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THE CODIMENSION FORMULA ON AF-COSUBMODULES**

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Abstract

Let M_1, M_2 be submodules of analytic Hilbert module X on $\Omega(\subset C^n)$ such that $M_1 \supseteq M_2$ and $\dim M_1/M_2 = k < \infty$. If M_2 is an AF-cosubmodule, then the codimension $\dim M_1/M_2$ of M_2 in M_1 equals the cardinality of zeros of M_2 related to M_1 by counting multiplicities. The codimension formula has some interesting applications. In particular, the author calculates out the dimension of Rudin quotient module, which is raised in [14].

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§1. Introduction

The Beurling's theorem shows that all submodules of the Hardy module $H^2(D)$ on the unit disk D are isomorphic[11]. In generalizing this result to several variables, one was somewhat surprised to find that the analogous result is false[2,5,6,10,11]. Due to the extreme complexity of the structure of analytic submodules of several variables, the additional assumption on finite codimension was naturally adopted as the first step towards a better understanding of their properties[1,6,7,8,9].

In [1], we study the structure of zero varieties of Hardy-submodules generated by polynomials. In [2], we developed the characteristic space theory of analytic Hilbert modules to study algebraic reduction and rigidity of Hilbert modules. This theory enables us to obtain a complete classification under unitary equivalence for Hardy submodules of several variables which are generated by polynomials (see [3]). An application to quasi-invariant subspaces of the Fock space can be found in [4]. In this note, using the characteristic space theory, we generalize the main results in [1] to the case of AF-cosubmodules of analytic Hilbert modules of several variables. Some interesting applications are obtained. In particular, we calculate out the dimension of Rudin quotient module, which is raised in [14].

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§2. Preliminary Notations

Let us recall some basic notations (see [2, 10]). Let Ω be a bounded nonempty open subset of C^n, $\mathrm{Hol}(\Omega)$ denote the ring of analytic functions on Ω, and X be Banach space contained in $\mathrm{Hol}(\Omega)$. We call X a reproducing Ω-space if X contains 1 and if for each $w \in \Omega$ the evaluation function, $E_w(f) = f(w)$, is a continuously linear functional on X. Write $\mathcal{C}$ for the ring of all polynomials on C^n. We call X a reproducing $\mathcal{C}$-module on Ω if X is a reproducing Ω–space, and for each polynomial p and each $x \in X$, $p\,x$ is contained in X. Note that, by a simple application of the closed graph theorem, the operator T_p defined to be multiplication by p is bounded on X for each $p \in \mathcal{C}$. Note also that $\mathcal{C} \subset X$ follows from the fact that 1 is in X. For $w \in C^n$, we call w a vertual point of X provided that the linear functional $f \mapsto f(w)$ defined on $\mathcal{C}$ extends to a bounded linear function on X. We use $vp(X)$ to denote the collection of all vertual points, then $vp(X) \supseteq \Omega$. We say that X is an analytic Hilbert module on Ω if the following conditions are satisfied:

(1) X is a reproducing $\mathcal{C}$-module on Ω;

(2) $\mathcal{C}$ is dense in X;

(3) $vp(X) = \Omega$.

For a polynomial $q = \sum a_{m_1\cdots m_n} z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$, let $q(D)$ denote the linear partial differential operator $\sum a_{m_1\cdots m_n} \frac{\partial^{m_1+m_2+\cdots+m_n}}{\partial z_1^{m_1}\partial z_2^{m_2}\cdots\partial z_n^{m_n}}$. Let M be a submodule of X, and $\lambda \in \Omega$. Set

$$M_\lambda = \{q \in \mathcal{C} \mid q(D)f|_\lambda = 0,\ \forall f \in M\},$$

where $q(D)f|_\lambda$ denotes $\big(q(D)f\big)(\lambda)$. By [1 or 2], M_λ is invariant under the action by basic partial differential operators $\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2} \cdots, \frac{\partial}{\partial z_n}\}$, and M_λ is called the characteristic space of M at λ. The envelope of M at λ, M_λ^e, is defined by

$$M_\lambda^e = \{f \in X \mid q(D)f|_\lambda = 0,\ \forall q \in M_\lambda\}.$$

Then M_λ^e is a submodule of X, and $M_\lambda^e \supseteq M$ (see [2]).

