Chin. Ann. of Math.
23B:4(2002),435-438.

A CONSTITUTIVE EQUATION SATISFYING THE NULL CONDITION FOR NONLINEAR COMPRESSIBLE ELASTICITY**

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Abstract

This paper constructs a polyconvex stored energy function, satisfying the null condition, for isotropic compressible elastic materials with given Lamé constants. The difference between this stored energy function and St Venant-Kirchhoff's is a three order term.

Keywords Compressible elasticity, Stored energy function, Null condition
2000 MR Subject Classification 35L70, 73B05, 73C50
Chinese Library Classification O175.27, O175.29, O343.5 **Document Code** A
Article ID 0252-9599(2002)04-0435-04

§1. Introduction

As indicated in [5], the global existence of classical solutions to the Cauchy problem for the nonlinear elastodynamic system is based essentially on two assumptions: the initial data must be small and the nonlinear term must obey the null condition. The omission of either of these two assumptions may lead to the blow-up phenomenon in finite time. Therefore, it is important to discuss if there are any stored energy functions for actual materials, which satisfy the null condition.

It is well-known that for any homogeneous, isotropic hyperelastic material, whose reference configuration is a natural state, its stored energy function is of the form

$$W(\mathbf{F}) = \frac{1}{2}\lambda(\mathrm{tr}\mathbf{E})^2 + \mu\mathrm{tr}\mathbf{E}^2 + \circ(\|\mathbf{E}\|^2), \tag{1.1}$$

where

$$\mathbf{E} = \frac{1}{2}(\mathbf{I} - \mathbf{F}^T\mathbf{F}), \tag{1.2}$$

$\mathbf{F}$ is the deformation gradient, λ and μ are the corresponding Lamé constants (see, for example, [2, 4]). Moreover, Sideris showed in [5] that the null condition places no restriction on the bulk and sheer moduli (equivalently, the Lamé constants λ and μ) of the equilibrium. Let λ and μ be given positive constants. The problem we shall discuss in this paper is if there exists a polyconvex stored energy function $W(\mathbf{F})$ with the given Lamé constants λ and μ, which satisfies Equation (1.1) and the null condition. Ciarlet and Geymonat proved that for any given Lamé constants λ and μ, there exist polyconvex stored energy functions of the form

$$W(\mathbf{F}) = a\|\mathbf{F}\|^2 + b\|\mathrm{cof}\mathbf{F}\|^2 + \Gamma(\det\mathbf{F}) + c, \tag{1.3}$$

where

$$\Gamma(\xi) = p\xi^2 - q\ln\xi \quad (p > 0, q > 0), \forall\xi > 0, \tag{1.4}$$

Manuscript received Feburary 22, 2002.
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**Project supported by the National Natural Science of Foundation of China (No.19871015).

$a > 0, b > 0$ and $c \in \mathbf{R}$, that satisfy Equation (1.1) (see [2, 3]). In this paper, we shall construct a family of simple functions $\Gamma(\xi)$ behaving as $\ln\xi$ and ξ^2 when $\xi \to 0$ and $\xi \to +\infty$ respectively, such that the corresponding $W(\mathbf{F})$ given by (1.3) is polyconvex and satisfies the null condition. Moreover, (1.1) still holds for such $W(\mathbf{F})$.

§2. Main Result and Proof

The main result in this paper is the following

Theorem 2.1. *Let λ and μ be two given Lamé constants satisfying that $\mu > 0$ and $\lambda + \mu > 0$. There exist polyconvex stored energy functions of the form*

$$W(\mathbf{F}) = a\|\mathbf{F}\|^2 + b\|\mathrm{cof}\mathbf{F}\|^2 + \Gamma(\det\mathbf{F}) + c, \tag{2.1}$$

where $a > 0, b > 0, c \in \mathbf{R}$ and

$$\Gamma(\xi) = p\xi^2 - q\xi\ln\xi - r\ln\xi, \forall\xi > 0 \tag{2.2}$$

with $p > 0, q > 0, r > 0$, such that the null condition is satisfied and

$$W(\mathbf{F}) = \frac{1}{2}\lambda(\mathrm{tr}\mathbf{E})^2 + \mu\mathrm{tr}\mathbf{E}^2 + \bigcirc(\|\mathbf{E}\|^3). \tag{2.3}$$

