory of dynamical systems to do qualitative analysis, all possible phase portraits in the parametric space for the traveling wave systems are obtained. It can be shown that the existence of a singular straight line in the traveling wave system is the reason why smooth solitary wave solutions converge to solitary cusp wave solution when parameters are varied. The different parameter conditions for the existence of solitary and periodic wave solutions of different kinds are rigorously determined.

**Keywords** Solitary wave, Periodic wave, Integrable system, Bifurcation of phase portraits, Smoothness of wave
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## §1. Introduction

In this paper, we consider the following class of nonlinear dispersion equations

$$K(m,n): \quad u_t + a(u^m)_x + (u^n)_{xxx} = 0, \quad m,n \geq 1, \tag{1.1}$$

where $m$, $n$ are integers, $a$ is a real parameter, $u(x,t)$ is the unknown function of the temporal variable $t$ and the spatial variable $x$. Equation (1.1) contains a nonlinear dispersion term $(u^n)_{xxx}$. Recently, (1.1) has been considered by P. Rosenau in [11-14] and in [10] (with J. M. Hyman), respectively. They hope to understand the role of nonlinear dispersion in pattern formation and to study the interaction between convection, dispersion and dissipation. For the aim and significance of the study of this class of equations and the background materials of model equations, we refer to the above papers and the references therein.

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In [13], the author stated that "a lack of proper mathematical tools makes this goal at the present time pretty much beyond our reach." In this paper, we shall show that the bifurcation theory of planar dynamical systems provides an available tool to understand qualitatively all traveling wave solutions of (1.1). In order to know when the smooth and non-smooth traveling wave solutions appear, what parameter conditions imply the appearance of so-called compactons and peakons of travelling wave solutions, and what are the relationship between traveling wave solutions of different types, the study on the global dynamical behaviour and the bifurcation set in the parametric space is very important for the traveling wave equation. We shall give a reasonable explanation for the compacton solutions and peakon solutions and correct some mistakes in [12–14] (see Remark 4.1 below).

It is well known that a traveling wave solution of Equation (1.1) with wave speed $c$ is a solution of the form $u = \phi(x - ct) = \phi(\xi)$ with $\xi = x - ct$. Substituting the traveling wave solution $u(x,t) = \phi(x - ct)$ with constant wave speed $c$ into (1.1), we have the following ordinary differential equation

$$-c\phi' + a(\phi^m)' + (\phi^n)''' = 0. \tag{1.2}$$

Integrating (1.2) once with respect to $\xi$ leads to

$$-c\phi + a\phi^m + n(n-1)\phi^{n-2}(\phi')^2 + n\phi^{n-1}\phi'' = g, \tag{1.3}$$

where $g \in R$ is the integral constant. Let $\phi' = y$. Then, we have the following planar autonomous system:

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{-n(n-1)\phi^{n-2}y^2 - a\phi^m + c\phi + g}{n\phi^{n-1}}, \tag*{(1.4)$_{(m,n)}$}$$

Systems (1.4) is a traveling wave system which corresponds to equation $K(m,n)$.

Obviously, the straight line $\phi = 0$, on which the vector field defined by (1.4) has no definition, is called a singular straight line. It is easy to see that the system (1.4) has a first integral

$$H(\phi, y) = \phi^n\Big(n\phi^{n-2}y^2 + \frac{2a}{m+n}\phi^m - \frac{2c}{n+1}\phi - \frac{2g}{n}\Big) = h. \tag{1.5}$$

A traveling wave solution of (1.1) is called a solitary wave if $\phi(\xi)$ has a well-defined limit as $|\xi|$ approachs to the infinity. Usually, a solitary wave solution corresponds to a homoclinic orbit of (1.4). Similarly, a periodic orbit of (1.4) corresponds to a periodic traveling wave of (1.1). In order to study these waves, we are going to find all period annuluses and their boundary curves for (1.4) and to describe all bifurcations of phase portraits on the $(\phi, y)$-phase plane and the bifurcation set on the parameter space for (1.4).

We emphasize that there are two methods to consider the relationship between different types of traveling waves. First, for a fixed parameter group of (1.4), by varying the invariant $h$ ( i.e., varying the initial value of the system), we get the convergence of periodic traveling wave solutions to a solitary wave. Second, by varying the parameter group of (1.4), so that the given parameters arrive at a bifurcation set, we get the convergence of smooth solitary traveling waves to a non-smooth traveling wave. We shall study the previous two types of convergences.

This paper is organized as follows. In Section 2, we describe the bifurcation set of the traveling wave equation (1.4). In Section 3, we discuss the effect of the singular straight line $\phi = 0$ to the smoothness of traveling wave solutions of (1.4). When the parameter pair $(g, c)$ lies on the bifucation curves, what happens about the dynamics of (1.4) will be analyzed in Section 4. In Section 5, we consider the existence and the relationship for the different types of solitary and periodic wave solutions.

## §2. Period Annuluses of Integrable System (2.2)

In this section, we shall study all possible periodic annuluses defined by the vector fields of (1.4) when the parameters $a$, $c$ and $g$ are varied. From (1.4), we see that

$$\frac{dy}{d\phi}=\frac{-n(n-1)\phi^{n-2}y^2-a\phi^m+c\phi+g}{ny\phi^{n-1}}. \tag{2.1}$$

Letting $d\xi=n\phi^{n-1}d\zeta$, consider the following integrable system

$$\frac{d\phi}{d\zeta}=ny\phi^{n-1}=\frac{1}{2\phi^{n-1}}\frac{\partial H}{\partial y},\quad \frac{dy}{d\zeta}=-n(n-1)\phi^{n-2}y^2-a\phi^m+c\phi+g=-\frac{1}{2\phi^{n-1}}\frac{\partial H}{\partial \phi}, \tag{2.2}$$

in which $\nu(\phi)=\frac{1}{2\phi^{n-1}}$ is an integral factor and the Hamiltonian $H(\phi,y)$ is defined by (1.5). It is easy to see from (2.1) that System (1.4) has the same topological phase portraits as (2.2), except on the straight line $\phi=0$. Now, $\phi=0$ is a straight line solution of (2.2). Thus, by using the topological phase portraits of (2.2), we can understand the dynamical behaviour of (1.4) except on the line $\phi=0$. In this section, we consider the phase orbits of System (2.2). In Section 3 we shall point out the different dynamics of the same orbits near the straight line $\phi=0$ with respect to the "time variables" $\zeta$ and $\xi$.

On the $(\phi,y)$-phase plane, the abscissas of equilibrium points of System (2.2) on the $\phi$-axis are the zeros of $E(\phi)=a\phi^m-c\phi-g$. When $n=2$, two equilibrium points of (2.2) are $(0,-\sqrt{0.5g})$ and $(0,\sqrt{0.5g})$ on $y$-axis if $g>0$. When $n>2$, System (2.2) has no equilibrium on the $y$-axis if $g\neq 0$. Noting that $E'(\phi)=am\phi^{m-1}-c$, for an odd $m$ and $ac>0$, $E'(\phi)$ has two zeros at $\widetilde{\phi}_{\pm}=\pm(\frac{c}{am})^{\frac{1}{m-1}}$; for an even $m$, $E'(\phi)$ has only one zero at $\widetilde{\phi}_{+}$. Clearly, $E(\widetilde{\phi}_{+})=-(\frac{m-1}{m}c\widetilde{\phi}_{+}+g)$. By using these information, we know the distributions of the zeros of $E(\phi)$ on the $\phi$-axis. Let $(\phi_e,y_e)$ be an equilibrium of (2.2). At this point, the determinant of the linearized system of (2.2) has the form

$$J(\phi_e,y_e)=-n^3(n-1)\phi_e^{2(n-2)}y_e^2+n\phi_e^{n-1}E'(\phi_e). \tag{2.3}$$

By the theory of planar dynamical systems, we know that if $J(\phi_e,y_e)>0$ (or $<0$), then the equilibrium $(\phi_e,y_e)$ is a center (or a saddle point); if $J(\phi_e,y_e)=0$ and the Poincaré index of $(\phi_e,y_e)$ is zero, then $(\phi_e,y_e)$ is a cusp. It is clear that for $n=2$, two equilibrium points on the $y$-axis are saddle points. As to the equilibrium $(\phi_e,0)$ on the $x$-axis, it is a center (or a saddle point), if $\phi_e^{n-1}E'(\phi_e)>0$ (or $<0$).

By using the above facts to do qualitative analysis, we obtain the following results.

