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ON THE SYMPLECTIC REDUCTIONS

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Abstract

A symplectic reduction method for symplectic G-spaces is given in this paper without using the existence of momentum mappings. By a method similar to the above one, the arthors give a symplectic reduction method for the Poisson action of Poisson Lie groups on symplectic manifolds, also without using the existence of momentum mappings. The symplectic reduction method for momentum mappings is thus a special case of the above results.

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§1. Introduction

Let $\sigma : G \times M \to M$ be a left Hamiltonian action of a Lie group G on a symplectic manifold (M, ω), then there must be a momentum mapping $J : M \to \mathcal{G}^*$ (dual of the Lie algebra $\mathcal{G}$ of G). When J is Ad^*-equivariant, $\mu \in \mathcal{G}^*$ a regular value of J and the isotropy group $K \subset G$ of μ under the Ad^* action on $\mathcal{G}^*$ acts on $J^{-1}(\mu)$ freely and properly, there is a unique symplectic structure ω_μ on $M_\mu = K \setminus J^{-1}(\mu)$ (space of orbits), such that (M_μ, ω_μ) is the symplectic reduction phase space of the action σ (see [1, 2]).

However momentum mapping on a Hamiltonian G-space is not unique. For each momentum mapping, there is a cocycle on G with its value in $\mathcal{G}$. All the cocycles belong to the same cohomology class. When the cohomology class is not zero, there is no Ad^*-equivariant momentum mapping for the action σ. In this case another action Ψ of G on $\mathcal{G}^*$ will take the place of the Ad^* action so that J is equivariant with respect to σ and Ψ and the Marsden -Weinstein symplectic reduction procedure can be carried out as usual. In this case, we choose K to be the isotropy group at $\mu \in \mathcal{G}^*$ under the action Ψ (see [1]).

If M is a general symplectic G-space instead of a Hamiltonian G-space, there does not exist any momentum mapping. Can symplectic reduction procedure still be carried out? An affirmative answer is given by Theorem 3.1.

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Suppose $\sigma : G \times M \to M$ is a Poisson action of Poisson Lie group G on symplectic manifold (M, ω). If there exists a momentum mapping $J : M \to G^*$ (dual Poisson Lie group of G) (see [3]), a Poisson tensor π_J on G^* and a left dressing action λ_J determined by π_J can be chosen such that J is equivariant with respect to σ and λ_J. Now let $\mu \in G^*$ be a regular value of J, K the isotropy group at μ under λ_J which acts freely and properly on $J^{-1}(\mu)$. Then there exists a symplectic structure ω_μ on $M_\mu = K \setminus J^{-1}(\mu)$ satisfying $p^*\omega_\mu = i^*\omega$ and (M_μ, ω_μ) is the symplectic reduced phase space of σ (see [3]). Here we also suppose the existence of a momentum mapping J. In [3] the existence of J is proved under the condition that M is simply connected. However in general we do not know if such a momentum mapping does exist.

Theorem 4.1 in this paper gives a symplectic reduction procedure under Poisson actions, which does not depend on any momentum mapping. Meanwhile Theorem 3.1 can be regarded as a special case of Theorem 4.1. Therefore Theorem 4.1 is the general symplectic reduction theorem. So Theorems 2.1, 2.2, 2.3, 3.1 are all the special cases of Theorem 4.1.

We briefly state the well-known facts about symplectic reduction and gather some preliminaries in §2. Symplectic reduction theorem for symplectic action and Poisson action are given in §3 and §4, respectively.

§2. Preliminaries

Let $\sigma : G \times M \to M$ be the left action of Lie group G on a manifold M and $\mathcal{G}$ the Lie algebra of G formed by the right invariant vector fields on G. Then the infinitesimal generator of the action σ corresponding to $X \in \mathcal{G}$, denoted by $X_M \in \chi(M)$, has the following properties:

Proposition 2.1. $\forall g \in G,\ X, Y \in \mathcal{G}$,

$$(Ad_g X)_M = \sigma_{g_*} X_M, \quad [X, Y]_M = [X_M, Y_M].$$

Denoting the orbit of σ through $x \in M$ by $G \cdot x$ and supposing the action σ to be free and proper, we have that the orbit space $G \setminus M$ is a manifold and canonical projection $p : M \to G \setminus M$ is a submersion[1].

If $\sigma : G \times M \to M$ is a Hamiltonian action of a Lie group G on a symplectic manifold (M, ω), i.e., $\forall X \in \mathcal{G}, X_M$ is a Hamiltonian vector field, we obviously have a momentum mapping $J : M \to \mathcal{G}^*$ such that $\langle J, X \rangle \in C^\infty(M)(\forall X \in \mathcal{G})$ and $X_M = H_{\langle J,X \rangle}$, where $H_{\langle J,X \rangle}$ is the Hamiltonian vector field determined by the function $\langle J, X \rangle$. A momentum mapping J is called Ad^*-equivariant if $J \circ \sigma_g = Ad_g^* \circ J$ $(\forall g \in G)$. For the above σ and J, we have the following symplectic reduction theorem.