In (2.1) *and* (1.3), $\mathrm{cof}\,\mathbf{F}$ *is the cofactor matrix of* $\mathbf{F}, \mathbf{E}$ *is given by* (1.2) *and* $\|\mathbf{F}\|^2 = \mathrm{tr}(\mathbf{F}^T\mathbf{F})$.

Remark 2.1. The stored energy function $W(\mathbf{F})$ given in the theorem behaves as $(\det\mathbf{F})^2$ when $\det\mathbf{F} \to +\infty$ and tends to $+\infty$ when $\det\mathbf{F} \to +0$. So the inequality of coerciveness

$$W(\mathbf{F}) \geq C_1(\|\mathbf{F}\|^2 + \|\mathrm{cof}\mathbf{F}\|^2 + (\det\mathbf{F})^2) + C_2 \tag{2.4}$$

holds with some constants $C_1 > 0$ and $C_2 \in \mathbf{R}$.

For isotropic hyperelastic materials, the stored energy function $W(\mathbf{F})$ can be expressed in terms of the principal invariants i_1, i_2, and i_3 of the matrix $\mathbf{F}^T\mathbf{F} - \mathbf{I}$. We have

Proposition 2.1. *Let the reference configuration be a stress-free state. The stored energy function $W(\mathbf{F})$ satisfies the null condition if and only if*

$$\Big(2\frac{\partial^3}{\partial i_1^3}W(\mathbf{F}) + 3\frac{\partial^2}{\partial i_1^2}W(\mathbf{F})\Big)_{i_1=i_2=i_3=0} = 0. \tag{2.5}$$

For the definition of the null condition and the proof of Proposition 2.1, see [1, 5] or [6].

Proposition 2.2. *The stored energy function of compressible Ogden's material*

$$\begin{aligned} W(\mathbf{F}) = a(\mu_1^\alpha + \mu_2^\alpha + \mu_3^\alpha - 3) &+ b((\mu_2\mu_3)^\beta + (\mu_3\mu_1)^\beta \\ &+ (\mu_1\mu_2)^\beta - 3) + \Gamma(\mu_1\mu_2\mu_3), \end{aligned} \tag{2.6}$$

where a, b, α and β are positive constants, satisfies the null condition (2.5) *if and only if*

$$a\alpha(\alpha-1)(\alpha-2) + 2b\beta(\beta-1)(\beta-2) + \Gamma'''(1) = 0, \tag{2.7}$$

where μ_1, μ_2, and μ_3 are the eigenvalues of the matrix $(\mathbf{F}^T\mathbf{F})^{\frac{1}{2}}$.

Proof. Let κ_1, κ_2, and κ_3 be the eigenvalues of the matrix $\mathbf{F}^T\mathbf{F} - \mathbf{I}$. Then $\mu_i = (\kappa_i + 1)^{\frac{1}{2}}$ and

$$\mu_i^\alpha = 1 + \frac{1}{2}\alpha\kappa_i + \frac{1}{8}\alpha(\alpha-2)\kappa_i^2 + \frac{1}{48}\alpha(\alpha-2)(\alpha-4)\kappa_i^3 + \circ(\kappa_i^3), \quad i = 1, 2, 3.$$