Let M be a submodule of X. We call M approximately finite codimensional (abbr., AF-cosubmodule) if M is equal to the intersection of all finite codimensional submodules containing M. The reason is that in this case, M is just the limit of decreasing net ($\supseteq$) of all finite codimensional submodules containing M. For a submodule M, the AF-envelope of M is defined by the intersection of all finite codimensional submodules containing M, and denoted by M^e. Clearly, the definition implies that the envelope of a submodule M is an AF-cosubmodule. In what follows we will use $Z(M)$ to denote the zero set of M, that is, $Z(M) = \{\lambda \in C^n \mid f(\lambda) = 0,\ \forall f \in M\}$. The next lemma illustrates some basic properties of characteristic space and envelope[2].

Lemma 2.1.[2] *Let X be an analytic Hilbert module, and M a submodule of X. Then we have*

(1) *if $Z(M) = \emptyset$, then $M^e = X$;*

(2) *if $Z(M) \neq \emptyset$, then $M \subseteq M^e \neq X$, $(M^e)^e = M^e$, and $Z(M) = Z(M^e)$;*

(3) *if $Z(M) \neq \emptyset$, then $M^e = \bigcap\limits_{\lambda \in Z(M)} M_\lambda^e$.*

In particular, let M_1, M_2 be two submodules of X, then $M_1^e = M_2^e$ if and only if $Z(M_1) = Z(M_2)$, and for every $\lambda \in Z(M_1)$, $M_{1\lambda} = M_{2\lambda}$.

§3. The Codimension Formula on AF-Cosubmodules

Following the notation in [1], we let M_1, M_2 be submodules of X, and $\lambda \in \Omega$. We call that

M_1, M_2 have the same multiplicity at λ if $M_{1\lambda}=M_{2\lambda}$. Let the symbol $Z(M_2)\backslash Z(M_1)$ denote the set of zeros of M_2 related to M_1, that is, $Z(M_2)\backslash Z(M_1)$ is defined by $\{\lambda\in Z(M_2)\mid M_{2\lambda}\neq M_{1\lambda}\}$. If $M_1\supseteq M_2$, the cardinality of zeros of M_2 related to M_1, $\operatorname{card}\big(Z(M_2)\backslash Z(M_1)\big)$, is defined by $\sum\limits_{\lambda\in Z(M_2)\backslash Z(M_1)}\dim M_{2\lambda}/M_{1\lambda}$.

Theorem 3.1. *Let M_1, M_2 be submodules of analytic Hilbert module X on Ω such that $M_1\supseteq M_2$ and $\dim M_1/M_2=k<\infty$. If M_2 is an AF-submodule, then we have*

(1) $Z(M_2)\backslash Z(M_1)=\sigma_p(M_{z_1},M_{z_2},\cdots,M_{z_n})\subset\Omega$,

(2) $M_2=\{h\in M_1\mid p(D)h|_\lambda=0,\ p\in M_{2\lambda},\ \lambda\in Z(M_2)\backslash Z(M_1)\}$,

(3) $\dim M_1/M_2=\sum\limits_{\lambda\in Z(M_2)\backslash Z(M_1)}\dim M_{2\lambda}/M_{1\lambda}=\operatorname{card}\big(Z(M_2)\backslash Z(M_1)\big)$,

where $(M_{z_1},M_{z_2},\cdots,M_{z_n})$ is an n-tuple of operators which are defined on the quotient module M_1/M_2 by $M_{z_i}\tilde{f}=\widetilde{(z_if)}$ for $i=1,\cdots,n$, and $\sigma_p(M_{z_1},\cdots,M_{z_n})$ denotes the joint eigenvalues of the n-tuple $(M_{z_1},M_{z_2},\cdots,M_{z_n})$. It is worth noticing that (3) *of Theorem* 3.1 *says the codimension* $\dim M_1/M_2$ *of M_2 in M_1 equals the cardinality of zeros of M_2 related to M_1. In this way,* (3) *is the codimension formula what we say.*

Proof. The proof is similar to that of [1, Theorem 2.4] (see also [1, Theorem 3.1]). For completeness, we give the details of the proof.