Theorem 2.1. *Let $\sigma : G \times M \to M$ be a Hamiltonian action of a Lie group G on a symplectic manifold (M, ω) with an Ad^*- equivariant momentum mapping $J : M \to \mathcal{G}^*$. Assume that $\mu \in \mathcal{G}^*$ is a regular value of J and that the isotropy group $K \subset G$ at μ under the Ad^* action on $\mathcal{G}^*$ acts on $J^{-1}(\mu)$ freely and properly. Then there exists a unique symplectic form ω_μ on $M_\mu = K \setminus J^{-1}(\mu)$ satisfying $p^*\omega_\mu = i^*\omega$, where $p : J^{-1}(\mu) \to M_\mu$ is the canonical projection and $i : J^{-1}(\mu) \to M$ is the inclusion. (M_μ, ω_μ) is the symplectic reduced phase space of the Hamiltonian G-space M* (see [1,2]).

Again we assume that $\sigma : G \times M \to M$ is a Hamiltonian action with a momentum mapping $J : M \to \mathcal{G}^*$. It follows that

$$\tau(g) \overset{\text{def.}}{=} J(\sigma_g(x)) - Ad_g^* J(x)$$

is a 1-cocycle on G with value in $\mathcal{G}$ when M is connected. All the 1-cocycles corresponding to the momentum mappings belong to the same cohomology class. It is clear that the action σ has no Ad^*-equivariant momentum mapping if the cohomology class is not zero. Let $\Psi : G \times \mathcal{G}^* \to \mathcal{G}^*$ be a map with $\Psi(g, \mu) = Ad_g^* \mu + \tau(g)$. Therefore J is equivariant with respect to σ and Ψ (see [1]). Replacing Ad^* and the isotropy group at μ under Ad^* by Ψ and the isotropy group at μ under Ψ (also denoted by K), respectively, we have symplectic reduction theorem[1].

Theorem 2.2. *Let $\sigma : G \times M \to M$ be a Hamiltonian action of a Lie group G on a connected symplectic manifold (M, ω) with a momentum mapping $J : M \to \mathcal{G}^*$. Then there exists an action Ψ of G on $\mathcal{G}^*$ such that J is equivariant with respect to σ and Ψ. If $\mu \in \mathcal{G}^*$ is a regular value of J and the isotropy group $K \subset G$ at μ under Ψ acts on $J^{-1}(\mu)$ freely and properly, then $M_\mu = K \setminus J^{-1}(\mu)$ has a unique symplectic form ω_μ satisfying*

$$p^* \omega_\mu = i^* \omega,$$

where $p : J^{-1}(\mu) \to M_\mu$ is the canonical projection and $i : J^{-1}(\mu) \to M$ is the inclusion.

In fact, in Theorems 2.1 and 2.2 the existence of the momentum mapping is unnecessary. So is the regular value μ of J. These will be discussed in §3 in detail.

We will now consider the case of Poisson action.

Definition 2.1. *A Lie group G with a Poisson structure π is called a Poisson Lie group if the multiplication in $G : G \times G \to G$ is a Poisson mapping.*

Let (G, π) be a Poisson Lie group. Then $\pi(e) = 0$, where e is identity in G. The dual $(\mathcal{G}^* \wedge \mathcal{G}^* \to \mathcal{G}^*)$ of the intrinsic derivative of π at e $(d_e\pi : \mathcal{G} \to \mathcal{G} \wedge \mathcal{G})$ defines a Lie algebra structure on $\mathcal{G}^*$. We denote the Lie algebra structure by $[\ ,\ ]_{\mathcal{G}^*}$ (see [4, 5]). Let G^* be a simply connected Lie group with Lie algebra $\mathcal{G}^*$. G^* is called the dual group of G which is itself a Poisson Lie group. The linearization of G^* at its identity e^* corresponds to the Lie algebra structure on $\mathcal{G}$ and the dual group of G^* is the universal covering group of G (see [3,5]).

In the 1-form space $\wedge^1(P)$ of a Poisson manifold (P, π), the Poisson structure π induces a bracket operation in the following way: $\forall \omega_1, \omega_2 \in \wedge^1(P)$,

$$\begin{aligned}\{\omega_1, \omega_2\} &= d\pi(\omega_1, \omega_2) + i_{\omega_1^\#} \circ d\omega_2 - i_{\omega_2^\#} \circ d\omega_1 \\ &= -d\pi(\omega_1, \omega_2) + L_{\omega_1^\#}\omega_2 - L_{\omega_2^\#}\omega_1,\end{aligned}$$

where $\# : \wedge^1(P) \to \chi(P)$ satisfies

$$\#(\omega_i) = \omega_i^\# = i_{\omega_i}\pi,$$

$i_{\omega_i}\pi$ denotes the contraction of π by ω_i and $L_{\omega_i^\#}$ is the Lie derivative.

