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I.I.D. STATISTICAL CONTRACTION OPERATORS AND STATISTICALLY SELF-SIMILAR SETS**

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Abstract

I.i.d. random sequence is the simplest but very basic one in stochastic processes, and statistically self-similar set is the simplest but very basic one in random recursive sets in the theory of random fractal. Is there any relation between i.i.d. random sequence and statistically self-similar set? This paper gives a basic theorem which tells us that the random recursive set generated by a collection of i.i.d. statistical contraction operators is always a statistically self-similar set.

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§1. Introduction

Hutchinson[5] constructed a class of (strictly) self-similar sets by contraction operators and obtained their Hausdorff dimension and Hausdorff exact measure function. Later, Mauldin and Williams[6], Graf[3] and Falconer[2] independently constructed the statistically self-similar measure and set in different way and investigated their probability properties and fractal properties. But no one has pointed out what is the probabilty character of statistically self-similar set.

In this paper, we will give a basic theorem which tells us that the random recursive set generated by a collection of i.i.d. statistical contraction operators is always a statistically self-similar set.

Let $N \geq 2$ be an integer, $C_0 = \{\emptyset\}$, $C_n = \{0, 1, \cdots, N-1\}^n$ $(n \geq 1)$, $D = \bigcup\limits_{n\geq 0} C_n$, $\{f_\sigma, \sigma \in D\}$ be a collection of statistical contraction operators. If $\sup\limits_{\sigma\in D} \mathrm{Lip}(f_\sigma) = \alpha < 1$, $\{(f_{(\sigma,0)}, \cdots, f_{(\sigma,N-1)}),\ \sigma \in D\}$ are i.i.d., then

$$K \triangleq \bigcap_{n=1}^{\infty} \bigcup_{(\sigma_1,\cdots,\sigma_n)\in C_n} \overline{f_{\sigma_1} \circ f_{(\sigma_1,\sigma_2)} \circ f_{(\sigma_1,\cdots,\sigma_n)}(E)}$$

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is always a statistically self-similar set, where $\mathrm{Lip}(\cdot)$ is the Lipschitz coefficient, (E,ρ) is a polish space, $\overline{A}$ is the closure of A.

§2. Notations and Preliminaries

Let $(\Omega,\mathcal{F},P)$ be a complete probability space, (E,ρ) be a separable complete metric space. $\mathcal{K}(E)$ denotes all non-empty compact sets in E,η is the Hausdorff metric on $\mathcal{K}(E)$, that is to say, $I,J\in\mathcal{K}(E)$,

$$\eta(I,J)=\sup\{\rho(x,I),\rho(y,J):\ x\in J,y\in I\},\ \rho(x,I)\ \text{ is the distance from } x \text{ to } I.$$

$(\mathcal{K}(E),\eta)$ is also a separable complete metric space.

For any subset A in a metric space, $\mathrm{diam}(A)$ denotes the diameter of A. $\overline{A}$ donotes the closure of A. Let $\mathbf{R}^d$ be the d-dimensional Euclidean space, $\mathbf{R}=\mathbf{R}^1$. We always denote the Borel field of topology space T by $\mathcal{B}(T)$, and all Borel probability measures by $\mathcal{P}(T)$. Let $f:E\to E,B\subset E,f(B)$ is the image of f on B.

Definition 2.1. *Let $f:E\to E$.*

$$\mathrm{Lip}(f)\triangleq\sup_{x\neq y,x,y\in E}\frac{\rho(f(x),f(y))}{\rho(x,y)}$$

is called the Lipschitz coefficient of f. If $\mathrm{Lip}(f)<1$, then we call f a contraction operator and denote the class of all contraction operators from E to E by $\mathrm{con}(E)$.

We always assume that $\mathrm{con}(E)$ carries the topology of pointwise convergence.

Propositon 2.1.[3] *let $f:E\to E$.*

(1) *If f is continuous, then $f(\mathcal{K}(E))\subset\mathcal{K}(E)$;*

(2) $I,J\in\mathcal{K}(E)\Rightarrow\eta(f(I),f(J))\le\mathrm{Lip}(f)\eta(I,J)$;

(3) $\mathrm{Lip}(f):\mathrm{con}(E)\to[0,1)$ *is lower-semicontinuous;*

(4) $f(J):\mathrm{con}(E)\times\mathcal{K}(E)\to\mathcal{K}(E)$ *is continuous;*

(5) $g(J_1,\cdots,J_m)=\bigcup\limits_{i=1}^{m}J_i:\mathcal{K}(E)^m\to\mathcal{K}(E)$ *is continuous;*

(6) $h(f_1,\cdots,f_m)=f_1\circ\cdots\circ f_m:\mathrm{con}(E)^m\to\mathrm{con}(E)$ *is continuous;*

(7) $I_i,J_i,\bigcup\limits_i I_i,\bigcup\limits_i J_i\in\mathcal{K}(E)\Rightarrow\eta\Big(\bigcup\limits_i I_i,\bigcup\limits_i J_i\Big)\le\sup\limits_i,\eta(I_i,J_i)$.

