Chin. Ann. of Math.
23B:4(2002),469-474.

REPRESENTATION OF SYMMETRIC SUPER-MARTINGALE MULTIPLICATIVE FUNCTIONALS***

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Abstract

The authors introduce concepts of even and odd additive functionals and prove that an even martingale continuous additive functional of a symmetric Markov process vanishes identically. A representation for symmetric super-martingale multiplicative functionals are also given.

Keywords Markov processes, Additive functional, Multiplicative functional
2000 MR Subject Classification 60J45
Chinese Library Classification O211.62 **Document Code** A
Article ID 0252-9599(2002)04-0469-06

§1. Introduction

Let E be a Lusin space with its Borel σ-algebra $\mathcal{B}(E)$, and m a σ-finite measure on $(E,\mathcal{B}(E))$. We denote by $\mathcal{B}(E\times E)$ the product σ-algebra on $E\times E$. Let $X=(\Omega,\mathcal{F},\mathcal{F}_t,\theta_t, X_t,P^x)$ be a Borel right process with state space E, lifetime ζ, transition semigroup $(P_t)_{t>0}$ and resolvent $(U^q)_{q>0}$. Throughout this paper we assume that X is m-symmetric; precisely

$$(f,P_tg)=(P_tf,g),\quad f,g\in L^2(m), \tag{1.1}$$

where $(u,v):=\int uvdm$ is the natural inner product in $L^2(m):=L^2(E;m)$. Then (P_t) may be extended into a symmetric operator semigroup on $L^2(m)$ and there is a quasi-regular Dirichlet form $(\mathcal{E},\mathcal{D})$ associated with it. A well-known consequence of symmetry (e.g., see [12]) is for $x\in E$,

$$P^x(\{\omega\in\Omega:\ X_{t-}(\omega)\text{ exists in }E\ \text{ for all }t<\zeta\})=1.$$

By an additive functional (abbreviated sometimes as AF), we mean an adapted real valued process on $[0,\zeta[$ with additivity and right continuity. Let $A=(A_t)$ be any additive functional of finite variation pathwise on any compact interval of $[0,\zeta[$. We can (and always do) take its perfected version.

Manuscript received May 10, 2001. Revised October 15, 2001

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***Project supported by the National Natural Science Foundation of China.

To describe the behavior of jumps of X, there exists a pair (N, H), which is usually called the Lévy system of X, with N a kernel on $(E, \mathcal{B}(E))$ satisfying $N(x, \{x\}) = 0$ for any $x \in E$ and H a positive continuous additive functional (abbreviated as PCAF in the sequel) of X with bounded 1-potential, such that for any Borel function ϕ on $E \times E$, the dual predictable projection (or compensator) of the homogeneous random measure

$$\kappa(\omega, dt) := \phi(X_{t-}(\omega), X_t(\omega))1_{\{X_{t-}(\omega)\neq X_t(\omega)\}}(dt)$$

is $\int_0^t N\phi(X_s)dH_s$, where

$$N\phi(x) := \int_{y\in E} N(x, dy)\phi(x, y).$$

The measure $J(dx, dy) := N(x, dy)\rho_H$ is called the jumping measure relative to m, where ρ_H is the Revuz measure of H.

In §2, we shall prove that any even continuous martingale additive functional vanishes identically and it generalizes a result of Fitzsimmons[3] in the diffusion case. Then we use this result to prove that any symmetric super-martingale multiplicative functional is of bounded variation in §3.

§2. Even and Odd Additive Functionals

Given a path $\omega \in \Omega$ with $\zeta(\omega) > t$, define the reversal path $r_t\omega$ at time t as

$$r_t\omega(s) := \begin{cases} \omega((t-s)-), & 0 \le s < t; \\ \omega(0), & s \ge t. \end{cases}$$

Then $r_t\omega(s-) = \omega(t-s)$ if $0 < s \le t$.

Lemma 2.1. *A necessary and sufficient condition for a right process X to be m-symmetric is that for any $t > 0$ and $G \in \mathcal{F}_t$,*

$$P^m(G \circ r_t;\ t < \zeta) = P^m(G;\ t < \zeta). \tag{2.1}$$

Proof. The proof is exactly the same as Lemma 5.7.1 in [6] by noting that for any fixed $t > 0$, $X_t = X_{t-}$ a.s.