**Case I** $n=2$, $m=2k$, $k=1,2,3,\cdots$, i.e., equation $K(2k,2)$.

(1) Suppose that $a>0$. Then on the $(g,c)$-parametric plane, there are 4 bifurcation curves:

$$L_1^{\pm}: g=0, c>0 \text{ or } c<0,\quad L_2^{\pm}: c=\pm 2k(a)^{\frac{1}{2k}}\left(\frac{|g|}{2k-1}\right)^{\frac{2k-1}{2k}}\quad (g<0). \tag{2.4}$$

These curves partition the $(g,c)$-plane into 4 regions: $(A_1)$, $(B_1)$, $(C_1)$, $(D_1)$. When $(g,c)\in(C_1)$, there is no center point of System (2.2). When $(g,c)\in L_2^{\pm}$, there is a double equilibrium (cusp). When $(g,c)\in L_1^{\pm}$, there exists an oval orbit of (2.2), which contacts $\phi=0$ at the origin of the phase plane and has the orbit equation

$$y^2+\frac{a}{2(k+1)}\phi^{2k}-\frac{c}{3}\phi=0. \tag{2.5}$$

Because for $n = 2$ there are two saddle points of (2.2) on the $y$-axis, all period annuluses with centers of (2.2) have finite boundary curves consisting of a homoclinic orbit or two (or three) heteroclinic orbits of (2.2). Fig. 2.1 shows the bifurcations of phase portraits of (2.2) and the bifurcation set on the $(g, c)$-parametric plane.

Fig.2.1. Bifurcations of System (2.2), when $n = 2$, $m = 2k$, $a > 0$.

(2) Suppose that $a < 0$. When $g \leq 0$, there is no center point of (2.2). We only consiser the right half $(g, c)$-plane. There are 4 bifurcation curves as follows:

$$L_3^{\pm} : c = \pm 2k|a|^{\frac{1}{2k}} \Big(\frac{g}{2k-1}\Big)^{\frac{2k-1}{2k}}, \tag{2.6}$$

$$L_4^{\pm} : c = \pm 6k\Big(\frac{|a|}{2(k+1)}\Big)^{\frac{1}{2k}} \Big(\frac{g}{2(2k-1)}\Big)^{\frac{2k-1}{2k}}. \tag{2.7}$$

Fig.2.2. Bifurcations of System (2.2), when $n = 2$, $m = 2k$, $a < 0$.

The $(g, c)$-half plane is partitioned by these curves into 5 regions: $(A_2)$, $(B_2)$, $(C_2)$, $(D_2)$, $(E_2)$. When $(g, c) \in (C_2)$, there is no center point of (2.2). When $(g, c) \in L_3^{\pm}$, there is a double equilibrium (cusp) of (2.2). When $(g, c) \in L_4^{\pm}$, there exists a period annuluas surrounded by a curve-triangle having the equations:

$$\phi = 0, \quad y = \pm\sqrt{\frac{2k-1}{2}\Big(\frac{k+1}{|a|}\Big)^{\frac{1}{2k-1}}\Big(\frac{c}{3k}\Big)^{\frac{2k}{2k-1}} + \frac{c}{3}\phi + \frac{|a|}{2k+2}\phi^{2k}}, \tag{2.8}$$

which are heteroclinic orbits of (2.2). The bifurcations of phase portraits of (2.2) and the bifurcation set on the $(g, c)$-parametric plane are shown in Fig.2.2.

**Case II** $n = 2$, $m = 2k + 1$, $k = 1, 2, 3, \cdots$, i.e., equation $K(2k + 1, 2)$.

(1) Suppose that $a > 0$. Then on the $(g, c)$-parametric plane, there are 5 bifurcation curves:

$$\begin{aligned} &L_1^{\pm} : g = 0, c > 0 \text{ or } c < 0, \quad L_5^{\pm} : g = \pm 2k(a)^{-\frac{1}{2k}}\Big(\frac{c}{2k+1}\Big)^{\frac{2k+1}{2k}}, \\ &L_6 : g = \frac{4k}{2k+3}(a)^{-\frac{1}{2k}}\Big(\frac{(2k+3)c}{3(2k+1)}\Big)^{\frac{2k+1}{2k}}. \end{aligned} \tag{2.9}$$

These curves partition the $(g, c)$-plane into 5 regions: $(A_3)$, $(B_3)$, $(C_3)$, $(D_3)$, $(E_3)$. When $(g, c) \in (E_3)$, there is no center point of (2.2). When $(g, c) \in L_1^-, L_5^{\pm}$, there is a multiple equilibrium, respectively. When $(g, c) \in L_1^+$, there exists an oval orbit of (2.2) which contacts $\phi = 0$ at the origin of the phase plane and has the orbit equation

$$y^2 + \frac{a}{2k+3}\phi^{2k+1} - \frac{c}{3}\phi = 0. \tag{2.10}$$

The algebraic curve (2.10) consists of two branches; one is an oval, while another is an open curve. When $(g, c) \in L_6$, there are two period annuluses surrounded by 4 heteroclinic orbits of (2.2) having the equation: $\phi = 0$ and

$$y^2 + \frac{a}{2k+3}\phi^{2k+1} - \frac{c}{3}\phi - \frac{g}{2} = 0. \tag{2.11}$$

Fig.2.3. Bifurcations of (2.2), when $n = 2$, $m = 2k + 1$, $a > 0$.

The bifurcations of phase portraits of (2.2) and the bifurcation set on the $(g, c)$-parametric plane are shown in Fig.2.3.

(2) For the case $a < 0$, a similar discussion gives Fig.2.4. The bifucation curves are $L_1^{\pm}$ and for $c < 0$,

$$L_7^{\pm} : g = \pm 2k(|a|)^{-\frac{1}{2k}}\Big(\frac{|c|}{2k+1}\Big)^{\frac{2k+1}{2k}}; \quad L_8 : g = \frac{4k}{2k+3}(|a|)^{-\frac{1}{2k}}\Big(\frac{(2k+3)|c|}{3(2k+1)}\Big)^{\frac{2k+1}{2k}}.$$

These curves partition the $(g, c)$-plane into 5 regions: $(A_4)$, $(B_4)$, $(C_4)$, $(D_4)$, $(E_4)$. When $(g, c) \in (E_4)$, there is no center point of (2.2).

Fig.2.4. Bifurcations of (2.2), when $n = 2$, $m = 2k + 1$, $a < 0$.

**Case III** $n = 3$, $m = 2k$, $k = 1, 2, 3, \cdots$, i.e., equation $K(2k, 3)$.

(1) For $a > 0$, on the $(g, c)$-parametric plane, there are 4 bifurcation curves defined by (2.4). These curves partition the $(g, c)$-plane into 4 regions: $(A_5)$, $(B_5)$, $(C_5)$, $(D_5)$. When $(g, c) \in (C_5)$, there is no center point of (2.2). When $(g, c) \in L_2^{\pm}$, there is a double equilibrium (cusp).

Fig.2.5. Bifurcations of (2.2), when $n = 2l + 1$, $m = 2k$, $a > 0$.

When $(g, c) \in L_1^+$, there exists a period annulus of arch of (2.2) surrounded by two heteroclinic orbits $\phi = 0$ and

$$3y^2 + \frac{2a}{2k+3}\phi^{2k-1} - \frac{c}{2} = 0. \tag{2.12}$$

When $(g, c) \in (A_5)$, there is a period annulus between the line $\phi = 0$ and the curve $3\phi y^2 + \frac{2a}{2k+3}\phi^{2k} - \frac{c}{2}\phi - \frac{2}{3}g = 0$. The periodic orbit in the annulus has the invariant $h \in (H(\phi_1, 0), 0)$, where $\phi_1$ is the abscissa of the center point. Since the second boundary curve is not closed, as $h \to 0$, the uper and lower parts of periodic orbits of (2.2) approach to the infinity. When $(g, c) \in (B_5)$ or $(D_5)$, there is a period annulus of (2.2) surrounded by a homoclinic orbit.

The bifurcations of phase portraits of (2.2) and the bifurcation set on the $(g, c)$-parametric plane are shown in Fig.2.5.

(2) For the case $a < 0$, the bifurcations of phase portraits of (2.2) and the bifurcation set on the $(g, c)$-parametric plane just can be shown by the reflection of Fig.2.5 with respect to the ordinate axis.