Definiton 2.2. *Let $f^{(\cdot)}:\Omega\to\mathrm{con}(E)$. We call f a statistical contraction operators, iff f is a random element from Ω to $\mathrm{con}(E)$, i.e.*

$$\{\omega\in\Omega,f^{(\omega)}\in A\}\in\mathcal{F}\ (\forall A\in\mathcal{B}(\mathrm{con}(E))).$$

We denote all statistical contraction operators by $\mathrm{con}(\Omega,E)$.

We write $f^{(\omega)}=f(\omega),f^{(\omega)}(x)=f(\omega,x)$ sometimes.

The proofs of the following propositions are straightforward and we leave them to the reader.

Propositon 2.2. *Let $(\overline{\Omega},\overline{\mathcal{F}})$ be a measurable space, $(\overline{\mathcal{K}},\overline{\eta})$ be a separable complete metric space. $\bar{f}(\omega,J):\overline{\Omega}\times\overline{\mathcal{K}}\to\overline{\mathcal{K}}$ satisfies: $\forall\omega\in\overline{\Omega},\bar{f}(\omega,\cdot)$ is uniformly continuous; $\forall J\in\overline{\mathcal{K}},\bar{f}(\cdot,J)$ is Borel measurable. Then $\bar{f}(\cdot,\cdot)$ is Borel measurable.*

Proposition 2.3. *If $f\in\mathrm{con}(\Omega,E)$, then $f(\omega,J):\Omega\times\mathcal{K}(E)\to\mathcal{K}(E)$ is Borel measurable; especially $f(\cdot,J)$ is Borel measurable for any fixed $J\in\mathcal{K}(E)$.*

Proposition 2.4. $\{f_1,\cdots,f_m\}\subset\mathrm{con}(\Omega,E)\Rightarrow f_1\circ\cdots\circ f_m\in\mathrm{con}(\Omega,E)$.

Proposition 2.5. $\{f_1,\cdots,f_m\}\subset\mathrm{con}(\Omega,E)\Rightarrow f_1\circ\cdots\circ f_2\circ\cdots\circ f_m(J)$ *is a random element from Ω to $\mathcal{K}(E)$ for any fixed $J\in\mathcal{K}(E)$.*

We always assume $\mathrm{diam}(E)<\infty$ in this paper.

Let us introduce some notations now. 1_A always denotes the indicator function on set $A, M(\Omega, \mathcal{K}(E))$ denotes the collection of all random elements from Ω to $\mathcal{K}(E)$.

Let $N \geq 2$ be an integer, $C_0 = \{\emptyset\}, C_n = C_n(N) = \{0, 1, \cdots, N-1\}^n, (n \geq 1), D = D(N) = \bigcup_{n\geq 0} C_n,\ C = \{(\sigma_0, \sigma_1, \cdots):\ 0 \geq \sigma_i < N\}$.

$\forall \sigma = (\sigma_0, \cdots, \sigma_{n-1}) \in C_n(N), \tau \in C(N) \bigcup D(N),\ \tau = (\tau_0, \tau_1, \cdots), |\sigma| = n =$ the length of $\sigma, \sigma * \tau = (\sigma_0, \cdots, \sigma_{n-1}, \tau_0, \tau_1, \cdots)$ is the juxtaposition of σ and $\tau, \tau|k = (\tau_0, \tau_1, \cdots, \tau_{k-1})$ (if $|\tau| \geq k$), $j|\tau = (\tau_j, \tau_{j+1}, \cdots)$ (if $|\tau| \geq j$). $\{0, 1, \cdots, N-1\}$ carries the discrete topology, and C carries the product topology, so C is compact.