Proposition 2.2. $\# : \wedge^1(p) \to \chi(P)$ *is the homomorphism of Lie algebras.*

It is natural that there is a Lie algebra structure in the space of 1-forms on Poisson Lie group (G, π) and the right invariant 1-forms on G consist of a Lie subalgebra of $\wedge^1(G)$. Confining $\wedge^1(G)$ to e^*, we induce a Lie algebra structure on $\mathcal{G}^*$ from that on $\wedge^1(G)$ and the Lie algebra structure on $\mathcal{G}^*$ coincides with the one induced on $\mathcal{G}^*$ by the linearization of π at e (see [5]).

$\forall \xi \in \mathcal{G}^*$ we denote by ξ^l and ξ^r the left and the right invariant 1-forms on G respectively such that $\xi^l(e) = \xi^r(e) = \xi$. By defining mappings

$$\begin{aligned} \lambda: \quad & \mathcal{G}^* \to \chi(G), \\ & \xi \longmapsto (\xi^l)^{\#}, \\ \rho: \quad & \mathcal{G}^* \to \chi(G), \\ & \xi \longmapsto (\xi^r)^{\#}, \end{aligned}$$

we obtain a Lie algebra homomorphism and a Lie algebra antihomomorphism.

Definition 2.2. *λ and ρ are called left infinitesimal dressing action and right infinitesimal dressing action, respectively. If they are integrable, we get left dressing action and right dressing action respectively.*

The orbits of left and right dressing action are just the symplectic leaves in G (see [5]).

Definition 2.3. *A left action $\sigma : G \times P \to P$ of Poisson Lie group (G, π_G) on a Poisson manifold (P, π_P) is called a Poisson action if σ is a Poisson mapping, where the product Poisson structure is attached to $G \times P$.*

Proposition 2.3. *Let (G, π_G) be a connected Poisson manifold. Then an action $\sigma : G \times P \to P$ is Poisson iff $L_{X_P}\pi_P = (d_e\pi_G(X))_P \quad (\forall X \in \mathcal{G})$, where $d_e\pi_G : \mathcal{G} \to \mathcal{G} \wedge \mathcal{G}$ is the intrinsic derivative of π_G at e and $(d_e\pi_G(X))_P$ the bivector field on P corresponding to $(d_e\pi_G)(X) \in \mathcal{G} \wedge \mathcal{G}$ under the infinitesimal action of σ* (see [3,5]).

Definition 2.4. *Let $\sigma : G \times P \to P$ be Poisson action. A C^∞-mapping $J : P \to G^*$ is called a momentum mapping of the action σ if $X_P = (J^*X^l)^{\#} \quad (\forall X \in \mathcal{G})$, where X^l is a left invariant 1-form on G^* with $X^l(e^*) = X$.*

In [3] Lu proved by a result from [6] that there is a unique momentum mapping J with $J(x_0) = \mu_0$ for any $x_0 \in P$ and $\mu_0 \in G^*$ if P is a simply connected symplectic manifold. While for a general Poisson action momentum mapping may not exist. Even if P is a symplectic manifold, it is still unknown whether a momentum mapping exists.

Proposition 2.4. *If $\sigma : G \times P \to P$ is an action of a Poisson Lie group G on a symplectic manifold with a momentum mapping $J : P \to G^*$, then there exists a Poisson structure π_J on G^* such that $J : P \to (G_*, \pi_J)$ is a Poisson mapping.*

Such a π_J is called an affine Poisson structure on G^* determined by J. Define a mapping $\lambda_J : \mathcal{G} \to \chi(G^*)$ with $\lambda_J(X) = (X^l)^{\#}_J$ $(\forall X \in \mathcal{G})$ where $(X^l)^{\#}_J = i_{X^l}\pi_J$. The integration of λ_J is called the left dressing action of G on G^* determined by π_J, and it is also denoted by λ_J.

Now we have the following symplectic reduction theorem[3].

Theorem 2.3. *Let $\sigma : G \times P \to P$ be a Poisson action of a Poisson Lie group G on a symplectic manifold (P, ω). Let $J : P \to G^*$ be a momentum mapping for σ. Then there is an affine Poisson structure π_J on G^* and a left dressing action λ_J of G on G^* determined by π_J such that J is a Poisson mapping and equivariant with respect to σ and λ_J. Moreover, if $\mu \in G^*$ is a regular value of J and the isotropy group K at μ under λ_J acts freely and properly on $J^{-1}(\mu)$, then there is a unique symplectic form ω_μ on $M_\mu = K\backslash J^{-1}(\mu)$ satisfying $p^*\omega_\mu = i^*\omega$, where $p : J^{-1}(\mu) \to M_\mu$ is the canonical projection and $i : J^{-1}(\mu) \to M$ is the inclusion.*

In §4 we will prove that the existence of the momentum mapping J in the above Theorem 2.3 can be omitted.