(1) Write

$$M_1\ =\ M_2\oplus R$$

and restrict $(M_{z_1},M_{z_2},\cdots,M_{z_n})$ on R. By [16], they can be simultaneously triangularized as

$$M_{z_i}=\begin{pmatrix}\lambda_i^{(1)} & & \star\\ & \ddots & \\ & & \lambda_i^{(k)}\end{pmatrix}.$$

Here $i=1,2,\cdots,n$, and $k=\dim M_1/M_2$, so that $\sigma_p(M_{z_1},M_{z_2},\cdots,M_{z_n})$ is equal to $\{\lambda^{(1)},\cdots,\lambda^{(k)}\}$. From the definition of analytic Hilbert module, the inclusion $\sigma_p(M_{z_1},M_{z_2},\cdots,M_{z_n})\subset\Omega$ is immediate. Writing

$$\mathcal{O}_{\lambda^{(j)}}\ =\ \{f\mid f\in\mathcal{C},\ \text{and}\ f\big(\lambda^{(j)}\big)\ =\ 0\},$$

$j=1,\cdots,k$, we have

$$\mathcal{O}_{\lambda^{(k)}}\cdots\mathcal{O}_{\lambda^{(2)}}\mathcal{O}_{\lambda^{(1)}}M_1\subseteq M_2\subseteq M_1.$$

Therefore, for $\lambda\in\Omega$ and $\lambda\notin\sigma_p(M_{z_1},M_{z_2},\cdots,M_{z_n})$, one has that $M_{1\lambda}=M_{2\lambda}$. This implies that

$$Z(M_2)\backslash Z(M_1)\subseteq\sigma_p(M_{z_1},M_{z_2},\cdots,M_{z_n}).$$

Let $\lambda\in\sigma_p(M_{z_1},M_{z_2},\cdots,M_{z_n})$. Since λ is a joint eigenvalue of $(M_{z_1},M_{z_2},\cdots,M_{z_n})$, there is a function $h\in M_1$, and $h\notin M_2$ such that $\mathcal{O}_\lambda h\subseteq M_2$. Let $M_2^\dagger$ be a submodule generated by M_2 and h. Then for $\lambda'\in\Omega$ and $\lambda'\neq\lambda$, we have $(M_2^\dagger)_{\lambda'}=M_{2\lambda'}$.

If $(M_2^\dagger)_\lambda=M_{2\lambda}$, then by Lemma 2.1(3), we have $(M_2^\dagger)^e=M_2^e=M_2$. This is clearly impossible. Hence, $M_{2\lambda}\supsetneqq(M_2^\dagger)_\lambda\supseteq M_{1\lambda}$. It follows that λ is in $Z(M_2)\backslash Z(M_1)$. We thus conclude that

$$Z(M_2)\backslash Z(M_1)=\sigma_p(M_{z_1},M_{z_2},\cdots,M_{z_n})\subset\Omega.$$

This completes the proof of (1).

(2) Set

$$M_2^\natural = \{h \in M_1 \mid p(D)h|_\lambda = 0,\ p \in M_{2\lambda},\ \lambda \in Z(M_2)\backslash Z(M_1)\}.$$

Then $M_2^\natural$ is an AF-cosubmodule which contains M_2. It is easy to see that for every $\lambda \in \Omega$, $(M_2^\natural)_\lambda = M_{2\lambda}$. Therefore, by Lemma 2.1(3), we have $(M_2^\natural)^e = M_2^e = M_2$. This implies that $M_2^\natural = M_2$. The proof of (2) is complete.

(3) The proof is by induction on numbers of points in $Z(M_2)\backslash Z(M_1)$. If $Z(M_2)\backslash Z(M_1)$ contains only one point λ, then by (2), M_2 can be written as

$$M_2 = \{h \in M_1 \mid p(D)h|_\lambda = 0,\ p \in M_{2\lambda}\}.$$

We define the pairing

$$[-,-]\ :\ M_{2\lambda}/M_{1\lambda} \times M_1/M_2 \to C$$

by $[\tilde{p}, \tilde{h}] = p(D)h|_\lambda$. Clearly, this is well-defined. From this pairing and the representation of M_2, it is not difficult to see that