Definition 2.3. *Let* $\{f_0, \cdots, f_{N-1}\} \subset \mathrm{con}(\Omega, E), Q \in \mathcal{P}(\mathcal{K}(E))$. *We call* Q *a* $P-(f_0, \cdots, f_{N-1})$ *statistically self-similar measure, iff* $\forall B \in \mathcal{B}(\mathcal{K}(E))$, *we have*

$$Q(B) = P \times Q, N\Big(\Big\{(\omega; K_0, \cdots, K_{N-1}) \in \Omega \times \mathcal{K}(E)^N : \bigcup_{i=0}^{N-1} f_i(\omega, K_i) \in B\Big\}\Big).$$

Definition 2.4. *Let* $K : \Omega \to \mathcal{K}(E)$ *be a random set (i.e.* $K^{-1}(\mathcal{B}(\mathcal{K}(E))) \subset \mathcal{F}$), $\{f_0, \cdots, f_{N-1}) \subset \mathrm{con}(\Omega, E)$. *We call* K *a* $P-(f_0, \cdots, f_{N-1})$ *statistically self-similar set, iff* $P \circ K^{-1}$, *the distribution of* K, *is a* $P-(f_0, \cdots, f_{N-1})$ *statistically self-similar measure.*

Proposition 2.6. *Let* $f_0, \cdots, f_{N-1} \subset \mathrm{con}(\Omega, E), K : \Omega \to \mathcal{K}(E)$ *be a random set. Then* K *is a* $P-(f_0, \cdots, f_{N-1})$ *statistically self-similar set iff*

$$K(\omega) \stackrel{d}{=} \bigcup_{i=0}^{N-1} f_i(\omega, K(\omega_i)).$$

where $\stackrel{d}{=}$ *means equal in distribution.*

Proof. It is easy to prove by the definition.

§3. Main Result

Theorem 3.1. *Let* $\{f_\sigma, \sigma \in D\} \subset \mathrm{con}(\Omega, E), f_{n,\sigma} = f_{\sigma|1} \circ \cdots \circ f_{\sigma|n} (\sigma \in C_n, n \geq 1), \Psi_n = \bigcup_{\sigma \in C_n} f_{n,\sigma}(E),\ K = \bigcap_{n=1}^{\infty} \overline{\Psi}_n$.

(1) *If* $\sup_{\sigma \in D} \mathrm{Lip}(f_\sigma) = \alpha < 1$ *a.s., then, for almost all* $\omega \in \Omega$,

$$\lim_{n\to\infty} \eta\Big(K, \bigcup_{\sigma \in C_n} f_{n,\sigma}(J_\sigma)\Big) = 0\ (\forall J_\sigma \in M(\Omega, \mathcal{K}(E))); K \in M(\Omega, \mathcal{K}(E)). \tag{3.1}$$

(2) *If the condition in* (1) *is satisfied, and* $\{(f_{\sigma*0}, \cdots, f_{\sigma*(N-1)}), \sigma \in D\}$ *are i.i.d. random elements from* Ω *to* $\mathrm{con}(E)^N$, *then* $P-(f_0, \cdots, f_{N-1})$ *statistically self-similar measure is unique, it is* $P_K \triangleq P \circ K^{-1}$, *so* K *is a statistically self-similar set.*

In order to prove this theorem, we need some lemmas.

We write $Q^{\#B} = Q^B$ for a measure Q and a set B sometimes. If Q is a set, Q^B has the same meaning as above.

Let

$$\begin{aligned} t_n(\omega; J_\sigma, \sigma \in C_n) &= \Psi_n^{(\omega)}(J_\sigma, \sigma \in C_n) \\ &= \bigcup_{\sigma \in C_n} f_{n,\sigma}^{(\omega)}(J_\sigma), (\omega \in \Omega, J_\sigma \in \mathcal{K}(E),\ n \geq 1), \end{aligned} \tag{3.2}$$

$$\begin{aligned} &T_P : \mathcal{P}(\mathcal{K}(E)) \to \mathcal{P}(\mathcal{K}(E)), \\ &\quad T_P(Q) = (P \times Q^N) \circ t_1^{-1},\ Q \in \mathcal{P}(\mathcal{K}(E)), \end{aligned} \tag{3.3}$$

$T_P^{(n)}$ is the n-compound mapping of T_p.

Lemma 3.1. *Let $\{f_\sigma, \sigma \in D\} \subset \mathrm{con}(\Omega, E)$, $f_{n,\sigma}$, Ψ_n, t_n, T_P and $T_p^{(n)}$ be defined as before. $\forall \{Q, Q_k,\ k \geq 1\} \subset \mathcal{P}(\mathcal{K}(E))$, if $Q_k \stackrel{W}{\to} Q$ as $k \to \infty$ (where $\stackrel{W}{\to}$ means weak convergence), then $T_P(Q_k) \stackrel{W}{\to} T_p(Q)$.*