**Case IV** $n = 3$, $m = 2k + 1$, $k = 1, 2, 3, \cdots$, i.e., equation $K(2k + 1, 3)$.

(1) For $a > 0$, on the $(g, c)$-parametric plane, there are 4 bifurcation curves: $L_1^\pm$ and $L_5^\pm$ defined as in Case II. These curves partition the $(g, c)$-plane into 4 regions: $(A_6)$, $(B_6)$, $(C_6)$, $(D_6)$. When $(g, c) \in L_5^\pm$, there is a double equilibrium (i.e., cusp) of (2.2). When $(g, c) \in L_1^+$, there exists two period annulus of (2.2) surrounded by three heteroclinic orbits $\phi = 0$ and

$$3y^2 + \frac{a}{2k+2}\phi^{2k} - \frac{c}{2} = 0. \tag{2.13}$$

When $(g, c) \in L_1^-$, (2.2) has no center point.

The bifurcations of phase portraits of (2.2) and the bifurcation set on the $(g, c)$-parametric plane are shown in Fig.2.6.

Fig.2.6. Bifurcations of (2.2), when $n = 2l + 1$, $m = 2k + 1$, $a > 0$.

(2) For $a < 0$, if $c > 0$, System (2.2) has no center point. In the lower half $(g, c)$-parametric

plane, there exist 3 bifurcation curve $L_1^-$ and $L_7^{\pm}$. These curves divide the $(g, c)$-parametric plane into 2 regions:$(A_7)$, $(B_7)$.

When $(g, c) \in L_7^{\pm}$, there is a double equilibrium (i.e., cusp) of (2.2). When $(g, c) \in L_1^-$, there is no center point of (2.2). When $(g, c) \in (A_7)$ or $(B_7)$, there exists a period annulus of (2.2) surrounded by a homoclinic orbit, respectively.

The bifurcations of phase portraits of (2.2) and the bifurcation set on the $(g, c)$-parametric plane are shown in Fig. 2.7.

**Case V** $n = 2l$ $(l \geq 2)$, $m = 2k$, $k = 1, 2, 3, \cdots$, i.e., equation $K(2k, 2l)$.

For $a > 0$, System (2.2) has the same bifurcation curves as (2.4). However, when $(g, c) \in (A_1)$ and $L_1^+$, all period annuluses have no finite boundary curve. The bifurcations of the phase portraits of (2.2) are shown in Fig.2.8.

Fig.2.7. Bifurcations of (2.2), when $n = 2l + 1$, $m = 2k + 1$, $a < 0$.

Fig.2.8. Bifurcations of (2.2), when $n = 2l$ $(l \geq 2)$, $m = 2k$, $a > 0$.

For $a < 0$, differently from the case I, there is no bifurcation curve $L_4^{\pm}$. So the right $(g, c)$-plane is divided into three regions: $(A_9)$, $(B_9)$, $(C_9)$. The bifurcations of phase portraits of (2.2) and the bifurcation set on the $(g, c)$-parametric plane are shown in Fig.2.9.

**Case VI** $n = 2l$ $(l \geq 2)$, $m = 2k + 1$, $k = 1, 2, 3, \cdots$, i.e., equation $K(2k + 1, 2l)$.

For $a > 0$, differently from the case II, there is no bifurcation curve $L_6$. However, there are still bifurcation curves $L_1^{\pm}$ and $L_5^{\pm}$. The $(g, c)$-parametric plane is divided into 4 regions: $(A_{10})$, $(B_{10})$, $(C_{10})$, $(D_{10})$.

The bifurcations of phase portraits of (2.2) and the bifurcation set on the $(g, c)$-parametric plane are shown in Fig.2.10.

Similarly, for $a < 0$, the $(g, c)$-parametric plane is divided into 4 regions: $(A_{11})$, $(B_{11})$, $(C_{11})$, $(D_{11})$. The bifurcations of phase portraits of (2.2) and the bifurcation set on the $(g, c)$-parametric plane are shown in Fig.2.11.

**Case VII** $n = 2l + 1$ $(l \geq 2)$, $m = 2k$, $k = 1, 2, 3, \cdots$, i.e., equation $K(2k, 2l + 1)$.

For $a > 0$, the system (2.2) has the same bifurcation curves as (2.4). But when $(g, c) \in L_1^{+}$, the period annulus has no finite boundary curve. The bifurcations of the phase portraits of (2.2) are shown in Fig.2.5.

For $a < 0$, the bifurcations of the phase portraits and the bifurcation set on the $(g, c)$-plane of (2.2) are the reflection of Fig. 2.5 with respect to the ordinate axis.

Fig.2.9. Bifurcations of (2.2), when $n = 2l$ $(l \geq 2)$, $m = 2k$, $a < 0$.

Fig.2.10. Bifurcations of (2.2), when $n = 2l$ ($l \geq 2$), $m = 2k+1$, $a > 0$.

Fig.2.11. Bifurcations of (2.2), when $n = 2l$ ($l \geq 2$), $m = 2k+1$, $a < 0$.

**Case VIII** $n = 2l+1$ ($l \geq 2$), $m = 2k+1$, $k = 1, 2, 3, \cdots$, i.e., equation $K(2k+1, 2l+1)$. In this case, the bifurcations of phase portraits of (2.2) and the bifurcation set on the $(g, c)$-parametric plane are shown in Fig.2.6 and Fig.2.7.

**Remark 2.1.** (i) For a planar polynomial system, say (2.2), by the theory of invariant manifolds, all periodic solutions and all stable and unstable manifolds connecting equilibrium points are analytic with respect to the "time variable" $\zeta$ (see [1,3,5,6]). However, for System (1.4), even though it has the same invariant curves as System (2.2), the smooth property of orbits (1.4) with respect to the "time variable" $\xi$ should be studied, since the straight line $\phi = 0$ is not an orbit of (1.4). We shall consider this problem in the next section.

Of course, if the orbits of (1.4) are slightly far off from the line $\phi = 0$, then the smooth property of these invariant manifolds will be maintained. For the solutions of (1.4), we prefer to say the smoothness instead of the analyticity.

(ii) We should notice that on the right half $(\phi, y)$-phase plane, System (1.4) has the same orbit direction as Systems (2.2). But on the left half $(\phi, y)$-phase plane, whether the orbit direction of (1.4) is the same as (2.2) or not depends on whether $n$ is odd or even. In above all phase portraits, we only drew the vector fields of (2.2) on the right half phase plane.

## §3. Rapid Jump of $\phi'(\xi)$ Near a Segment of the Straight Line $\phi$=0

In this section we consider the dynamics of orbits of System (1.4). From Remark 2.1, in order to understand the dynamical behaviour of solutions of (1.4) with respect to the "time" variable $\xi$, we need to study the property of orbits close to the singular straight line $\phi = 0$. In other words, we have to know what distinguishes the singular straight line $\phi = 0$ of (1.4) geometrically from the straight line solution $\phi = 0$ of (2.2).

For short statement, we suppose that on the right side of the line $\phi = 0$, there is a periodic annulus with center $P_1(\phi_1, 0)$, which has boundary curves having the equation: $H(\phi, y) = 0$ defined by (1.5). From Figs.2.1–2.6, Fig.2.8 and Fig.2.10, we see that whether these boundary curves are closed or open depends on whether $n = 2$ and $n = 3$, $g = 0$ or $n = 3$, $g \neq 0$ and $n > 3$, because from $H(\phi, y) = 0$ and $g > 0$ we have

$$\phi = 0, \quad y^2 = \frac{\frac{2g}{n} + \frac{2c}{n+1}\phi - \frac{2a}{m+n}\phi^m}{n\phi^{n-2}} \quad \text{and} \quad \lim_{\phi \to 0} y^2 = \infty \quad \text{for } n \geq 3. \tag{3.1}$$

Fig.3.1. Three kinds of boundary curves of periodic annuluses.

Hence, there are three kinds of finite boundary curves of period annuluses as shown in Fig. 3.1 (i),(ii) and (iii). An orbit $\gamma$ in the family of periodic orbits on a period annulus of $P_1$ is a closed branch of the invariant curves $H(\phi, y) = h_\gamma$, where $h_\gamma \in (H(\phi_1, 0), 0)$.

From Fig.3.1 we see that suppose a point from a point $C(\phi_{_M}, 0)$ near the right boundary curve starts along a periodic orbit $\gamma$ whose invariant $h_\gamma$ is close to 0, as $\xi$ increasing, it will move to the left until it passes the "turning point" $B_\gamma$ at the intersection point with the horizontal inclination curve $\Gamma_H : n(n-1)\phi^{n-2}y^2 + a\phi^m - c\phi - g = 0$ of the vector field defined by (1.4). Then it will rise, almost vertically, to the upper arc of $\gamma$.