§3. Symplectic Reduction of Symplectic Actions

Let $\sigma : G \times M \to M$ be a left action of a Lie group G on a symplectic manifold (M, ω), $\mathcal{G}$ the Lie algebra of G. If $\forall g \in G, \sigma_g^* \omega = \omega$, then σ is a symplectic action. In this case, $\forall X \in \mathcal{G}$, infinitesimal generator X_M of the action σ corresponding to X is a symplectic vector field, i.e., $L_{X_M}\omega = 0$, where L_{X_M} is Lie derivative. From canonical isomorphism $\chi(M) \to \wedge^1(M)$ on the symplectic manifold, there exists a unique 1-form $\theta(X)$ such that $i_{X_M}\omega = \theta(X)$ and $\theta(X)$ is a closed 1-form, where $\chi(M)$ is the C^∞ vector field space and $\wedge^1(M)$ is 1-form space on M. We denote the inverse of the canonical isomorphism by the map $\sharp : \wedge^1(M) \to \chi(M)$. Then $\sharp(\theta(X)) = \theta(X)^\sharp = X_M$. Therefore it induces a linear map $\theta : g \to \wedge^1(M)$ such that $\forall X \in \mathcal{G}, \theta(X)^\sharp = X_M$. In the case of Poisson actions, θ is called a premomentum map in [3]. In the case of symplectic actions, we also call θ a premomentum map. Obviously, every symplectic action has a unique premomentum map. General Poisson actions, however, may not have premomentum map. If existed, it is not unique.

Proposition 3.1. *Premomentum map $\theta : \mathcal{G} \to \wedge^1(M)$ is a Lie algebra homomorphism.*

Proof. According to Proposition 2.2, $\forall X, Y \in \mathcal{G}$, by

$$\theta([X,Y])^\sharp = [X,Y]_M = [X_M, Y_M] = [\theta(X)^\sharp, \theta(Y)^\sharp] = \{\theta(X), \theta(Y)\}^\sharp,$$

we complete our proof.

Let $\theta_x : \mathcal{G} \to T_x^*M$ be the restriction of θ to the point x, i.e., $\forall X \in \mathcal{G},\ \theta_x(X) = \theta(X)|_x$. Then the image space $\theta_x(\mathcal{G})$ of θ_x is a subspace of T_x^*M. Its annihilator space $\theta_x(\mathcal{G})^\perp \subset T_xM$ is written as $\triangle^\theta(x)$. The tangent space of the orbit $G \cdot x$ of the action σ at the point x is denoted by

$$T_x(G \cdot x) = \{X_M(x) | X \in \mathcal{G}\}.$$

Obviously, we can obtain the following properties.

Proposition 3.2. *In the tangent space T_xM of M at any point x, $T_x(G \cdot x)$ and $\triangle^\theta(x)$ are symplectically orthogonal complemented spaces to each other, i.e., $T_x(G \cdot x) \perp_\omega \triangle^\theta(x)$ and*

$$\dim T_x(G \cdot x) + \dim \triangle^\theta(x) = \dim M.$$

Obviously, $\{T_x(G \cdot x) | x \in M\}$ is completely integrable C^∞ distribution on M (it is generally nonhomogeneous). If $x \in M$ such that there exists a neighborhood U of x and the distribution is homogeneous on U, then the point x is called a regular point of the action σ. If x is a regular point and $\dim T_x(G \cdot x) = t$, the point x is called a regular point with rank t. It is easy to see that all of the regular points in M form an open dense subset of M and the regular points with rank t form an open subset. Obviously $t \leq \dim G$.

Proposition 3.3. *Let x be a regular point of the action σ with rank t, U maximal connected open set which consists of the regular points with rank t and contains the point x. Then $\triangle^\theta|_U$ is an $(n-t)$ dimensional completely integrable C^∞ distribution on U.*

Proof. Obviously, $\triangle^\theta|_U$ is an $(n-t)$ dimensional C^∞ distribution on U. Then we need only to prove that it is involutive. $\forall V, W \in \triangle^\theta|_U$ and $X \in \mathcal{G}, x \in U$,

$$\omega(x)(X_M, [V,W]) = \langle \theta(X), [V,W] \rangle (x)$$
$$= -d\theta(X)(V,W)(x) + V(\langle \theta(X), W \rangle)(x) - W(\langle \theta(X), V \rangle)(x) = 0,$$

where $d\theta(X) = 0$ is used. By Frobenius theorem, $\triangle^\theta|_U$ is completely integrable.

Let N_x be the maximal integral submanifold of $\triangle^\theta|_U$ through the point x. It is an $(n-t)$ dimensional submanifold. Let $K = \{g \in G | \sigma_g(N_x) \subset N_x\}$, then K is a Lie subgroup of G. It is called a stable subgroup of N_x. When the action σ is restricted to K and N_x, we have $\sigma|_{K \times N_x} : K \times N_x \to N_x$. Suppose that this action is free and proper, then the orbit space $N_K = K \setminus N_x$ of K is a C^∞ manifold and the canonical projection map $p : N_x \to N_K$ is a submersion[1]. A 2-form ω_K is defined on N_K : $\forall y \in N_x$, $p^*\omega_K(p(y)) = i^*\omega(y)$, where $i : N_X \to M$ is the inclusion map.