Proof. $\forall g : \mathcal{K}(E) \to R$, if g is continuous and bounded, then

$$\begin{aligned}\int_{\mathcal{K}(E)} g(J) T_p(Q_k)(dJ) &= \int_{\mathcal{K}(E)} g(J)[(P \times Q_k^N) \circ t_1^{-1}](dJ) \\ &= \int_{\Omega \times \mathcal{K}(E)^N} g(t_1(\omega; J_i, i \in C_1))(P \times Q_k^N)(d\omega; dJ_i, i \in C_1) \\ &= \int_\Omega P(d\omega)\Big(\int_{\mathcal{K}(E)^N} g(t_1(\omega; J_i, i \in C_1)) Q_k^N(dJ_i, i \in C_1)\Big).\end{aligned}$$

Since $Q_k \stackrel{W}{\to} Q$, we have $Q_k^N \stackrel{W}{\to} Q^N$ by Theorem 3.1 in Chapter 1 in [1]. But for any fixed $\omega \in \Omega$, $g(t_1(\omega; \cdot, \cdots, \cdots)) : \mathcal{K}(E)^N \to R$ is continuous and bounded by Proposition 2.1, hence

$$\begin{aligned}&\lim_{k\to\infty} \int_{\mathcal{K}(E)^N} g(t_1(\omega; J_i, i \in C_1)) Q_k^N(dJ_i, i \in C_1) \\ &= \int_{\mathcal{K}(E)^N} g(t_1(\omega; J_i, i \in C_1)) Q^N(dJ_i, i \in C_1).\end{aligned}$$

It follows from the bounded convergence theorem that

$$\begin{aligned}\lim_{k\to\infty} \int_{\mathcal{K}(E)^N} g(J) T_p(Q_k)(dJ) &= \int_\Omega P(d\omega) \int_{\mathcal{K}(E)^N} g(t_1(\omega; J_i, i \in C_1)) Q^N(dJ_i, i \in C_1) \\ &= \int_{\mathcal{K}(E)} g(J) T_p(Q)(dJ).\end{aligned}$$

Lemma 3.1 is proved.

Lemma 3.2. *Let $\{f_\sigma, \sigma \in D\}, f_{n,\sigma}, \Psi_n, t_n, T_P$ and $T_p^{(n)}$ be difined as in Lemma 3.1. If*

$$\begin{aligned}&P^2\Big(\Big\{(\omega, \bar{\omega}) : \bigcup_{i \in C_1} f_i^{(\omega)}(t_n(\bar{\omega}; J_{i*\sigma}, \sigma \in C_n)) \in A\Big\}\Big) \\ &= P(\{\omega : t_{n+1}(\omega; J_\tau, \tau \in C_{n+1}) \in A\}) \\ &\quad (\forall A \in \mathcal{B}(\mathcal{K}(E)), J_\tau \in \mathcal{K}(E), \tau \in C_{n+1},\ n \geq 1),\end{aligned} \tag{3.4}$$

then

$$T_p^{(n)}(Q) = (P \times Q^{C_n}) \circ t_n^{-1}\ (n \geq 1,\ Q \in \mathcal{P}(\mathcal{K}(E))). \tag{3.5}$$

Proof. We prove (3.5) by induction. When $n = 1$, (3.5) is true by the definiton of T_P. If (3.5) is true for n, $\forall A \in \mathcal{B}(\mathcal{K}(E))$, let

$$\begin{aligned}F(A) &= \Big\{(\omega; J_i, i \in C_1) : \bigcup_{i \in C_1} f_i^{(\omega)}(J_i) \in A\Big\} = t_1^{-1}(A), \\ M(A) &= \Big\{(\omega, \bar{\omega}; J_\tau, \tau \in C_{n+1}) : \bigcup_{i \in C_1} f_i^{(\omega)}(t_n(\bar{\omega}; J_{i*\sigma}, \sigma \in C_n)) \in A\Big\},\end{aligned}$$

then

$$M(A) = \{(\omega, \bar{\omega}; J_\tau, \tau \in C_{n+1}) : (\omega; t_n(\bar{\omega}; J_{i*\sigma}, \sigma \in C_n), i \in C_1) \in F(A)\},$$

hence

$$1_{F(A)}(\omega; t_n(\bar\omega; J_{i*\sigma}, \sigma \in C_n), i \in C_1) = 1_{M(A)}(\omega, \bar\omega; J_\tau, \tau \in C_{n+1}). \tag{3.6}$$