After it passes through another "turning point" $A_\gamma$, it will move to the right, and finishes his motion of one period to $C$. Let $(\phi_{tp}, \pm y_{tp})$ be the coordinates of $A_\gamma$ and $B_\gamma$. Then we have

$$H(\phi_{tp}, \pm y_{tp}) = h_\gamma, \quad y_{tp}^2 = \frac{g + c\phi_{tp} - a\phi_{tp}^m}{n(n-1)\phi_{tp}^{n-2}}. \tag{3.2}$$

Suppose that $B_{\gamma\epsilon}$ and $A_{\gamma\epsilon}$ are two points in a left neighbourhood of $B_\gamma$ and $A_\gamma$ on the orbit $\gamma$, respectively. A question is asked: what is the motion velocity of $\phi(\xi)$ on the segment $B_{\gamma\epsilon}A_{\gamma\epsilon}$ with respect to the "time variable" $\xi$? The following theorems will reply this problem.

**Theorem 3.1.** *When $h \to 0$, the periodic orbits of the periodic annulus surrounding $P_1$ approach to the boundary curves. Let $(\phi, y = \phi')$ be a point on a periodic orbit $\gamma$ of* (1.4). *Then along the segment $B_{\gamma\epsilon}A_{\gamma\epsilon}$ near the straight line $\phi = 0$, $y = \phi'(\xi)$ rapidly jumps in very short time interval of $\xi$.*

**Proof.** The first conclusion follows from the continuity of the level set $H(\phi, y) = h$ given by (1.5). We next show the second conclusion. Near the segment $\phi = 0$, assume that $\phi = \epsilon \ll 1$. Thus we can rewrite System (1.4) as the following relaxation oscillation system (see [4, 8]):

$$\frac{d\phi}{d\xi} = y, \quad \epsilon\frac{dy}{d\xi} = -(n-1)y^2 + \frac{g + c\phi - a\phi^m}{n\phi^{n-2}}. \tag{3.3}$$

We notice from (3.3) that for $n = 2$, $g \neq 0$,

$$(g + c\phi - a\phi^m - 2y^2)\frac{d\phi}{dy} = 2\epsilon y; \tag{3.4}$$

for $n=3$, $g=0$, $c\neq 0$,

$$(c-a\phi^{m-1}-6y^2)\frac{d\phi}{dy}=3\epsilon y; \tag{3.5}$$

and for $n>3$, $g\neq 0$,

$$\Big(\frac{g+c\phi-a\phi^m}{n\phi^{n-2}}-(n-1)y^2\Big)\frac{d\phi}{dy}=n\epsilon y. \tag{3.6}$$

Thus, as $\epsilon\to 0$, a segment $B_{\gamma\epsilon}A_{\gamma\epsilon}$ of every periodic orbit will tend to the line $\phi=0$.

Now we consider the periodic orbit $\gamma$ near the invariant level $h=0$ (see Fig.3.1). According to the first equation of (3.3), the period of $\gamma$ satisfies

$$T=\oint_\gamma\frac{d\phi}{y}=\Big(\int_{B_{\gamma\epsilon}A_{\gamma\epsilon}}+\int_{A_{\gamma\epsilon}CB_{\gamma\epsilon}}\Big)\frac{d\phi}{y}=T_1+T_2. \tag{3.7}$$

The contribution over the curve segment $y\in B_{\gamma\epsilon}A_{\gamma\epsilon}$ is

$$T_1=\int_{y_{B_{\gamma\epsilon}}}^{y_{A_{\gamma\epsilon}}}\frac{d\phi}{y}=\int_{y_{B_{\gamma\epsilon}}}^{y_{A_{\gamma\epsilon}}}\frac{d\phi dy}{ydy}=\int_{y_{B_{\gamma\epsilon}}}^{y_{A_{\gamma\epsilon}}}\frac{n\epsilon dy}{\frac{g+c\phi-a\phi^m}{n\phi^{n-2}}-(n-1)y^2}=O(\epsilon). \tag{3.8}$$

Because of $g+c\phi-a\phi^m-n(n-1)\phi^{n-2}y^2\neq 0$ along the curve segment from $B_{\gamma\epsilon}$ to $A_{\gamma\epsilon}$, (3.8) implies that $\phi'(\xi)$ jumps rapidly in very short time interval of $\xi$ (see Fig.3.2 (i)).

Fig.3.2. The profiles of $y=\phi'(\xi)$ and $\phi(\xi)$.

To understand the change of $y$ with the "time" $\xi$ when $(\phi,y)$ is close to the segment $\phi=0$, $y\in(-y_{\gamma\epsilon},y_{\gamma\epsilon})$, where $y_{\gamma\epsilon}$ is the $y$-coordinate of $A_{\gamma\epsilon}$, $y_{\gamma\epsilon}=y_{tp}-\epsilon$, we introduce the fast time scale $\tau=\frac{\xi}{\epsilon}$, and note (3.2), then System (3.3) becomes

$$\frac{d\phi}{d\tau}=\epsilon y,\quad \frac{dy}{d\tau}=(n-1)((y_{tp}-O(\epsilon))^2-y^2). \tag{3.9}$$

From the second equation of (3.9) we have

$$\begin{aligned}y&=\pm(y_{tp}-O(\epsilon))\tanh((n-1)(y_{tp}-O(\epsilon))\tau)\\&=\pm(y_{tp}-O(\epsilon))\tanh\Big((n-1)(y_{tp}-O(\epsilon))\frac{\xi}{\epsilon}\Big),\end{aligned} \tag{3.10}$$

where the positive sign on the right hand of (3.10) corresponds to $y\geq 0$. Using the language of the theory of singular perturbations, $\phi=\epsilon$ and $y$ defined by (3.10) is an asymptotic inner solution of the periodic orbit $\gamma$ of (1.4) on the segment $B_{\gamma\epsilon}A_{\gamma\epsilon}$.

Clearly, when $\tau \to \infty$, $y(\tau) \to \pm(y_{tp} - O(\epsilon))$. Hence, along the segment $B_{\gamma\epsilon}A_{\gamma\epsilon}$ of periodic orbit $\gamma$, $\gamma$ being very close to the segment of $\phi = 0$, when $y = \phi'(\xi)$ goes from the point $B_{\gamma\epsilon}$ to point $A_{\gamma\epsilon}$, the "time" $\xi$ undergoes only an interval of $O(\epsilon)$, i.e., the sign of $\phi'(\xi)$ changes very swiftly from $-$ to $+$. Therefore, on the periodic orbit $\gamma$, the corresponding traveling wave solution coordinate $\phi(\xi)$ undergoes two quite slow changes with $\xi$: decreasing from $\phi_M$ to $\phi_{tp}$ along the lower arc of $A_{\gamma\epsilon}CB_{\gamma\epsilon}$ and increasing from $\phi_{tp}$ to $\phi_M$ along the upper arc of $A_{\gamma\epsilon}CB_{\gamma\epsilon}$, followed by a very rapid change of $y = \phi'(\xi)$ on the segment $A_{\gamma\epsilon}B_{\gamma\epsilon}$: the value of $\phi(\xi)$ quite slowly changes in the "time interval" $O(\epsilon)$ of $\xi$, but the motion direction of $\phi(\xi)$ changes rapidly (see Fig.3.2 (ii)). Thus, we have

**Theorem 3.2.** *If there is a segment* $B_{\gamma\epsilon}A_{\gamma\epsilon}$ *(which does not degenerate to a point) on a periodic orbit* $\gamma$ *of* (1.4), $\gamma$ *being close to a segment of the straight line* $\phi = 0$, *then corresponding to this orbit* $\gamma$, *the traveling wave of* (1.1) *is a periodic cusp wave* (*Fig*.3.2 (ii)).

We notice that for a fixed parameter group of $(a, g, c)$, corresponding to the family of periodic orbits surrounding the center $P_1$ of (1.4) with $h \in (H(\phi_1, 0), 0)$, the periodic traveling wave solutions of equation $K(m, n)$ are not all cusp waves. As $h$ increases from $H(\phi_1, 0)$ to 0, the traveling wave solutions will gradually change from smooth waves to non-smooth cusp waves.

## §4. Smooth and Non-Smooth Traveling Wave Solutions on the Bifurcation Set

In this section, we shall study bounded traveling wave solutions when the parameter pair $(g, c)$ lies on a bifurcation curve on the $(g, c)$-parametric plane. We shall show that the compactons (see [11]), so called $K(m, n)$ compact solutions, just correspond to the oval orbits (smooth periodic wave solutions) of (2.2) when $n = 2$, $m \in Z^+$ and $(g, c) \in L_1^{\pm}$, where $Z^+$ denotes the set of positive integers. As to the peakons, they correspond to the boundary curve-triangle of periodic annuluses of (1.4) when $n = 2$, $m \in Z^+$ and $(g, c) \in L_4^{\pm}$ or $L_6$, $L_8$, respectively.