Now we prove that the definition of ω_K is reasonable.

For any $V_1, V_2, W \in T_yN_x = \triangle^\theta(y)$ such that $p_*V_1 = p_*V_2 = V$, we have

$$\omega_K(p(y))(p_*V_1, p_*W) - \omega_K(p(y))(p_*V_2, p_*W)$$
$$= (p^*\omega_K)(y)(V_1, W) - (p^*\omega_K)(y)(V_2, W)$$
$$= (i^*\omega)(y)(V_1 - V_2, W) = \omega(y)(V_1 - V_2, W).$$

Since $V_1 - V_2 \in T_y(K \cdot y)$, i.e.,$V_1 - V_2$ is tangent to the K-orbit, the right hand side is 0, which shows that at each point $y \in N_x, \omega_K(p(y))(V, p_*W)$ is independent of the choice of the vector in $p_*^{-1}(V)$.

In addition, $\forall Y \in \mathcal{K}$, the Lie algebra of the stable subgroup K, which is a subalgebra of $\mathcal{G}$, the infinitesimal generator for Y under the action $\sigma|_{K \times N_x}$ is $Y_M|_{N_x}$. It is clear that $L_{Y_M|_{N_x}}(i^*\omega) = 0$, from which we conclude that $\sigma_k^*(i^*\omega) = i^*\omega$, $\forall k \in K$. Now for any $V(y), W(y) \in T_yN_x$, set $V(ky) = \sigma_{k*}V(y)$, $W(ky) = \sigma_{k*}W(y)$. Since $p_* \circ \sigma_{k*} = p_*$, $p_*(V(ky)) = p_*(V(y)), p_*(W(ky)) = p_*(W(y))$. So

$$(i^*\omega)(ky)(V(ky), W(ky)) = [(\sigma_k^* \circ i^*)\omega](y)(V(y), W(y)) = (i^*\omega)(y)(V(y), W(y)).$$

It is easy to see that $\omega_K(p(y))$ is independent of the choice of the points in $p^{-1}(p(y))$. It follows that ω_K is well defined.

Now we prove that ω_K is a symplectic form on N_K.

$$p^*(d\omega_K) = d(p^*\omega_K) = d(i^*\omega) = i^*(d\omega) = 0.$$

Since p is a submersion, p^* is an injection, hence $d\omega_K = 0$. Suppose $V' \in T_{p(y)}N_K$ and $\omega_K(p(y))(V', W') = 0$, $\forall W' \in T_{p(y)}N_K$. Let $V, W \in T_yN_x$ and $p_*V = V'$, $p_*W = W'$. Then

$$i^*\omega(y)(V,W) = p^*\omega_K(y)(V,W) = \omega_K(p(y))(V', W') = 0.$$

Hence $V \in T_y(K \cdot y)$, i.e., $V' = p_*V = 0$. It follows that ω_K is a closed nondegenerate 2-form on N_K, i.e., a symplectic form on N_K. Now we have proved:

Theorem 3.1. *Suppose $\sigma : G \times M \to M$ is a symplectic action of a Lie group G on a symplectic manifold (M, ω) and $x \in M$ is a regular point with rank t. Then there exists a connected $n-t$ dimensional submanifold N_x passing x with the property that the tangent space of N_x at each point is the symplectic orthogonal complement space of the tangent space of the G-orbit at the same point. Besides we assume $K \subset G$ is the stable subgroup of N_x*

such that the action $\sigma|_{K\times N_x} : K \times N_x \to N_x$ *is both free and proper. Then on the orbit space* $N_K = K \setminus N_x$ *there exists a unique symplectic form* ω_K *such that* (N_K, ω_K) *is the symplectic reduced phase space of* (M, ω)*, where* ω_K *satisfies* $p^*\omega_K = i^*\omega$*,* $p : N_x \to N_K$ *is the canonical projection and* $i : N_x \to M$ *is the inclusion.*

Particularly, suppose $\sigma : G \times M \to M$ is a Hamiltonian action with an Ad^*-equivariant momentum mapping $J : M \to \mathcal{G}^*$. Suppose $\dim G = m$, $\dim M = n$ and $\mu \in \mathcal{G}^*$ is a regular value of J. Then

$$T_x J^{-1}(\mu) = ker J_* \quad (\forall x \in J^{-1}(\mu)),$$

which is an $(n-m)$ dimensional vector subspace of $T_x M$. Noting that $V \in T_x J^{-1}(\mu)$ is equivalent to $\omega(x)(Y_M, V) = 0 \quad (\forall Y \in \mathcal{G})$, i.e., $V \in T_x J^{-1}(\mu)$ is equivalent to $V \in \triangle^\theta(x) = T_x(G\cdot x)^\perp$, we have $\dim T_x(G \cdot x) = m$. It is obvious that x is a regular point of the action σ with rank $n-m$. By Theorem 3.1 for every point y on the $(n-m)$ dimensional submanifold N_x, $T_y N_x$ is the symplectic orthogonal complement space of $T_y(G \cdot y)$.