It follows from the definitions of $T_p^{(n)}$ and t_n and (3.6) that

$$\begin{aligned}
&T_p^{(n+1)}(Q)(A)\\
&= T_p((P \times Q^{C_n}) \circ t_n^{-1})(A)\\
&= (P \times [(P \times Q^{C_n}) \circ t_n^{-1}]^N)(t_1^{-1}(A))\\
&= \int_\Omega P(d\omega) \prod_{i \in C_1} \int_{\mathcal{K}(E)} [(P \times Q^{C_n}) \circ t_n^{-1}](dJ_i) \cdot 1_{F(A)}(\omega; J_i, i \in C_1)\\
&= \int_\Omega P(d\omega) \prod_{i \in C_1} \int_{\Omega \times \mathcal{K}(E)^{C_n}} (P \times Q^{C_n})(d\bar\omega, dJ_{i*\sigma}, \sigma \in C_n)\\
&\quad \cdot 1_{F(A)}(\omega; t_n(\bar\omega; J_{i*\sigma}, \sigma \in C_n), i \in C_1)\\
&\overset{(3.6)}{=} \prod_{i \in C_1} \int_\Omega P(d\omega) \int_{\Omega \times \mathcal{K}(E)^{C_n}} (P \times Q^{C_n})(d\bar\omega, dJ_{i*\sigma}, \sigma \in C_n) \cdot 1_{M(A)}(\omega, \bar\omega; J_\tau, \tau \in C_{n+1})\\
&= \prod_{\tau \in C_{n+1}} \int_{\mathcal{K}(E)} Q(dJ_\tau) \int_{\Omega^2} P^2(d\omega, d\bar\omega) \cdot 1_{M(A)}(\omega, \bar\omega; J_\tau, \tau \in C_{n+1}).
\end{aligned} \tag{3.7}$$

But, by the definition of $M(A)$ and (3.4),

$$\begin{aligned}
&\int_{\Omega^2} P^2(d\omega, d\bar\omega) \cdot 1_{M(A)}(\omega, \bar\omega; J_\tau, \tau \in C_{n+1})\\
&= P^2\Big(\Big\{(\omega, \bar\omega) : \bigcup_{i \in C_1} f_i^{(\omega)}(t_n(\bar\omega; J_{i*\sigma}, \sigma \in C_n)) \in A\Big\}\Big)\\
&= P(\{\omega : t_{n+1}(\omega, J_\tau, \tau \in C_{n+1}) \in A\}).
\end{aligned}$$

It follows from the above equation and (3.7) that

$$T_P^{(n+1)}(Q)(A) = (P \times Q^{C_{n+1}}) \cdot t_{n+1}^{-1}(A).$$

Lemma 3.2 is proved.

Lemma 3.3. *If the conditions in Lemma* 3.2 *are satisfied, and*

$$\sup_{\sigma \in D} \mathrm{Lip}(f_\sigma) = \alpha < 1 \quad \text{a.s.},$$

then

(1) $T_P^{(n)}(Q) \overset{W}{\to} P_k \triangleq P \circ K^{-1}$ *as* $n \to \infty$ $(\forall Q \in \mathcal{P}(\mathcal{K}(E)))$, K *is defined as in Theorem* 3.1;

(2) $P - (f_0, \cdots, f_{N-1})$ *statistically self-similar measure is unique, it is* P_k, *so* K *is a statistically self-similar set.*

Proof. (1) It is enough to prove (1) that for any open set $G \subset \mathcal{K}(E)$ and closed set $F \subset \mathcal{K}(E)$ we have

$$\liminf_{n \to \infty} T_P^{(n)}(Q)(G) \geq P_K(G), \tag{3.8}$$

$$\limsup_{n \to \infty} T_P^{(n)}(Q)(F) \leq P_K(F). \tag{3.9}$$

We only prove (3.9). By Fubini theorem and Lemma 3.2 we have

$$T_P^{(n)}(Q)(F) = P \times Q^D \Big(\Big\{(\omega; J_\sigma, \sigma \in D) \in \Omega \times \mathcal{K}(E)^D : \bigcup_{\sigma \in C_n} f_{n,\sigma}^{(\omega)}(J_\sigma) \in F\Big\}\Big),$$

hence

$$\begin{aligned}&\limsup_{n\to\infty} T_P^{(n)}(Q)(F)\\ &\le P\times Q^D\Big(\bigcap_{m=1}^{\infty}\bigcup_{n\ge m}\Big\{(\omega;J_\sigma,\sigma\in D)\in\Omega\times\mathcal{K}(E)^D:\bigcup_{\sigma\in C_n} f_{n,\sigma}^{(\omega)}(J_\sigma)\in F\Big\}\Big)\\ &\le P\times Q^D\Big(\Big\{(\omega;J_\sigma,\sigma\in D)\in\Omega\times\mathcal{K}(E)^D:\lim_{n\to\infty}\bigcup_{\sigma\in C_n} f_{n,\sigma}^{(\omega)}(J_\sigma)\in F\Big\}\Big)\\ &= P\times Q^D(\{(\omega;J_\sigma,\sigma\in D)\in\Omega\times\mathcal{K}(E)^D:K(\omega)\in F\})\\ &= P(\{\omega\in\Omega:K(\omega)\in F\})=P_k(F).\end{aligned}$$

(3.9) is proved.