Let $A = (A_t(\omega))$ be any adapted and right continuous real process. Set $r(A)_t := \lim_{s\downarrow t} A_s \circ r_s$, whenever the limit of right side exists. It follows from the fact that $r_t\omega(u) = r_s \circ \theta_{t-s}\omega(u)$ for $0 < u \le s \le t$ that if A is an additive functional, $r(A) := (r(A)_t)$ is also an additive functional if $r(A)_t < \infty$ a.s. for $t < \zeta$. We say A is even if $r(A) = A$, and odd if $r(A) = -A$. More detailed properties of even and odd additive functional in diffusion case are given in [3].

Example 2.1. Let $u \in \mathcal{D}$ and be quasi-continuous. Then $t \mapsto u(X_t)$ is right continuous and $t \mapsto u(X_{t-})$ left continuous. Define

$$A_t^{[u]} := u(X_t) - u(X_0),$$

which is an additive functional of X. Then, it is easy to check that

$$A_t^{[u]} \circ r_t = u(X_0) - u(X_{t-}).$$

Hence $A^{[u]}$ is odd.

Example 2.2. By a result of [12], any increasing continuous AF (or PCAF) is even. Consequently any continuous AF of bounded variation is even. On the other hand, using a proof in [3] without any modification, we see that any CAF of zero energy is even.

Example 2.3. Let ϕ be a Borel function on $E \times E$ and $A := \sum_{s \leq \cdot} \phi(X_{s-}, X_s)$. Then A is an AF as long as $|A_t| < \infty$ a.s. for $t < \zeta$. It is easily seen that

$$A_t \circ r_t = \sum_{s \leq t} \phi(X_{(t-s)}, X_{(t-s)-}) = \sum_{s < t} \phi(X_s, X_{s-}).$$

Hence $r(A)_t = \sum_{s \leq t} \hat{\phi}(X_{s-}, X_s)$, where $\hat{\phi}(x, y) := \phi(y, x)$. It is now obvious that A is even if ϕ is symmetric: $\hat{\phi} = \phi$ and odd if anti-symmetric: $\hat{\phi} = -\phi$. This example may justify the names 'even' and 'odd' more concretely.

We also define the even part of A as $A^{\text{even}} := \frac{1}{2}(A + r(A))$ and the odd part as $A^{\text{odd}} := \frac{1}{2}(A - r(A))$. Then

$$A = A^{\text{even}} + A^{\text{odd}}.$$

Hence the Fukushima's decomposition can be reformulated as

$$u(X.) - u(X_0) = (M^{[u]})^{\text{odd}}, \quad -N^{[u]} = (M^{[u]})^{\text{even}}.$$

We now prove that any even continuous martingale AF of X vanishes identically. This was proved in [3], where X is a symmetric diffusion.

Lemma 2.2. *Let A and B be AF's and semi-martingale on $[0, \zeta[$. Suppose that A is even, B is odd and at least one of them is continuous. Then the square bracket process $[A, B]$ vanishes identically on $[0, \zeta[$.*

Proof. Since $\Delta[A, B] = \Delta A \cdot \Delta B$, $[A, B]$ is a continuous AF of bounded variation and therefore even by a result of J. Walsh. On the other hand, we may choose a sequence of symmetric partitions $\{\Delta_n = \{t_k^n\}\}$ on $[0, t]$ with mesh sizes tending to zero, such that

$$[A, B]_t = \lim_n \sum_{\Delta_n} (A_{t_k^n} - A_{t_{k-1}^n})(B_{t_k^n} - B_{t_{k-1}^n}).$$

When $t < \zeta$,

$$\Big[\sum_{\Delta_n} (A_{t_k^n} - A_{t_{k-1}^n})(B_{t_k^n} - B_{t_{k-1}^n})\Big] \circ r_t$$
$$= -\sum_{\Delta_n} (A_{t_k^n} - A_{t_{k-1}^n})(B_{t_k^n} - B_{t_{k-1}^n}).$$

Then it is easily seen that $[A, B]$ is odd. Hence $[A, B]$ vanishes identically.

Theorem 2.1. *If M is an even local martingale continuous additive functional of X, then M vanishes identically.*

Proof. Take $f \in L^1(m)$ bounded, positive, and define $u := U^q f$ with $q > 0$. Then $Au = qu - f$ and $N_t^{[u]} = \int_0^t Au(X_s)ds$ is a continuous AF of bounded variation. Thus $[M, N^{[u]}] = 0$ and by the lemma above, $[M, A^{[u]}] = 0$. It follows that $[M, M^{[u]}] = 0$. Set

$$Z_t := \int_0^t e^{-qs} dM_s^{[u]}, \ L_t := e^{M_t - \frac{1}{2}\langle M \rangle_t}.$$