$$\dim M_1/M_2 = \dim M_{2\lambda}/M_{1\lambda}.$$

Now let $l > 1$, and assume that (3) has been proved for $Z(M_2)\backslash Z(M_1)$ containing points less than l. Let $Z(M_2)\backslash Z(M_1) = \{\lambda_1, \cdots, \lambda_l\}$; here $\lambda_i \neq \lambda_j$ for $i \neq j$. Writing

$$M_2^\star = \{h \in M_1 \mid p(D)h|_{\lambda_1} = 0,\ p \in M_{2\lambda_1}\},$$

we have $(M_2^\star)_{\lambda_1} = M_{2\lambda_1}$. Similarly to the preceding proof, we have

$$\dim M_1/M_2^\star = \dim M_{2\lambda_1}/M_{1\lambda_1}.$$

Write $M_{2\lambda_1} = M_{1\lambda_1} \dot{+} R$ with $\dim R = \dim M_{2\lambda_1}/M_{1\lambda_1}$, and let $\sharp R$ denote the linear space of polynomials generated by R such that it is invariant under the action by $\{\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}\}$. Put

$$\mathcal{Q}_{\mathcal{R}} = \{p \in \mathcal{C} \mid\ q(D)p|_{\lambda_1} = 0;\ q \in \sharp R\}.$$

Then it is easily verified that $\mathcal{Q}_{\mathcal{R}}$ is a finite codimensional ideal of $\mathcal{C}$ with only zero point λ_1 because $\sharp R$ is finite dimensional. From the definition of $M_2^\star$, the following inclusions are easily verified:

$$\mathcal{Q}_{\mathcal{R}} M_1 \subseteq M_2^\star \subseteq M_1.$$

Consequently, for $\lambda \neq \lambda_1$, $M_{1\lambda} = (M_2^\star)_\lambda$. So,

$$Z(M_2)\backslash Z(M_2^\star) = \{\lambda_2, \cdots, \lambda_l\}.$$

By the induction hypothesis, we have

$$\dim M_2^\star/M_2 = \sum_{j=2}^{l} \dim M_{2\lambda_j}/(M_2^\star)_{\lambda_j} = \sum_{j=2}^{l} \dim M_{2\lambda_j}/M_{1\lambda_j}.$$

It follows that

$$\begin{aligned}\dim M_1/M_2 = \dim M_1/M_2^\star + \dim M_2^\star/M_2 &= \sum_{j=1}^{l} \dim M_{2\lambda_j}/M_{1\lambda_j} \\ &= \operatorname{card}\big(Z(M_2)\backslash Z(M_1)\big).\end{aligned}$$

The proof of Theorem 3.1 is thus completed.

Remark 3.1. It is worth noticing that the assumption is necessary in Theorem 3.1 that M_2 is approximately finite codimensional. In fact, by [12], we know that there exists a submodule M of the Bergman module $L_a^2(D)$ of the unit disk D such that $\dim M/zM = 2$, while $Z(zM)\backslash Z(M) = \{0\}$ and $\dim (zM)_0/M_0 = 1$.

Now assume that M is a finite codimensional submodule of X. Then Theorem 3.1 implies the following

Corollary 3.1. *Let M be a finite codimensional submodule of X, then we have*

(1) $Z(M) = \sigma_p(M_{z_1}, M_{z_2}, \cdots, M_{z_n}) \subset \Omega$,

(2) $M = \bigcap\limits_{\lambda\in Z(M)} M_\lambda^e$,

(3) $\operatorname{codim} M = \sum\limits_{\lambda\in Z(M)} \dim M_\lambda = \operatorname{card}\bigl(Z(M)\bigr)$.

Notice that (3) of Corollary 3.1 says that the codimension codim M of M in X equals the cardinality of zeros of M by counting multiplicities.

As in the proof of Theorem 3.1, we also have the following

Theorem 3.2. *Let M_1, M_2 be submodules of analytic Hilbert module X on Ω such that $M_1 \supseteq M_2$ and $\dim M_1/M_2 = k < \infty$. If M_1 is an AF-cosubmodule, then we have*

$$\dim M_1/M_2 \geq \operatorname{card}\bigl(Z(M_2)\backslash Z(M_1)\bigr),$$

and the equalily holds if and only if M_2 is an AF-cosubmodule.