(2) Now we want to show that $P-(f_0,\cdots,f_{N-1})$ statistically self-similar measure is unique, it is P_K. It follows from Lemma 3.1 and (1) that

$$P_K=\lim_{n\to\infty}T_P^{(n+1)}(P_k)=\lim_{n\to\infty}T_P(T_P^{(n)}(P_K))=T_P(\lim_{n\to\infty}T_P^{(n)}(P_k))=T_P(P_K),$$

this means that P_k is a statistically self-similar measure.

Suppose Q is any $P-(f_0,\cdots,f_{N-1})$ statistically self-similar measure, then

$$Q=T_P(Q)=\cdots=T_P^{(n)}(Q).$$

Let $n\to\infty$. By (1) in this lemma, we get

$$Q=\lim_{n\to\infty}T_P^{(n)}(Q)=P_K.$$

Lemma 3.3 is proved.

Lemma 3.4. *Let* $\{f_\sigma,\sigma\in D\}\subset\mathrm{con}(\Omega,E),\{(f_{\sigma*0},\cdots,f_{\sigma*(N-1)}),\ \sigma\in D\}$ *be a class of i.i.d. random elements from* Ω *to* $\mathrm{con}(E)^N$. *Let*

$$\tilde f_\sigma^{(\omega,\bar\omega)}=\begin{cases} f_\sigma^{(\omega)}, & \text{when } \sigma\in C_1,(\omega,\bar\omega)\in\Omega^2,\\ f_\sigma^{\bar\omega}, & \text{when } \sigma\in\bigcup_{k\ge2}C_k,\ (\omega,\bar\omega)\in\Omega^2.\end{cases}$$

Then

(1) $\{(\tilde f_{\sigma*0},\cdots,\tilde f_{\sigma*(N-1)},\sigma\in D\}$ *is a class of i.i.d. random elements from* Ω^2 *to* $\mathrm{con}(E)^N$ *and*

$$(\tilde f_0,\cdots,\tilde f_{N-1})\overset{d}{=}(f_0,\cdots,f_{N-1}),$$

$$\Big(\tilde f_\sigma,\sigma\in\bigcup_{k=2}^{n+1}C_k\Big)\overset{d}{=}\Big(f_\sigma,\sigma\in\bigcup_{k=2}^{n+1}C_k\Big)\ (n\ge1),$$

where $\overset{d}{=}$ *means equal in distribution.*

(2) *For any fixed* $\{J_i,J_\sigma,i\in C_1,\sigma\in\bigcup_{k=2}^{n+1}C_k\}\subset\mathcal{K}(E)$, $\{(f_{\sigma*0}(J_0),\cdots,f_{\sigma*(N-1)}(J_{N-1})),$ $\sigma\in D\}$ *is a class of i.i.d. random elements from* Ω *to* $\mathcal{K}(E)^N$, *and* $\{(\tilde f_{\sigma*0}(J_0),\cdots,$ $\tilde f_{\sigma*(N-1)}(J_{N-1})),\ \sigma\in D\}$ *is a class of i.i.d. random elements from* Ω^2 *to* $\mathcal{K}(E)^N$, *and*

$$(\tilde f_0(J_0),\cdots,\tilde f_{N-1}(J_{N-1}))\overset{d}{=}(f_0(J_0),\cdots,f_{N-1}(J_{N-1})),\tag{3.10}$$

$$\Big(\tilde f_\sigma,(J_\sigma),\sigma\in\bigcup_{k=2}^{n+1}C_k\Big)\overset{d}{=}\Big(f_\sigma(J_\sigma),\sigma\in\bigcup_{k=2}^{n+1}C_k\Big).\tag{3.11}$$