Without loss of generality we may suppose $\exists V \in \chi(M)$ with $V|_{N_x}$ being tangent to N_x and its flow $\varphi(t)$ with $\varphi_0(x) = x, \varphi_1(x) = y$. Since $V(\varphi_t(x)) \in \triangle^\theta(\varphi_t(x))$, we have

$$J_*(V_{\varphi_t(x)}) = 0,$$

i.e.,$J(\varphi_t(x)) = J(x) = \mu$, namely, $J(y) = \mu$. Hence $N_x \subset J^{-1}(\mu)$. Conversely, if $J^{-1}(\mu)$ is connected, then $J^{-1}(\mu) \subset N_x$.

Therefore N_x is the connected component of $J^{-1}(\mu)$ containing x. Now consider the stable subgroup K of N_x. $\forall g \in G,\ g \in K \Longleftrightarrow \sigma_g(N_x) \subset N_x$. Since J is Ad^*-equivariant, we have

$$J(\sigma_g(y)) = Ad_g^*(J(y)) = Ad_g^*\mu \quad (\forall y \in N_x).$$

Hence $g \in K \Longleftrightarrow Ad_g^*\mu = \mu$, which shows that K is the isotropy group at μ under the coadjoint action. It follows that Theorem 2.1 is actually a special case of Theorem 3.1.

For a Hamiltonian action $\sigma : G \times M \to M$ whose momentum mapping $J : M \to \mathcal{G}^*$ is not Ad^*-equivariant, there is an action ψ of G on $\mathcal{G}^*$ such that J is equivariant with respect to σ and ψ. When $\mu \in \mathcal{G}^*$ is a regular value of J, each $x \in J^{-1}(\mu)$ is a regular point for σ with rank $n-m$. By the same reason as above N_x is the connected component of $J^{-1}(\mu)$ containing x. Now the stable subgroup K of N_x is just the isotropy group at μ under the action ψ. Hence Theorem 2.2 is a special case of Theorem 3.1, too.

Remark 3.1. For a general symplectic G-space we are not sure about the existence of momentum mapping. Even if the momentum mapping does exist, it may have no regular value. If this happens, Theorems 2.1 and 2.2 will be of no effect, while Theorem 3.1 ensures the existence of the symplectic reduced phase space. If there exists a momentum mapping J and the regular value μ of J, $J^{-1}(\mu)$ or its connected component is an $(n-m)$ dimensional submanifold. The submanifold has the lowest dimension among the $N_x's$ obtained in Theorem 3.1.

§4. The Case of Poisson Action

Let $\sigma : G \times M \to M$ be a left Poisson action of a Poisson Lie group (G, π_G) on a symplectic manifold (M, ω). Then σ has a pre-momentum mapping $\theta : \mathcal{G} \to \wedge^1(M)$ such that $\theta(X)^\# = X_M$ $(\forall X \in \mathcal{G})$. Propositions 3.1 and 3.2 still hold in this case.

By Proposition 2.3, we have the following

Proposition 4.1. *Let (G, π_G) be a Poisson Lie group and (M, ω) a symplectic manifold. Then a left action $\sigma : G \times M \to M$ is a Poisson action iff $d\theta(X)(\omega_1^\#, \omega_2^\#) = \langle[\sigma^*\omega_1, \sigma^*\omega_2]_{\mathcal{G}^*}, X\rangle$ $(\forall \omega_1, \omega_2 \in \wedge^1(M), X \in \mathcal{G})$, where $\sigma^* : \wedge^1(M) \to C^\infty(M, \mathcal{G}^*)$ is determined by the dual of the infinitesimal action for σ, i.e. $\langle\sigma^*\omega, X\rangle = \langle\omega, X_M\rangle \in C^\infty(M)$ $(\forall \omega \in \wedge^1(M), X \in \mathcal{G})$ and $[\ ,\ ]_{\mathcal{G}^*}$ is Lie bracket on $\mathcal{G}^*$ determined by the Poisson tensor π_G.*

Proof. By Proposition 2.3, σ is a Poisson action $\Longleftrightarrow L_{X_M}\pi_M = (d_e\pi_G(X))_M$ $(\forall X \in \mathcal{G})$. Now

$$\begin{aligned} L_{X_M}\pi_M(\omega_1, \omega_2) &= \langle X_M, \{\omega_1, \omega_2\}\rangle - \omega_1^\#(\langle X_M, \omega_2\rangle) + \omega_2^\#(\langle X_M, \omega_1\rangle) \\ &= -\langle\{\omega_1, \omega_2\}^\#, \theta(X)\rangle + \omega_1^\#\langle\omega_2^\#, \theta(X)\rangle - \omega_2^\#\langle\omega_1^\#, \theta(X)\rangle \\ &= d\theta(X)(\omega_1^\#, \omega_2^\#). \end{aligned}$$

On the other hand,

$$(d_e\pi_G(X))_M(\omega_1, \omega_2) = \langle\sigma^*\omega_1 \wedge \sigma^*\omega_2, d_e\pi_G(X)\rangle = \langle[\sigma^*\omega_1, \sigma^*\omega_2]_{\mathcal{G}^*}, X\rangle,$$

and this completes the proof.