### 4.1. Smooth Periodic Traveling Waves: Compactons

Suppose that $(g, c) \in L_1^+$ (i.e., $g = 0$, $c > 0$) and $n = 2$, $m \in Z^+$. Then System (1.4) becomes

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{-2y^2 - a\phi^m + c\phi}{2\phi}, \tag{4.1}$$

which has the first integral

$$H_2(\phi, y) = \phi^2(2y^2 + \frac{2a}{m+2}\phi^m - \frac{2c}{3}\phi) = h. \tag{4.2}$$

We consider the case $a > 0$. Then System (4.1) has a center point $P_1(\phi_1, 0) = ((\frac{c}{a})^{\frac{1}{m-1}}, 0)$. When $h \in (H_2(\phi_1, 0), 0)$, where $H_2(\phi_1, 0) = -\frac{2(m-1)c}{3(m+2)}(\frac{c}{a})^{\frac{3}{m-1}}$, there is a family of periodic orbits of (4.1) with the level set $H_2(\phi, y) = h$ defined by (4.2). As $h \longrightarrow 0$, these periodic orbits approach to the limit oval orbit which is a branch of the curve $H_2(\phi, y) = 0$. Using (4.2) with $h = 0$ and the first equation of (4.1), we obtain the parametric representations of some limit oval orbits as follows.

$$m = 2, \quad \phi(\xi) = \frac{4c}{3a}\cos^2\Big(\frac{\sqrt{a}\xi}{4}\Big) = \frac{2c}{3a}\Big(1 + \cos\Big(\frac{\sqrt{a}\xi}{2}\Big)\Big), \quad y(\xi) = -\frac{c}{3\sqrt{a}}\sin\Big(\frac{\sqrt{a}\xi}{2}\Big). \tag{4.3}$$

Clearly, $\phi(\xi)$ is a periodic function with period $T_{(2,2)} = \frac{4\pi}{\sqrt{a}}$.

$$m = 3, \quad \phi(\xi) = \sqrt{\frac{5c}{3a}} cn^2\Big(\Big(\frac{ac}{60}\Big)^{\frac{1}{4}}\xi, \frac{1}{\sqrt{2}}\Big), \quad y(\xi) = \phi'(\xi), \tag{4.4}$$

where $cn(u,k)$ is the Jacobian elliptic function with modulus $k$. Obviously, $\phi(\xi)$ is a periodic function with period $T_{(3,2)} = 2(\frac{60}{ac})^{\frac{1}{4}} K(\frac{1}{\sqrt{2}})$, where $K(k)$ is the complete elliptic integral of the first kind.

$$m = 4, \quad \phi(\xi) = \Big(\frac{2c}{a}\Big)^{\frac{1}{3}} \frac{1 - cn(\omega_0\xi + \eta_0, k_0)}{(\sqrt{3}+1) + (\sqrt{3}-1)cn(\omega_0\xi + \eta_0, k_0)}, \quad y(\xi) = \phi'(\xi), \tag{4.5}$$

where $\omega_0 = (\frac{1}{3})^{\frac{1}{4}}(\frac{ac^2}{2})^{\frac{1}{6}}$, $\eta_0 = \frac{(3\pi^2)^{\frac{1}{4}}\Gamma(\frac{1}{6})}{3\Gamma(\frac{2}{3})}$, $k_0 = \frac{\sqrt{3}-1}{2\sqrt{2}}$. By (4.5), $\phi(\xi)$ has the period $T_{(4,2)} = \frac{4K(k_0)}{\omega_0}$.

For $m \geq 5$, let $\phi_0 = (\frac{(m+2)c}{3a})^{\frac{1}{m-1}}$ be the right intersection point of the limit oval with $\phi$-axis. We have

$$\sqrt{\frac{c}{3}}\xi = \int_{\phi_0}^{\phi} \frac{ds}{\sqrt{s(1 - \frac{3a}{(m+2)c}s^{m-1})}}. \tag{4.6}$$

This is the implicit parametric expression of limt oval orbit of (4.1). The classical mathematical analysis can not give the integral formula for $m \geq 5$. But by means of the numerical integral for solving initial value problems of (1.4), we may obtain various profiles of smooth periodic traveling waves (compactons) of $K(m,2)$, for example, the cases of $m = 7$ and $m = 8$ are shown in Fig. 4.1 (i), (ii), respectively. Unlike Theorem 3.1, here the limit oval orbit of (4.1) contacts the line $\phi = 0$ at $(0,0)$, the change of $y = \phi'(\xi)$ is continuous without a jump. Thus, we have a periodic traveling wave.

Fig.4.1. Compactons of $K(7,2)$ and $K(8,2)$,
Parameters: $m = 7$, $a = 3$, $c = 64$; $m = 8$, $a = 1$, $c = 38.4$.

Similarly, if $(g,c) \in L_1^-$, then when $a > 0$, $m = 2k$, $n = 2$ or $a < 0$, $m = 2k+1$, $n = 2$, by the results of Section 2, there is a limit oval orbit on the left of $\phi = 0$, which has the curve equation $H_2(\phi, y) = 0$. It gives similar parametric representations as (4.3)–(4.6). To sum up, we obtain

**Theorem 4.1** *Suppose that $g = 0$.*

(i) *If $a > 0$ and $c \neq 0$, for $h \in (H_2(\phi_1, 0), 0)$, corresponding to the family of periodic orbits of $(1.4)_{(m,2)}$, Equation $K(m, 2)$ $(m \in Z^+)$ has infinitely many smooth periodic traveling wave solutions. When $h \to 0$, these periodic traveling wave solutions converge to a smooth periodic traveling wave solution (compacton).*

(ii) *If $a < 0$ and $c < 0$, for Equation $K(2k + 1, 2)$, the similar conclusions hold.*

We notice that even though the above oval is a degenerate homoclinic orbit of (2.2) for the "time" variable $\zeta$, for System (4.1) it is a periodic orbit with respect to the "time" variable $\xi$. This is an important difference for the dynamical behaviour between (1.4) and (2.2).

**4.2. Periodic Cusp traveling Waves**

Suppose that $(g, c) \in L_1^+$ and $n = 3$, $m \in Z^+$. Then System (1.4) has the form

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{-6y^2 - a\phi^{m-1} + c}{3\phi}, \tag{4.7}$$

which has the first integral

$$H_3(\phi, y) = \phi^4\Big(3y^2 + \frac{2a}{m+3}\phi^{m-1} - \frac{c}{2}\Big) = h. \tag{4.8}$$

For the case $a > 0$, it is easy to see that for $m = 2k$ $(k = 1, 2, 3, \cdots)$, there is one center point $P_1(\phi_1, 0)$ of (4.7). System (4.7) has a family of periodic orbits with the invariant $h \in (H_3(\phi_1, 0), 0)$, where $H_3(\phi_1, 0) = -\frac{(m-1)c}{2(m+3)}\big(\frac{c}{a}\big)^{\frac{4}{m-1}}$. The boundary curves of period annuluses of (4.7) consist of the segment $\phi = 0$, $y \in (-\sqrt{\frac{c}{6}}, \sqrt{\frac{c}{6}})$ and an arch of the curve defined by (2.12) (see Fig.3.1 (i)). By a similar discussion for (3.9), near the above segment, an arc of a periodic orbit $\gamma$ of (4.7) has the asymptotic representation

$$\phi = \epsilon, \quad y = \phi' = \pm\Big(\sqrt{\frac{c}{6}} - O(\epsilon)\Big)\tanh\Big(2\Big(\sqrt{\frac{c}{6}} - O(\epsilon)\Big)\frac{\xi}{\epsilon}\Big). \tag{4.9}$$

Thus, by Theorems 3.1 and 3.2, as $\epsilon \to 0$, the family of periodic orbits of (4.7) approaches to a limit periodic orbit which corresponds to the boundary curves of the period annulus. Using (4.7) and (4.8), we obtain the following parametric representations of the periodic cusp waves.

$$m = 2, \quad \phi(\xi) = \frac{5c}{4a} - \frac{a}{30}\xi^2 \quad \text{for } \xi \in \Big[-\frac{5\sqrt{3c}}{\sqrt{2}a}, \frac{5\sqrt{3c}}{\sqrt{2}a}\Big], \tag{4.10}$$

where the motion defined by Equation (4.10) has the period $T_{(2,3)} = \frac{5\sqrt{6c}}{a}$.

$$m = 4, \quad \phi(\xi) = \Big(\frac{7c}{4a}\Big)^{\frac{1}{3}}\frac{(\sqrt{3}+1)cn(\omega_1\xi, k_1) - (\sqrt{3}-1)}{cn(\omega_1\xi, k_1) + 1} \quad \text{for } \xi \in [-\widetilde{\xi}_0, \widetilde{\xi}_0], \tag{4.11}$$

where $\omega_1 = \frac{(2ca^2)^{\frac{1}{6}}}{3^{\frac{1}{4}}7^{\frac{1}{3}}}$, $k_1 = \frac{\sqrt{3}+1}{2\sqrt{2}}$, $\widetilde{\xi}_0 = \frac{1}{\omega_1}F\Big(\arcsin\Big(\frac{2(3)^{\frac{1}{4}}}{\sqrt{3}+1}\Big), k_1\Big)$, $F(\cdot, k)$ is the elliptic integral of the first kind. The wave defined by (4.11) has the period $T_{(4,3)} = 2\widetilde{\xi}_0$.