So we have

$$\begin{aligned} \mu_1^\alpha + \mu_2^\alpha + \mu_3^\alpha - 3 &= \frac{1}{2}\alpha(\kappa_1 + \kappa_2 + \kappa_3) + \frac{1}{8}\alpha(\alpha-2)(\kappa_1^2 + \kappa_2^2 + \kappa_3^2) \\ &\quad + \frac{1}{48}\alpha(\alpha-2)(\alpha-4)(\kappa_1^3 + \kappa_2^3 + \kappa_3^3) + \circ(|\kappa|^3) \\ &= \frac{1}{2}\alpha i_1 + \frac{1}{8}\alpha(\alpha-2)(i_1^2 - 2i_2) \\ &\quad + \frac{1}{48}\alpha(\alpha-2)(\alpha-4)(i_1^3 - 3i_1i_2 + 3i_3) + \circ(|\kappa|^3), \end{aligned} \tag{2.8}$$

where $i_1 = \kappa_1 + \kappa_2 + \kappa_3, i_2 = \kappa_2\kappa_3 + \kappa_3\kappa_1 + \kappa_1\kappa_2$ and $i_3 = \kappa_1\kappa_2\kappa_3$.

In a similar way, we can get

$$\begin{aligned}
&(\mu_2\mu_3)^\beta + (\mu_3\mu_1)^\beta + (\mu_1\mu_2)^\beta - 3 \\
=& \frac{1}{2}\beta(2i_1 + i_2) + \frac{1}{4}\beta(\beta-2)(i_1^2 - i_2 + i_1 i_2 - 3i_3) \\
&+ \frac{1}{48}\beta(\beta-2)(\beta-4)(2i_1^3 - 3i_1 i_2 - 3i_3) + \circ(|\ \kappa\ |^3),
\end{aligned} \tag{2.9}$$

$$\Gamma(\det \mathbf{F}) = \Gamma(\mu_1\mu_2\mu_3) = \Gamma((1 + i_1 + i_2 + i_3)^{\frac{1}{2}}). \tag{2.10}$$

Using (2.10), it is easy to verify that

$$\frac{\partial^2}{\partial i_1^2}\Gamma\Big|_{i_1=i_2=i_3=0} = \frac{1}{4}(\Gamma''(1) - \Gamma'(1)), \tag{2.11}$$

$$\frac{\partial^3}{\partial i_1^3}\Gamma\Big|_{i_1=i_2=i_3=0} = \frac{1}{8}(\Gamma'''(1) - 3\Gamma''(1) + 3\Gamma'(1)). \tag{2.12}$$

Then, it follows immediately from (2.8)–(2.12) that

$$\Big(2\frac{\partial^3}{\partial i_1^3}W(\mathbf{F}) + 3\frac{\partial^2}{\partial i_1^2}W(\mathbf{F})\Big)_{i_1=i_2=i_3=0} = \frac{1}{4}a\alpha(\alpha-1)(\alpha-2) + \frac{1}{2}b\beta(\beta-1)(\beta-2) + \frac{1}{4}\Gamma'''(1).$$

This completes the proof of the proposition.

Lemma 2.1. *Let* $p \geq r/2$ *and* $q = 2r > 0$. *Then the function* $\Gamma(\xi) = p\xi^2 - q\xi\ln\xi - r\ln\xi$ *is convex on* $(0, +\infty)$.

Proof. It suffices to show that

$$\Gamma''(\xi) \geq 0, \quad \forall \xi \in (0, +\infty). \tag{2.13}$$

In fact, noting $q = 2r$, we have

$$\Gamma'''(\xi) = 2r\xi^{-2}(1 - \xi^{-1}). \tag{2.14}$$

Therefore, $\xi = 1$ is the unique stationary point of the function $\Gamma''(\xi)$ on $(0, +\infty)$. Noting that $p \geq r/2$ and $\Gamma''''(1) = 2r\xi^{-3}(3\xi^{-1} - 2)\ |_{\xi=1} = 2r > 0$, we have $\min\limits_{\xi\in(0,+\infty)} \Gamma''(\xi) = \Gamma''(1) \geq 0$, then inequality (2.13) holds.