Assume that G is connected, $\dim G = m$ and $\dim M = n$. Let $x \in M$ be a regular point for σ with rank $t \le m$. By a discussion similar to that in §3 we get a connected open set U containing x such that

$$\dim T_y(G \cdot y) = t, \quad \dim \triangle^\theta(y) = n - t\ (\forall y \in U),$$

and $\triangle^\theta(y)$ is a symplectic orthogonal complement space of $T_y(G \cdot y)$. Now Proposition 3.3 still holds but its proof is slightly different from that in §3.

Proposition 4.2. *For a Poisson action $\sigma : G \times M \to M$ of Poisson Lie group G on symplectic manifold (M, ω), $\triangle^\theta|_U$ is an $(n - t)$ dimensional completely integrable C^∞ distribution on U.*

Proof. For any $V, W \in \triangle^\theta|_U$, taking $\omega_1, \omega_2 \in \wedge^1(U)$ with $\omega_1^\# = V, \omega_2^\# = W$, we have

$$\omega(X_M, \omega_i^\#) = 0 \quad (\forall X \in \mathcal{G}, i = 1, 2).$$

Therefore

$$\langle\sigma^*\omega_i, X\rangle = \langle\omega_i, X_M\rangle = 0.$$

Hence $\sigma^*\omega_i = 0 (i = 1, 2)$ since $X \in \mathcal{G}$ is arbitrary. By Proposition 4.1, we have

$$d\theta(X)(\omega_1^\#, \omega_2^\#) = \langle[\sigma^*\omega_1, \sigma^*\omega_2]_{\mathcal{G}^*}, X\rangle = 0.$$

It follows that

$$0 = d\theta(X)(V, W) = V(\langle\theta(X), W\rangle) - W(\langle\theta(X), V\rangle) - \langle\theta(X), [V, W]\rangle = -\langle\theta(X), [V, W]\rangle.$$

Hence $[V, W] \in \triangle^\theta|_U$, i.e., $\triangle^\theta|_U$ is involutive. So $\triangle^\theta|_U$ is completely integrable by Frobenius's theorem.

Next we will follow the procedure in §3. Suppose N_x is the maximal integral submanifold passing through x for $\triangle^\theta|_U$ and $K \subset G$ the stable subgroup of N_x. Confine the action σ to $K \times N_x : \sigma|_{K\times N_x} : K \times N_x \to N_x$ and suppose that $\sigma|_{K\times N_x}$ is free and proper. Then the orbit space $N_K = K \setminus N_x$ is a C^∞-manifold and the canonical projection $p : N_x \to N_K$ is a submersion. Now we define a 2-form ω_K on N_x such that

$$p^*(\omega_K(p(y))) = i^*\omega(y) \quad (\forall y \in N_x).$$

Repeating the procedure in §3 we can prove that ω_K is well defined and a symplectic form on N_K. Now we come to the conclusion that (N_K, ω_K) is the symplectic reduced phase space of (M, ω) under σ. So we have proved the following

Theorem 4.1. *If $\sigma : G \times M \to M$ is a left Poisson action of a connected Poisson Lie group G on a symplectic manifold (M, ω) and $x \in M$ is a regular point for σ with rank t, then there exists a connected $(n-t)$ dimensional submanifold N_x through x such that the tangent space of N_x at each point is the symplectic orthogonal complement space of the tangent space of the G-orbit at the same point. Furthermore suppose $K \subset G$ is the stable subgroup of N_x such that the action $\sigma|_{K\times N_x} : K \times N_x \to N_x$ is free and proper, then there is a unique symplectic form ω_K on the orbit space $N_K = K \setminus N_x$ such that (N_K, ω_K) is the symplectic reduced phase space of (M, ω) with $p^*\omega_K = i^*\omega$, where $p : N_x \to N_K$ is the canonical projection and $i : N_x \to M$ is the inclusion.*

Assume that the conditions in Theorem 2.2 hold. Let $\dim G = m$ and $\dim M = n$. If $\mu \in G^*$ is a regular value of J, $x \in J^{-1}(\mu)$ is a regular point for σ with rank m. The $(n-m)$ dimensional manifold N_x obtained in Theorem 4.1 is just the connected component containing x in $J^{-1}(\mu)$ and the stable subgroup K of N_x is just the isotropy group at μ under the left dressing action λ_J of G on G^*. Hence we see that Theorem 2.2 is a special case of Theorem 4.1. Finally, Theorem 3.1 can also be regarded as a special case of Theorem 4.1 for a Poisson action is a symplectic action in case of the 0-Poisson structure is attached to G. Therefore Theorem 4.1 includes all the results obtained hitherto about symplectic reduction theory of the actions of Lie groups on symplectic manifolds.