For $m = 2k + 1$, System (4.7) has two center points $(-\phi_1, 0)$ and $(\phi_1, 0)$. There exist two corresponding period annuluses enclosed by curve (2.13) and the segment $\phi = 0, y \in (-\sqrt{\frac{c}{6}}, \sqrt{\frac{c}{6}})$. The two families of periodic orbits have the same invariant level $h \in (H_3(\phi_1, 0), 0)$.

The boundary curves of two period annuluses correspond to two periodic traveling cusp waves which have the parametric representations as follows.

$$m=3,\quad \phi(\xi)=\pm\sqrt{\frac{3c}{2a}}\cos\Big(\frac{\sqrt{a}\xi}{3}\Big)\quad \text{for } \xi\in\Big[-\frac{3\pi}{2\sqrt{a}},\frac{3\pi}{2\sqrt{a}}\Big], \tag{4.12}$$

this periodic cusp wave has the period $T_{(3,3)}=\frac{3\pi}{\sqrt{a}}$.

$$m=5,\quad \phi(\xi)=\pm\Big(\frac{2c}{a}\Big)^{\frac{1}{4}}cn\Big(\omega_2\xi,\frac{1}{\sqrt{2}}\Big)\quad \text{for } \xi\in\Big[-\frac{K(\frac{1}{\sqrt{2}})}{\omega_2},\frac{K(\frac{1}{\sqrt{2}})}{\omega_2}\Big], \tag{4.13}$$

where $\omega_2=(\frac{4ac}{72})^{\frac{1}{4}}$. This periodic cusp wave has the period $T_{(5,2)}=\frac{2K(\frac{1}{\sqrt{2}})}{\omega_2}$.

$$m=7,\quad \phi(\xi)=\pm\Big(\frac{5c}{2a}\Big)^{\frac{1}{6}}\frac{1-cn(\omega_3\xi-\eta_0,k_0)}{(\sqrt{3}+1)+(\sqrt{3}-1)cn(\omega_3\xi-\eta_0,k_0)}$$

$$\text{for } \xi\in\Big[-\frac{\eta_0}{\omega_3},\frac{\eta_0}{\omega_3}\Big], \tag{4.14}$$

where $\eta_0$, $k_0$ are the same as in (4.5), $\omega_3=(\frac{ac^2}{540})^{\frac{1}{6}}$. This periodic traveling cusp wave has the period $T_{(7,3)}=\frac{2\eta_0}{\omega_3}$.

For $m\geq 8$, let $\widetilde{\phi_0}=(\frac{(m+3)c}{4a})^{\frac{1}{m-1}}$ be the positive intersection points of the limit curve of the periodic family with the $\phi$-axis. We have

$$\sqrt{\frac{c}{6}}\xi=\int_{\widetilde{\phi_0}}^{\phi}\frac{ds}{\sqrt{s(1-\frac{4a}{(m+3)c}s^{m-1})}}. \tag{4.15}$$

This is the implicit parametric expression of the right limit arc orbit of (4.1). The classical mathematical analysis can not give the integral formula for $m\geq 8$. But by means of the numerical integral for solving initial value problems of (1.4), we may obtain various profiles of periodic cusp traveling waves of $K(m,3)$, for example, the cases of $m=8$ and $m=9$ are shown in Fig.4.2 (i), (ii), respectively.

Fig.4.2. Periodic cusp traveling waves of $K(8,3)$ and $K(9,3)$.
Parameters: $m=8$, $a=1.1$, $c=51.2$; $m=9$, $a=3$, $c=256$.

Thus, we have

**Theorem 4.2.** *Suppose that $g=0$. If $a>0$ and $c\neq 0$, for $h\in(H(\phi_1,0),0)$, corresponding to the two families of periodic orbits of $(1.4)_{(m,3)}$, Equation $K(m,3)$ $(m\in Z^+)$ has two* *families of infinitely many periodic traveling wave solutions. When $h$ is gradually changed from $H_3(\phi_1, 0)$ to 0, two series of periodic traveling wave solutions will gradually change their smoothness and converge to two periodic cusp traveling wave solutions, respectively.*

### 4.3. Solitary Cusp Traveling Waves: Peakons and Valleyons

Suppose that $(g, c) \in L_4^{\pm}$, $n = 2$, $m = 2k$ and $(g, c) \in L_6$ or $L_8$, $n = 2$, $m = 2k + 1$. Under these assumptions, there is a periodic annulus $P(\phi_c, 0)$ enclosed by a curve triangle (see Figs.2.2, 2.3 and 2.4) whose one border is a segment of $\phi = 0$, other two borders intersect at a finite equilibrium point $S(\phi_s, 0)$ of (1.4). Without loss of generality, we consider the cases $(g, c) \in L_4^-$ and $L_8$.

For $a < 0$ and $c < 0$, when $(g, c) \in L_4^-$, we have the phase portrait of (1.4) as shown in Fig.3.1 (ii). When $m = n = 2$, the upper and lower straight lines of the boundary triangle of the periodic annulus with center $(\frac{c}{3a}, 0)$ are $y = \mp\frac{\sqrt{|a|}}{2}(\phi - \frac{2c}{3a})$, which have the following parametric representations:

$$W^s: \quad \phi(\xi) = \frac{2c}{3a}\Big(1 - \exp\Big(-\frac{\sqrt{|a|}}{2}\xi\Big)\Big) \quad \text{for } \xi \in (0, \infty), \tag{4.16$_1$}$$

$$W^u: \quad \phi(\xi) = \frac{2c}{3a}\Big(1 - \exp\Big(\frac{\sqrt{|a|}}{2}\xi\Big)\Big) \quad \text{for } \xi \in (-\infty, 0). \tag{4.16$_2$}$$

It is clear that along $W^u$ (resp. $W^s$), as $\xi \to -\infty$ (resp. $\infty$), $\phi \to \frac{2c}{3a}$. By the same discussion as in Section 3, when $\xi$ approaches to 0 along $W^u$, $\phi'(\xi)$ will rapidly changes its sign from $-$ to $+$, but $\phi(\xi)$ is almost unchanged. Thus, this boundary triangle gives a solitary cusp traveling wave solution of equation $K(2, 2)$. Equation (4.16) just denotes the so-called "a dark peakon" in [13, p.312]. Here we have given its determinated geometrical pattern which corresponds to a boundary curve triangle (which contains a singular line and a finite equilibrium) of a period annulus of the traveling wave system $(1.4)_{(2,2)}$.

When $m = 4$, System $(1.4)_{(4,2)}$ has a saddle point $S(\phi_{s0}, 0)$, where $\phi_{s0} = (\frac{c}{2a})^{\frac{1}{3}}$. The upper and lower boundary curves of the period annulus with center $C$ are

$$y = \pm\sqrt{\frac{a}{6}}(\phi_{s0} - \phi)\sqrt{\phi^2 + 2\phi_{s0}\phi + 3\phi_{s0}^2}, \tag{4.17}$$

which have the parametric representations:

$$W^s: \quad \phi(\xi) = \Big(\frac{c}{2a}\Big)^{\frac{1}{3}} \frac{\Big((3\sqrt{2} + 4)\exp\Big(-\Big(\frac{|a|c^2}{4}\Big)^{\frac{1}{6}}\xi\Big) - 4\Big)^2 - 18}{\Big((3\sqrt{2} + 4)\exp\Big(-\Big(\frac{|a|c^2}{4}\Big)^{\frac{1}{6}}\xi\Big) + 2\Big)^2 - 6} \quad \text{for } \xi \in (0, \infty), \tag{4.18$_1$}$$

$$W^u: \quad \phi(\xi) = \Big(\frac{c}{2a}\Big)^{\frac{1}{3}} \frac{\Big((3\sqrt{2} + 4)\exp\Big(\Big(\frac{|a|c^2}{4}\Big)^{\frac{1}{6}}\xi\Big) - 4\Big)^2 - 18}{\Big((3\sqrt{2} + 4)\exp\Big(\Big(\frac{|a|c^2}{4}\Big)^{\frac{1}{6}}\xi\Big) + 2\Big)^2 - 6} \quad \text{for } \xi \in (-\infty, 0). \tag{4.18$_2$}$$

Similarly, as in the case $m = 2$, (4.18) defines a solitary cusp traveling wave solution of equation $K(4, 2)$.