Using Ito's formula, we have

$$Z_t = e^{-qt}u(X_t) - u(X_0) + \int_0^t e^{-qs} f(X_s)ds + [e^{-q\lambda}, M^{[u]}]_t,$$

where $\lambda_t = t$. However,

$$[e^{-q\lambda}, M^{[u]}]_t = -\Big[\int_0^{\cdot} e^{-qs}ds, M^{[u]}\Big]_t = -\int_0^t e^{-qs}d[\lambda, M^{[u]}]_s.$$

Since λ_t is a PCAF, it follows from Lemma 2.2 that

$$[\lambda, M^{[u]}] = [\lambda, A^{[u]}] = 0.$$

Therefore

$$Z_t = e^{-qt}u(X_t) - u(X_0) + \int_0^t e^{-qs}f(X_s)ds.$$

Define $dP_L^x := L_t dP^x$ on $\mathcal{F}_t$ for any $t > 0$. Since

$$[L, Z]_t = \int_0^t e^{-qs}L_s d[M, M^{[u]}]_s = 0,$$

Z is a P_L^x-local martingale by Girsanov theorem on $[0, \zeta)$. Since Z is uniformly bounded on compact intervals, it is a true P_L^x-martingale. Consequently $P_L^x(Z_t) = 0$. Letting t tend to zero, we have $U_L^q f = U^q f$, where (U_L^q) is the potential operators associated with P_L^x. It follows that $P_L^x = P^x$ and $L \equiv 1$ on $[0, \zeta)$. Therefore $M \equiv 0$, being a continuous local martingale of bounded variation.

§3. Representation of Symmetric Multiplicative Functionals

We are now given a super-martingale multiplicative functional $M = (M_t)$ of X. Roughly speaking, M satisfies the following conditions

(i) $M_t \geq 0$ a.s. for all $t > 0$;

(ii) $M_{t+s} = M_t \cdot M_s \circ \theta_t$ a.s. for all $s, t > 0$;

(iii) $P^x(M_t) \leq 1$ for all $t > 0$ and $x \in E$;

(iv) $t \mapsto M_t$ is right continuous a.s.

Without loss of generality we assume that $M_0 = 1$ a.s., since we can always appeal a killing of a hitting time to reach this. Let $Y = (\Omega, \mathcal{F}, \mathcal{F}_t, Y_t, \theta_t, Q^x)$ be the right process transformed by M, precisely it is a right process with the same state space E and transition kernel (Q_t) defined by

$$Q_t f(x) := P^x(M_t f(X_t)),\ x \in E$$

for $f \in p\mathcal{B}(E)$. The process Y is locally absolutely continuous with respect to X; i.e., $Q^x|_{\mathcal{F}_t} << P^x|_{\mathcal{F}_t}$ for any $t > 0$ and $x \in E$. It is well-known that the converse aeertion holds too. Therefore the transformation of super-martingale multiplicative functionals is one of great importance in the theory of Markov processes and it includes many useful ones such as killing transform, Doob's h-transform, drift transformation. We denote the respective resolvent by $(V^q)_{q>0}$. Notice that Y is realized on the same sample space as X, and that

$$X_t(\omega) = Y_t(\omega) = \omega(t).$$

The processes are distinguished actually by their respective laws P^x and Q^x, but we use Y_t for emphasis when dealing with Q^x. We say M is (m-)symmetric if Y is also an (m-) symmetric Markov process E. The following lemma is easy to check by Lemma 2.1.

Lemma 3.1. *A necessary and sufficient condition for M to be symmetric is that M is even: $M_t \circ r_t = M_t$ a.s. for each $t < \zeta$.*

Now we state the main result of this section. Though it assumes M never vanishes, the general case where M may vanish may be reduced to the non-vanishing case by the work on killing of the terminal time S shown in [13].

Theorem 3.1. *If M is symmetric and never vanishes, then there exist a symmetric Borel function $\phi > -1$ on $E \times E$ which vanishes on diagonal and a PCAF A such that for all $t > 0$,*

$$M_t = e^{L_t - A_t} \prod_{0<s\le t} [1 + \phi(X_{s-}, X_s)] e^{-\phi(X_{s-}, X_s)}, \ t < \zeta, \tag{3.1}$$

where L is the compensated martinagle of $\sum_{0<s\le \cdot} \phi(X_{s-}, X_s)$. The PCAF A is determined by M up to P^m-evanescence. In particular, ϕ is uniquely determined by M modulo null sets of the measure J. Moreover,

$$\int_0^t N(1_{\{|\phi|\le 1\}}\phi^2 + 1_{\{\phi>1\}}\phi)(X_s)\, dH_s < +\infty \tag{3.2}$$

for all $t < \zeta$, P^m-a.s.