Proof of Theorem 2.1. We can verify that the stored energy function $W(\mathbf{F})$ given by (2.1) satisfies (2.3) if and only if the following equations hold:

$$3a + 3b + \Gamma(1) + c = 0, \tag{2.15}$$

$$2a + 4b + \Gamma'(1) = 0, \tag{2.16}$$

$$2b + \frac{1}{2}\Gamma'(1) + \frac{1}{2}\Gamma''(1) = \frac{1}{2}\lambda, \tag{2.17}$$

$$-2b - \Gamma'(1) = \mu \tag{2.18}$$

(see [2, 3]).

(2.15) can be satisfied by a suitable choice of constant c. Moreover, it is easy to see that (2.16)–(2.18) are equivalent to

$$\Gamma''(1) - \Gamma'(1) = \lambda + 2\mu, \tag{2.19}$$

$$2a - \Gamma'(1) = 2\mu, \tag{2.20}$$

$$2b + \Gamma'(1) = -\mu. \tag{2.21}$$

Then, noting (2.20)–(2.21), we see that $a > 0, b > 0$ are equivalent to

$$-2\mu < \Gamma'(1) < -\mu. \tag{2.22}$$

We now choose positive constants p, q such that $\Gamma(\xi)$ satisfies (2.19),(2.22),

$$\Gamma''(\xi) \geq 0 \quad \text{on} \quad (0, +\infty), \tag{2.23}$$

$$\Gamma'''(1) = 0. \tag{2.24}$$

Conditions (2.23)–(2.24), which hold if $p \geq r/2 > 0$ and $q = 2r$ (see Lemma 2.3), imply that the stored energy function $W(\mathbf{F})$ is polyconvex and satisfies the null condition (by Proposition 2.2).

Taking $q = 2r$, (2.24) follows from (2.14), and

$$\Gamma'(1) = 2p - 3r, \Gamma''(1) = 2p - r. \tag{2.25}$$

Then, $p \geq r/2 > 0$ is equivalent to

$$\Gamma''(1) - \Gamma'(1) > 0, \tag{2.26}$$

$$\Gamma''(1) \geq 0. \tag{2.27}$$

(2.26) is satisfied automatically, moreover, (2.27) is equivalent to

$$-(\lambda + 2\mu) \leq \Gamma'(1) \tag{2.28}$$

provided that (2.19) holds. Let

$$\eta = \min\{2\mu, \lambda + 2\mu\}. \tag{2.29}$$

It is clear that $\eta > \mu$, and if

$$-\eta < \Gamma'(1) < -\mu, \tag{2.30}$$

then both (2.22) and (2.27) are satisfied.

Noting (2.25), it comes from (2.19) that

$$r = \frac{1}{2}(\Gamma''(1) - \Gamma'(1)) = \frac{1}{2}\lambda + \mu. \tag{2.31}$$

Then, noting (2.25), we see that (2.30) is equivalent to

$$\frac{1}{2}(-\eta + \frac{3}{2}\lambda + 3\mu) < p < \frac{1}{2}(\frac{3}{2}\lambda + 2\mu). \tag{2.32}$$

Finally, using (2.20), (2.21), (2.25), (2.31), (2.32) and $q = 2r$ we get

$$a = -\frac{3}{4}\lambda - \frac{1}{2}\mu + p, \quad b = \frac{3}{4}\lambda + \mu - p,$$
$$p \in \Big(\frac{1}{2}\Big(- \eta + \frac{3}{2}\lambda + 3\mu\Big), \frac{1}{2}\Big(\frac{3}{2}\lambda + 2\mu\Big)\Big),$$
$$q = \lambda + 2\mu, r = \frac{1}{2}(\lambda + 2\mu).$$

The proof of the theorem is completed.

Acknowledgements. The author would like to thank Professor Li Ta-tsien for helpful discussions and suggestions.

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