We next suppose that $(g, c) \in L_8$, $m = 2k + 1$. From Fig.2.4, there are two periodic annuluses with centers $P_1$ and $P_2$. The boundary curves of two annulues are shown in Fig.3.1 (iii). For $m = 3$, System $(1.4)_{(3,2)}$ has a saddle point $S_+(\phi_s, 0) = (\sqrt{\frac{5c}{9a}}, 0)$ and two centers $P_1(\phi_{c1},0)$, $P_2(\phi_{c2},0)$, where

$$\phi_{c1}=\Big(\frac{\sqrt{21}-\sqrt{5}}{6}\Big)\sqrt{\frac{c}{a}},\quad \phi_{c2}=-\Big(\frac{\sqrt{21}+\sqrt{5}}{6}\Big)\sqrt{\frac{c}{a}}.$$

The upper and lower boundary curves of the boundary curve triangle of the period annulus of $P_1$ can be written as follows:

$$y=\pm\sqrt{\frac{a}{5}}(\phi_s-\phi)\sqrt{\phi+2\phi_s}, \tag{4.19}$$

which have the parametric representations, respectively,

$$W^s:\quad \phi(\xi)=\sqrt{\frac{5c}{9a}}\Big(3\frac{(1-(\sqrt{3}+\sqrt{2})^2\exp(-(\frac{c}{5a})^{\frac14}\xi))^2}{(1+(\sqrt{3}+\sqrt{2})^2\exp(-(\frac{c}{5a})^{\frac14}\xi))^2}-2\Big)\quad \text{for } \xi\in(0,\infty), \tag{4.20$_1$}$$

$$W^u:\quad \phi(\xi)=\sqrt{\frac{5c}{9a}}\Big(3\frac{(1-(\sqrt{3}+\sqrt{2})^2\exp((\frac{c}{5a})^{\frac14}\xi))^2}{(1+(\sqrt{3}+\sqrt{2})^2\exp((\frac{c}{5a})^{\frac14}\xi))^2}-2\Big)\quad \text{for } \xi\in(-\infty,0). \tag{4.20$_2$}$$

Hence, similarly to the case for equation $K(2,2)$, (4.20) defines a solitary cusp traveling wave solution of $K(3,2)$.

On the other hand, for the boundary curve arch of the period annulus with center $P_2$, Equation (4.20) with $\xi\in(-2(\frac{5a}{c})^{\frac14}\ln(\sqrt{3}+\sqrt{2}),2(\frac{5a}{c})^{\frac14}\ln(\sqrt{3}+\sqrt{2}))$ determines a periodic cusp traveling wave solution of $(1.4)_{(3,2)}$ with period $4(\frac{5a}{c})^{\frac14}\ln(\sqrt{3}+\sqrt{2})$.

For $n=2$, $m\geq 5$, the upper and lower boundary curves of the periodic annulus of the center $P$ satisfy

$$y=\pm\sqrt{\frac{g}{2}+\frac{c}{3}\phi-\frac{a}{m+2}\phi^m},\quad g=(-1)^{m-1}\frac{2(m-1)}{m+2}(|a|)^{-\frac{1}{m-1}}\Big(\frac{(m+2)|c|}{3m}\Big)^{\frac{m}{m-1}}.$$

Thus, we have the implicit parametric representation

$$\xi=\pm\int_0^{\phi}\frac{ds}{\sqrt{\frac{g}{2}+\frac{c}{3}s-\frac{a}{m+2}s^m}}. \tag{4.21}$$

It gives the implicit formula of solitary cusp traveling wave solution of $K(m,2)$. By means of the numerical integral for solving initial value problems of (1.4), we may obtain various profiles of the solitary cusp traveling waves of $K(m,2)$, for example, the cases of $m=6$ and $m=5$ are shown in Fig.4.1 (i), (ii), respectively.

Fig.4.3. A valleyon of $K(6,2)$ and a peakon of $K(5,2)$,

Parameters: $m = 6$, $a = -5832$, $c = -1728$, $g = 640$; $m = 5$, $a = 1$, $c = \frac{15}{7}$, $g = \frac{8}{7}$.

We point out that when $(g, c) \in L_4^-$ and $L_8$, we have $0 < \phi_c < \phi_s$, the profiles of the solitary traveling waves of $K(m, 2)$ are waves of valley form (see Fig.4.3 (i)). When $(g, c) \in L_4^+$ and $L_6$, the inequality $\phi_s < \phi_c < 0$ holds, similarly, the profiles of the solitary traveling waves of $K(m, 2)$ are waves of peak form (see Fig.4.3 (ii)). Thus, these two profies are called valleyons and peakons, respectively.

Thus, we obtain

**Theorem 4.3.** (i) *If $n = 2$, $m = 2k$, $(k = 1, 2, 3, \cdots)$, and $(g, c) \in L_4^\pm$, then corresponding to the boundary curve triangle of the period annulus with center $P$ of $(1.4)_{(2k,2)}$, equation $K(2k, 2)$ has a solitary cusp traveling wave solution. When $h \in (H(\phi_c, 0), 0)$, corresponding to the family of the periodic orbits of $(1.4)_{(2k,2)}$, equation $K(2k, 2)$ has infinitely many periodic cusp traveling wave solutions. As $h \to 0$, these periodic cusp waves converge to a solitary cusp wave.*

(ii) *If $n = 2$, $m = 2k + 1$, $(g, c) \in L_6, L_8$, then when $h \in (H(\phi_{c1}, 0), 0)$ or $h \in (H(\phi_{c2}, 0), 0)$, corresponding to two families of periodic orbits of $(1.4)_{(2k+1,2)}$, equation $K(2k + 1, 2)$ has two families of periodic cusp traveling wave solutions. As $h \to 0$, one family of periodic cusp waves converges to a solitary cusp wave (which corresponds to the boundary curve triangle of a periodic annulus of $(1.4)_{(2k+1,2)}$). Another family of periodic waves converges to a periodic cusp wave (which corresponds to the boundary arch of a periodic annulus of $(1.4)_{(2k+1,2)}$).*

**Remark 4.1.** In [12–14], the author claimed that "the prototype dispersive model $K(m, n)$ supports for $a > 0$ compact solitary traveling structures which for $n = m$ take very simple form,

$$u = \phi(\xi) = \Big(\frac{2cn}{n+1}\cos^2\Big(\frac{(n-1)\xi}{2n}\Big)\Big)^{\frac{1}{n-1}} \quad \text{for } |\xi| \le \frac{2n\pi}{n-1}, \tag{4.22}$$

and otherwise zero." It is easy to check that this result is incorrect. By a simple calculation, for $m = n > 3$, (4.22) does not satisfy the traveling wave equation (1.2) and (1.5) with $h = g = 0$. On the other hand, we see from Section 2 that when $m = n > 3$, $g = 0$, equation $(1.4)_{(m,m)}$ has only periodic cusp wave solutions. So the motion defined by (4.22) can not be a solitary wave solution of $K(m, m)$. When $m = n = 2$, (4.3) gives a smooth periodic wave of $K(2, 2)$. When $m = n = 3$, (4.12) defines a periodic cusp wave solution, which is not a solitary wave solution with the form of (4.22).

## §5. Existence and Convergence of Smooth and Non-Smooth Traveling Wave Solutions as Parameters Vary

In this section, we use the results of Section 2 to discuss the existence and the relationship of the traveling waves of different types when the parameter pair $(g, c)$ lies on the different regions on the $(g, c)$-parametric plane. By Remark 2.1 (i), when the orbits of (1.4) are far from the straight line $\phi = 0$, the solutions of (1.4) are smooth with respect to $\xi$. Corresponding to the smooth homoclinic orbits of (1.4), there are two types of solitary waves: the wave of peak form and the wave of valley form. In fact, suppose that a homoclinic orbit $\gamma_0(\xi) = (\phi_0(\xi), y_0(\xi))$ encloses the center $(\phi_c, 0)$ and is homoclimic to the saddle point $(\phi_s, 0)$ satisfying $\phi_c > \phi_s$, then $\gamma_0(\xi)$ is a solitary wave of peak form. Otherwise, if $\phi_c < \phi_s$, it is a solitary wave of valley form (see Fig.5.1 (i),(ii)).