For a Poisson action there exists Poisson reduction[3,7,8].

Proposition 4.3. *Let $\sigma : G \times P \to P$ be a left Poisson action of Poisson Lie group G on Poisson manifold P. If the orbit space $G \setminus P$ is a manifold, then there exists a unique Poisson structure on $G \setminus P$ such that the canonical projection $p : P \to G \setminus P$ is a Poisson mapping.*

Proof. $\forall f, g \in C^\infty(G \setminus P)$, $p^*f, p^*g \in C^\infty_G(P)$, where $C^\infty_G(P)$ is the G-invariant function space on P, which is a Lie subalgebra of $C^\infty(P)$. Now we define a bracket operation on $C^\infty(G \setminus P)$, satisfying

$$p^*\{f, g\} = \{p^*f, p^*g\}_P.$$

It is obvious that this $\{\ ,\ \}$ is the Poisson structure on $G \setminus P$ satisfying the condition in Proposition 4.3.

$G \setminus P$ with this Poisson structure is called the Poisson reduction phase space of P under the action σ.

Example 4.1. Let G be a Lie group. Using the multiplication of G, we can get the action of G on G. There is a natural symplectic structure on T^*G such that T^*G becomes a symplectic manifold. The action of G on G can be lifted to an action of G on T^*G. The action is a symplectic action. On reduction phase space $G \setminus T^*G \cong \mathcal{G}^*$, there is a Poisson structure such that projection $T^*G \to G \setminus T^*G$ is a Poisson map.

In the case that P is a symplectic manifold and $\sigma : G \times P \to P$ is still a Poisson action, the following theorem shows the relation between the symplectic reduction phase space obtained in Theorem 4.1 and the above Poisson reduction phase space.

Theorem 4.2. *Suppose that* (M, ω) *is a symplectic manifold and* $\sigma : G \times M \to M$ *is a Poisson action with Poisson reduction phase space* $G \setminus P$. *Then each symplectic reduction phase space under* σ *is a symplectic leaf in* $G \setminus P$. *Conversely, each symplectic leaf in* $G \setminus P$ *is a symplectic reduction phase space.*

Proof. Let $x \in M$ be a regular point of the action σ. The N_x in Theorem 4.1 is a submanifold of M and $\forall y \in N_x$, the K-orbit $K \cdot y = G \cdot y \cap N_x$. Let $p_K : N_x \to M_K$ be the canonical projection in the symplectic reduction and $p : M \to G \setminus M$ be the canonical projection in the Poisson reduction. Obviously, $p_K(y) = p(y)$. If $G \setminus M$ is a manifold, then M_K is a submanifold of $G \setminus M$.

Let $v \in T_{p(y)}(G \setminus M)$ belong to the characteristic distribution of $G \setminus M$, i.e. there exists $f \in C^\infty(G \setminus M)$, such that the Hamiltonian vector field H_f satisfies $H_f(p(y)) = v$. Since p is a Poisson mapping, $p_*(H_{f \circ p}(y)) = v$. We know that $f \circ p$ is a G-invariant function on M and $H_{f \circ p}(y) \in \triangle^\theta(y)$, i.e. $H_{f \circ p}(y)$ is tangent to N_x. It follows that $v = p_*(H_{f \circ p}(y)) \in T_{p(y)} M_K$. Conversely, if $v \in T_{p_K(y)} M_K$, then there exists $v' \in T_y N_x \subset T_y M$ such that $p_* v' = v$. Since v' is symplectically orthogonal to $T_y(G \cdot y)$ in $T_y M$, there exists G-invariant function f on M satisfying $H_f(y) = v'$. It follows that $p_*(H_f(y)) = v$ belongs to the characteristic distribution at $p(y) \in G \setminus M$. Hence as a submanifold M_K is exactly a symplectic leaf in $G \setminus M$.

$\forall f, g \in C^\infty(G \setminus M), y \in N_x$,

$$\begin{aligned}\{f, g\}(p(y)) &= \{p^* f, p^* g\}_M(y) = \omega(y)(H_{g \circ p}, H_{f \circ p}) \\ &= p^* \omega_K(y)(H_{g \circ p}, H_{f \circ p}) = \omega_K(p(y))(H_g, H_f).\end{aligned}$$

This means that ω_K is just the symplectic structure induced by the Poisson structure on $G \setminus M$ on the symplectic leaf M_K. Finally, for any symplectic leaf S in $G \setminus P$, take any point $p(x) \in S$. Since $G \setminus P$ is a manifold, x must be a regular point under σ. Therefore the symplectic reduction phase space obtained in Theorem 4.1 is the symplectic leaf through $p(x)$ in $G \setminus P$, that is S.

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