Proof. It is known that M, as a super-martingale MF of X, admits a representation (refer to [1] or [10]). There is a local martingale AF L, a PCAF A, and a Borel function $\phi : E \times E_\Delta \to (-1, +\infty)$ such that

$$\Delta L_t := L_t - L_{t-} = \phi(X_{t-}, X_t), \quad \forall t < \zeta, \ P^m\text{-a.s.},$$

such that

$$M_t = e^{L_t - \frac{1}{2}\langle L^c\rangle_t - A_t} \prod_{0<s\le t} [1 + \phi(X_{s-}, X_s)] e^{-\phi(X_{s-}, X_s)}, \quad \forall t < \zeta, \ P^m\text{-a.s.} \tag{3.3}$$

The AF L and the PCAF A are determined by Z up to P^m-evanescence. In particular, ϕ is uniquely determined by Z modulo null sets of the measure J and stisfies the integrability in the theorem. Since M is symmetric, it is easily seen that its Dolean-Dade logarithm $\mathrm{Log}M$ is an even additive functional. Clearly

$$\mathrm{Log}M_t = L_t^c + \sum_{s\le t} \phi(X_{s-}, X_s) - \int_0^t N\phi(X) dH - A_t.$$

Hence

$$L_t^c - r(L^c)_t + \sum_{s\le t} (\phi - \hat{\phi})(X_{s-}, X_s) = 0.$$

It follows from the continuity of L^c that $\sum_{s\le \cdot} \phi(X_{s-}, X_s)$ is even and therefore ϕ can be taken to be symmetric. Finally it follows that L^c is an even continuous martingale additive functional and vanishes identically by Theorem 2.1.

Remark 3.1. The result tells that a symmetric super-martingale MF has a rather simple representation and is actually of bounded variation. Actually we may prove that any even semi-martingale AF is of bounded variation by a similar approach. Notice that a general even additive functional is not of bounded variation. For example, for any element u in the associated Dirichlet space, $N^{[u]}$, the CAF of zero energy in Fukushima's decomposition, is not of bounded variation in general.

Remark 3.2. For a σ-finite measure μ, it is not hard to see that the super-martingale

MF's which transform X to μ-symmetric Markov processes are unique up to a μ-symmetric super-martingale MF, which is of the simple form as above.

Acknowledgement. The authors would like to thank Professor P. Fitzsimmons for helpful comments.

References

[1] Chen, Z., Fitzsimmons, P., Takeda, M., Ying, J. & Zhang, T., Absolute continuity of symmetric Markov processes [R], preprint, 2001.
[2] Dellacherie, C., Potentiels de Green et fonctionneles additives [M], Sém. de Probabilités IV, Lecture Notes in Math., **124**(1970), Springer-Verlag.
[3] Fitzsimmons, P. J., Absolute continuity of symmetric diffusion [J], *Ann. Probab.,* **25:**1(1997), 230–258.
[4] Fitzsimmons, P. J., Even and odd continuous additive functionals [A], In Dirichlet forms and stochastic processes, Proc. Internat. Conf. on Dirichlet Forms and Stochastic Processes [C] (Beijing, 1993), 139–154, de Gruyter, Berlin, 1995.
[5] Fitzsimmons, P. J. & Getoor, R. K., Revuz measures and time changes [J], *Math. Z.,* **199**(1988), 233–256.
[6] Fukushima, M., Oshima, Y. & Takeda, M., Dirichlet forms and symmetric Markov processes [M], Walter de Gruyter, Berlin-New York, 1994.
[7] Fukushima, M. & Takeda, M., A transformation of a symmetric Markov process and the Donsker-Varadhan theory [J], *Osaka J. Math.,* **21**(1984), 311–326.
[8] Getoor, R. K., Excessive measures [M], Birkhauser, 1990.
[9] Getoor, R. K. & Sharpe, M. J., Naturality, standardness, and weak duality for Markov processes [J], *Z. W.,* **67**(1984), 1–62.
[10] Kunita, H., Absolute continuity of Markov processes [J], *Lecture Notes in Mathematics,* **511**(1976), 44–77.
[11] Sharpe, M. J., General theory of Markov processes [M], Academic Press, 1988.
[12] Walsh, J. B., Markov processes and their functionals in duality [J], *Z. Wahrsch. Verw. Gebiete,* **24**(1972), 229–246.
[13] Ying, J., Bivariate Revuz measures and the Feynman-kac formula [J], *Ann. Inst. Henri Poincare,* **32:**2(1996), 251–287.