Fig.5.1. the smooth solitary wave of valley form and smooth solitary wave of peakon form of $K(2,2)$; Parameters: (i) $a=1$, $c=-4$, $g=-1$; (ii) $a=1$, $c=4$, $g=-1$.

Thus, by using Figs.2.1-2.11 and the results in Section 3 we have

**Theorem 5.1.** (i) *Suppose that* $a>0$. *Then*

(1) *If* $(g,c)\in(B_1)$, *equation* $K(2k,2l)$ $(k,l\geq 1)$ *has a family of smooth periodic traveling wave solutions and a smooth solitary solution of peak form. If* $(g,c)\in(D_1)$, *equation* $K(2k,2l)$ *has a family of smooth periodic traveling wave solutions and a smooth solitary solution of valley form.*

(2) *If* $(g,c)\in(B_5),(D_5)$, *equation* $K(2k,2l+1)$ $(k,l\geq 1)$ *has a family of smooth periodic traveling wave solutions and a smooth solitary solution of peak form.*

(3) *If* $(g,c)\in(D_3)$, *equation* $K(2k+1,2)$ $(k\geq 1)$ *has a family of smooth periodic traveling wave solutions and a smooth solitary solution of peak form. If* $(g,c)\in(B_3)$, *equation* $K(2k+1,2)$ *has two families of periodic traveling wave solutions and a smooth solitary solution of peak form; moreover, one of the families of periodic waves is smooth, while the other contains periodic cusp waves.*

(4) *If* $(g,c)\in(C_6)$, *equation* $K(2k+1,2l+1)$ $(k,l\geq 1)$ *has two families of periodic traveling wave solutions and a smooth solitary solution of peak form; moreover, one of the families of periodic waves is smooth, while the other contains periodic cusp waves. If* $(g,c)\in(B_6)$, *equation* $K(2k+1,2l+1)$ *has two families of periodic traveling wave solutions and a smooth solitary solution of valley form; moreover, one of the families of periodic waves is smooth, while the other contains periodic cusp waves.*

(5) *If* $(g,c)\in(C_{10})$, *equation* $K(2k+1,2l)$ $(k,l\geq 1)$ *has a family of smooth periodic traveling wave solutions and a smooth solitary solution of peak form. If* $(g,c)\in(B_{10})$, *equation* $K(2k+1,2l)$ *has two families of periodic traveling wave solutions and a smooth solitary solution of peak form; moreover, one of the families of periodic waves is smooth, while the other contains periodic cusp waves.*

(ii) *Suppose that* $a<0$. *Then*

(1) *If* $(g,c)\in(B_2),(A_9)$ *equation* $K(2k,2l)$ $(k,l\geq 1)$ *has a family of smooth periodic traveling wave solutions and a smooth solitary solution of valley form. If* $(g,c)\in(D_2),(C_9)$, *equation* $K(2k,2l)$ *has a family of smooth periodic traveling wave solutions and a smooth solitary solution of peak form.*

(2) *If* $(g,c) \in (\overline{B}_5), (\overline{D}_5)$, *equation* $K(2k, 2l+1)$ $(k, l \geq 1)$ *has a family of smooth periodic traveling wave solutions and a smooth solitary solution of valley form, where* $(\overline{B}_5), (\overline{D}_5)$ *are the reflection regions of* $(B_5), (D_5)$ *with respect to the c-axis on the* $(g,c)$*-parametric plane.*

(3) *If* $(g,c) \in (D_4)$, *equation* $K(2k+1, 2)$ $(k \geq 1)$ *has a family of smooth periodic traveling wave solutions and a smooth solitary solution of valley form. If* $(g,c) \in (B_4)$, *equation* $K(2k+1, 2)$ *has two families of periodic traveling wave solutions and a smooth solitary solution of valley form; moreover, one of the families of periodic waves is smooth, while the other contains periodic cusp waves.*

(4) *If* $(g,c) \in (B_7)$, *equation* $K(2k+1, 2l+1)$ $(k, l \geq 1)$ *has a family of periodic traveling wave solutions and a smooth solitary solution of peak form. If* $(g,c) \in (A_7)$, *equation* $K(2k+1, 2l+1)$ *has a family of periodic traveling wave solutions and a smooth solitary solution of valley form.*

(5) *If* $(g,c) \in (D_{11})$, *equation* $K(2k+1, 2l)$ $(k, l \geq 1)$ *has a family of smooth periodic traveling wave solutions and a smooth solitary solution of valley form. If* $(g,c) \in (A_{11})$, *equation* $K(2k+1, 2l)$ *has two families of periodic traveling wave solutions and a smooth solitary solution of valley form; moreover, one of the families of periodic waves is smooth, while the other contains periodic cusp waves.*

There exist a lot of parametric regions in the $(g,c)$-parametric plane such that equation $K(m,n)$ has one or two families of periodic cusp traveling wave solutions. From the results of Sections 2 and 3 we have

**Theorem 5.2.** (i) *Suppose that* $a > 0$. *Then*

(1) *If* $(g,c) \in (A_1)$, *equation* $K(2k, 2l)$ $(k, l \geq 1)$ *has two families of periodic cusp traveling wave solutions.*

(2) *If* $(g,c) \in (A_5)$, *equation* $K(2k, 2l+1)$ $(k, l \geq 1)$ *has a family of periodic cusp traveling wave solutions.*

(3) *If* $(g,c) \in$ (*resp.* $A_3$) (*or* $(C_3)$, *equation* $K(2k+1, 2)$ $(k \geq 1)$ *has a family* (*resp. two families*) *of periodic cusp traveling wave solutions.*

(4) *If* $(g,c) \in (A_6), (D_6)$ *and* $L_5^{\pm}$, *equation* $K(2k+1, 2l+1)$ $(k, l \geq 1)$ *has one family of periodic cusp traveling wave solutions.*

(5) *If* $(g,c) \in (A_{10}), L_1^+$ *and* $L_5^+$ *in Fig.2.10, equation* $K(2k+1, 2l)$ $(k, l \geq 1)$ *has a family of periodic cusp traveling wave solutions.*

(ii) *Suppose that* $a < 0$. *Then*

(1) *If* $(g,c) \in (A_2), (E_2)$, *equation* $K(2k, 2)$ $(k, l \geq 1)$ *has one family of periodic cusp traveling wave solutions.*

(2) *If* $(g,c) \in (\overline{A}_5)$, *equation* $K(2k, 2l+1)$ $(k, l \geq 1)$ *has a family of periodic cusp traveling wave solutions, where* $(\overline{A}_5)$ *is the reflection region of* $(A_5)$ *with respect to the c-axis on the* $(g,c)$*-plane.*

(3) *If* $(g,c) \in (A_4)$ (*resp.* $(C_4)$), *equation* $K(2k+1, 2)$ $(k \geq 1)$ *has a family* (*resp. two families*) *of periodic cusp traveling wave solutions.*

(4) *If* $(g,c) \in (B_{11})$, *equation* $K(2k+1, 2l)$ $(k, l \geq 1)$ *has a family of periodic cusp traveling wave solutions.*

When the parameter pair $(g,c)$ in a parameter region approaches to a boundary curve (so called the bifurcation set), what happen about the dynamics of solutions of equation $K(m,n)$? By the continuity of the first integral with respect to the parameters $g$, $a$ and $c$, we have

**Theorem 5.3.** (i) *If $a > 0$, $(g, c) \in (B_1)$ (resp. $(A_1)$), then as $(g, c) \to L_1^+$, the series of smooth solitary wave solutions (resp. the series of periodic cusp waves) of $K(2k, 2)$ converges to a smooth periodic wave defined by* (4.6).

(ii) *If $a < 0$, $(g, c) \in (B_2)$ (resp. $(A_2)$), then as $(g, c) \to L_4^-$, the series of smooth solitary wave solutions (resp. the series of periodic cusp waves) of $K(2k, 2)$ converges to the solitary cusp wave solution defined by* (4.21).

(iii) *If $a > 0$, $(g, c) \in (B_5)$, then the series of smooth solitary wave solutions of $K(2k, 3)$ converges to a periodic cusp wave defined by* (4.15).

We point out that the results on the qulitative analysis in Sections 2 and 3 have given us all information of the dynamical behaviour for the traveling wave system (1.4) of $K(m, n)$. By these knowledges, we can make corresponding numerical study for equation $K(m, n)$ when $m, n > 5$.

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