Proof. (1) is obvious, we only need to prove (2). Let μ be a probability measure on $\mathcal{B}(\mathcal{K}(E)^N)$ such that $\mu(A) = 1$ or 0 according to $(J_0, \cdots, J_{N-1}) \in A$ or not. Let

$$\begin{aligned}
&g_\sigma : \Omega \times \mathcal{K}(E)^N \to \text{con}(E)^N \times \mathcal{K}(E)^N, \sigma \in D,\\
&g_\sigma(\omega; \tilde{J}_\sigma, \cdots, \tilde{J}_{N-1}) = (f^{(\sigma)}_{\sigma*0}, \cdots, f^{(\sigma)}_{\sigma*N-1}; \tilde{J}_0, \cdots, \tilde{J}_{N-1}),\\
&h : \text{con}(E)^N \times \mathcal{K}(E)^N \to \mathcal{K}(E)^N,\\
&h(r_0, \cdots r_{N-1}; \tilde{J}_0, \cdots, \tilde{J}_{N-1}) = (r_0(\tilde{J}_0), \cdots, r_{N-1}(\tilde{J}_{N-1})).
\end{aligned}$$

Then, by Proposition 2.1, $\{g_\sigma, \sigma \in D\}$ is a class of i.i.d. random elements from probability space $(\Omega \times \mathcal{K}(E)^N, \mathcal{F} \times \mathcal{B}(\mathcal{K}(E)^N), P \times \mu)$ to $\text{con}(E)^N \times \mathcal{K}(E)^N$, h is a continuous operator. Hence

$$\{(f^{(\omega)}_{\sigma*0}(\tilde{J}_0), \cdots, f^{(\omega)}_{\sigma*(N-1)}(\tilde{J}_{N-1})) = h(g_\sigma(\omega; \tilde{J}_0, \cdots, \tilde{J}_{N-1})),\ \sigma \in D\}$$

is a class of i.i.d. random elements from $(\Omega \times \mathcal{K}(E)^N, \mathcal{F} \times B(\mathcal{K}(E)^N), P \times \mu)$ to $\mathcal{K}(E)^N$; especially, for fixed $(J_0, \cdots, J_{N-1})$,

$$\{(f^{(\omega)}_{\sigma*0}(J_0) \cdots, f^{(\omega)}_{\sigma*(N-1)}(J_{N-1})) = h(g_\sigma(\omega; J_0, \cdots, J_{N-1})), \sigma \in D\}$$

is a class of i.i.d. random elements from $(\Omega, \mathcal{F}, P)$ to $\mathcal{K}(E)^N$.

We can prove $\{(\tilde{f}_{\sigma*0}(J_0), \cdots, \tilde{f}_{\sigma*(N-1)}(J_{N-1})), \sigma \in D\}$ is a class of i.i.d. random elements from $(\Omega^2, \mathcal{F}^2, P^2)$ to $\mathcal{K}(E)^N$ in the same way. (3.10) and (3.11) are obvious. Lemma 3.4 is proved.

Now let us prove Theorem 3.1. (1) is a special case of Theorem 2.1 in [4]. In order to prove the conclusion (2) of Theorem 3.1, using Lemma 3.3, it is enough to prove (3.4) is true under the conditions in the Theorem 3.1. Let $B(n+1) = \bigcup_{k=1}^{n+1} C_k$,

$$\begin{aligned}
&q : \text{con}(E)^{B(n+1)} \times \mathcal{K}(E)^{B(n+1)} \to \mathcal{K}(E),\\
&q(S_\tau; J_\tau, \tau \in B(n+1)) = \bigcup_{i \in C_1} \bigcup_{\sigma \in C_n} S_i \circ S_{i*(\sigma|1)} \circ \cdots \circ S_{i*(\sigma|n)}(J_{i*\sigma}).
\end{aligned}$$

Then q is a continuous operator by Proposition 2.1. Hence $q(\tilde{f}^{(\omega,\tilde{\omega})}_\tau; J_\tau, \tau \in B(n+1))$ and $q(f^{(\omega)}_i, f^{(\tilde{\omega})}_{i*\sigma}; J_i, J_{i*\sigma}, i \in C_1,\ \sigma \in B(n))$ are random elements from $(\Omega^2, \mathcal{F}^2, P^2)$ to $\mathcal{K}(E)$ for fixed $J_\tau, (\tau \in B(n+1))$.

Since $\{(f^{(\omega)}_{\sigma*0}, \cdots, f^{(\omega)}_{\sigma*(N-1)}), \sigma \in D\}$ are i.i.d., it follows that

$$\forall \Big\{A_i, B_\tau, i \in C_1, \tau \in \bigcup_{k=2}^{n+1} C_k\Big\} \subset \mathcal{B}(\text{con}(E))$$

we have

$$\begin{aligned}
&P^2(f^{(\omega)}_i \in A_i, f^{(\bar{\omega})}_{i*(\sigma|j)} \in B_{i*(\sigma|j)}, i \in C_1, \sigma \in C_n, 1 \le j \le n)\\
&= P^2(f^{(\omega)}_i \in A_i, i \in C_1, f^{(\bar{\omega})}_{\sigma*j} \in B_{\sigma*j}, \sigma \in B(n), j \in C_1)\\
&= P^2(f^{(\omega)}_i \in A_i, i \in C_1) P^2(f^{(\bar{\omega})}_{\sigma*j} \in B_{\sigma*j}, \sigma \in B(n), j \in C_1)\\
&= P^2(f^{(\omega)}_i \in A_i, i \in C_1) P^2(f^{(\bar{\omega})}_{(|\tau|-1)|\tau} \in B_\tau, \tau \in \bigcup_{k=2}^{n+1} C_k).
\end{aligned} \tag{3.12}$$