§4. Applications

In this section we give three examples to show applications of the results in this note.

Example 4.1. Let I be a finite codimensional ideal of polynomials on C^n. Then I has an irredundant primary decomposition in $\mathcal{C}$ (see [13]); $I = \bigcap\limits_{j=1}^{m} I_j$, where I_j is primary for a maximal ideal of evaluation at some point λ_j. Since $I_i + I_j = \mathcal{C}$ for $i \neq j$, it follows that $I = \prod\limits_{j=1}^{m} I_j$. This derives that for any natural number k,

$$I^k = \prod_{j=1}^{m} I_j^k = \bigcap_{j=1}^{m} I_j^k.$$

So

$$\operatorname{codim} I^k = \sum_{j=1}^{m} \operatorname{codim} I_j^k.$$

As is well known, for a large integer k, $\operatorname{codim}(I_j^k)$ is a polynomial of k with the degree n which is called the Hilbert-Samuel polynomial of I_j (see [13]). Therefore, for each finite codimensional ideal I, the codimension of I^k, $\operatorname{codim}(I^k)$, is a polynomial of k with the degree n, which is said to be a Hilbert-Samuel polynomial of I, and denoted by $\mathcal{P}_I(k)$. Let X be an analytic Hilbert module on Ω, and I be an ideal of $\mathcal{C}$. From [2], I can be uniquely decomposed into $I_\Omega \bigcap I_{\Omega^c}$ such that each algebraic component of I_Ω meets Ω nontrivially, and each of I_{Ω^c} does not. Write $[I]$ for the closure of I in X, then $[I]$ is a submodule of X. Let I be a finite codimensional ideal. Then for any natural number k, one has $[I^k] = [I_\Omega^k]$. From Corollary 2.8 in [10], the equality $\operatorname{codim}[I^k] = \operatorname{codim} I_\Omega^k$ is immediate. From Corollary 3.1, we have the following: Let I be a finite codimensional ideal. Then for a large integer k, the cardinality of zeros of $[I^k]$, $\operatorname{card}\bigl(Z([I^k])\bigr)$, is a polynomial of k with the degree n, more precisely, $\operatorname{card}\bigl(Z([I^k])\bigr) = \mathcal{P}_{I_\Omega}(k)$.

Example 3.2. Recall that Rudin's submodule M of $H^2(D^2)$ over the bidisk is defined to be the collection of all functions in $H^2(D^2)$ which have a zero of order greater than or equal to

n at $(0, 1-n^{-3})$ for $n = 1, 2. \cdots$. Douglas and Yang[14] showed that $M \ominus (zM + wM)$ is finite dimensional, while $M \ominus (zM + wM)$ is not a generating set of M. They raised the question what $\dim\big(M \ominus (zM + wM)\big)$ is equal to. It is easy to check that both M and $\overline{zM + wM}$ are AF-cosubmodules, and $Z(\overline{zM + wM}) \setminus Z(M) = \{(0, 0)\}$, $\mathrm{card}\big(Z(\overline{zM + wM}) \setminus Z(M)\big) = 2$. Theorem 3.1 thus implies that $\dim\big(M \ominus (zM + wM)\big)$ is equal to 2.

Example 3.3. Let M be a submodule of the Bergman module $L_a^2(D)$ over the disk algebra. By Aleman, Richter and Sunderberg's work[15], we know that $M \ominus zM$ is a generating set for M. It is easy to see that $\dim(M \ominus zM) \le \mathrm{rank}(M)$. One thus concludes that $\dim(M \ominus zM) = \mathrm{rank}(M)$. Therefore, for any natural number n, unlike the Hardy module $H^2(D)$, the Bergman module $L_a^2(D)$ has a submodule M with rank n (see [12]). Let M be an AF-cosubmodule of $L_a^2(D)$ with $\mathrm{rank}(M) < \infty$. This implies that zM also is an AF-cosubmodule. It is easy to check that $\mathrm{card}\big(Z(zM) \setminus Z(M)\big) = 1$, and hence $\dim(M \ominus zM) = 1$. We conclude that $\mathrm{rank}(M) = 1$. It follows that every AF-cosubmodule with finite rank is generated by a single function.

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