Similarly, we also have

$$P^2(f_i^{(\omega)} \in A_i, f_{\sigma|j}^{(\bar\omega)} \in B_{i*(\sigma|j)}, \sigma \in C_n, i \in C_1, 1 \le j \le N)$$
$$= P^2(f_i^{(\omega)} \in A_i, i \in C_1) P^2\Big(f_{(|\tau|-1)|\tau}^{(\bar\omega)} \in B_\tau,\ \tau \in \bigcup_{k=2}^{n+1} C_k\Big). \tag{3.13}$$

But, by the definition of $\{\tilde f_\sigma, \sigma \in D\}$ and Lemma 3.4, we have

$$(f_0^{(\omega)}, \cdots, f_{N-1}^{(\omega)}, f_{i*(\sigma|1)}^{(\bar\omega)}, \cdots, f_{i*\sigma}^{(\bar\omega)}, i \in C_1, \sigma \in C_n)$$
$$\equiv (\tilde f_0^{(\omega,\bar\omega)}, \cdots, \tilde f_{N-1}^{(\omega,\bar\omega)}, \tilde f_{i*(\sigma|1)}^{(\omega,\bar\omega)}, \cdots, \tilde f_{i*\sigma}^{(\omega,\bar\omega)}, i \in C_1, \sigma \in C_n)$$
$$\overset{d}{=} (f_0^{(\omega)}, \cdots, f_{N-1}^{(\omega)}, f_{i*(\sigma|1)}^{(\omega)}, \cdots, f_{i*\sigma}^{(\omega)}, i \in C_1, \sigma \in C_n). \tag{3.14}$$

It follows from (3.12)–(3.14) that

$$((f_i^{(\omega)}, f_{\sigma|1}^{(\bar\omega)}, \cdots, f_\sigma^{(\bar\omega)}), \sigma \in C_n, i \in C_1)$$
$$\overset{d}{=} ((f_i^{(\omega)}, f_{i*(\sigma|1)}^{(\omega)}, \cdots, f_{i*\sigma}^{(\omega)}),\ \sigma \in C_n,\ i \in C_1). \tag{3.15}$$

Hence

$$q(f_i^{(\omega)}, f_{i*(\sigma|j)}^{(\omega)}; J_i, J_{i*(\sigma|j)},\ i \in C_1,\ \sigma \in C_n,\ 1 \ge j \le n)$$
$$\overset{d}{=} q(f_i^{(\omega)}, f_{\sigma|j}^{(\bar\omega)}; J_i, J_{i*(\sigma|j)},\ i \in C_1,\ \sigma \in C_n,\ 1 \le j \le n). \tag{3.16}$$

But

$$q(f_i^{(\omega)}, f_{\sigma|j}^{(\bar\omega)}; J_i, J_{i*(\sigma|j)},\ i \in C_1,\ \sigma \in C_n,\ 1 \le j \le n)$$
$$= \bigcup_{i \in C_1} f_i^{(\omega)}\Big(\bigcup_{\sigma \in C_n} f_{\sigma|1}^{(\bar\omega)} \circ \cdots \circ f_{\sigma|n}^{(\bar\omega)}(J_{i*\sigma})\Big)$$
$$= \bigcup_{i \in C_1} f_i^{(\omega)}(t_n(\bar\omega; J_{i*\sigma}, \sigma \in C_n)),$$
$$q(f_i^{(\omega)}, f_{i*(\sigma|j)}^{(\omega)}; J_i, J_{i*(\sigma|j)},\ i \in C_1, \sigma \in C_n,\ 1 \le j \le n)$$
$$= \bigcup_{i \in C_1} \bigcup_{\sigma \in C_n} f_i^{(\omega)} \circ f_{i*(\sigma|1)}^{(\omega)} \circ \cdots \circ f_{i*\sigma}^{(\omega)}(J_{i*\sigma})$$
$$= t_{n+1}(\omega, J_\tau, \tau \in C_{n+1}),$$

hence (3.16) means (3.4) is true. The theorem is proved.

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