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THE DETERMINANT REPRESENTATION OF THE GAUGE TRANSFORMATION OPERATORS****

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Abstract

The determinant representation of the gauge transformation operators is establised. In this process, the generalized Wronskian determinant is introduced. As a simple application, the authors present a construction of the special τ-function obtained firstly by Chau et al. (Commun. Math. Phys., 149(1992), 263), which involves the generalized Wronskian determinant. Also, some properties of this determinant are given.

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§1. Introduction

The KP hierarchy [1−3] and the cKP hierarchy [4−8], in terms of the pseudo-differential operators, are important topics in research of soliton theory. Of course, how to find a class of the solution of these integrable hierarchies is still an attractive problem. Recently, two kinds of gauge transformation operators have been successfully applied to solve the KP hierarchy and cKP hierarchy[9−17]. One of them is differential type (i.e., Ψ_D or T_D) and the other is integral type (i.e., Ψ_I or T_I) (see [9,15]). Chau et al[9,15] have obtained a new and more universal determinant expression (see (3.17) in [9]) of the τ function by successive application of such two gauge transformation operators. This determinant[9] resembles the well-known Wronskian determinant except that elements in the first some rows are in the integral form. We call it the generalized Wronskian determinant. So far no one, to the best of our knowledge, has provided a direct and simple proof of the appearance for the generalized Wronskian determinant and discussed properties analogous to the Wronskian determinant. On the other hand, to obtain the expression of each component in multi-component cKP hierarchy by using gauge transformation method, the key object is the determinant representation of the gauge transformation operators.

The main aim of the present letter is to establish the determinant representation of the gauge transformation operators. As a simple and direct application, we will answer the first question above-mentioned in detail.

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§2. The Determinant Representation

For the KP hierarchy and cKP hierarchy , there are two kinds of the gauge transformation operators[9,15]:

$$\text{Type } \mathbf{I}: \qquad T_D(\chi) = \chi \circ \partial \circ \chi^{-1}, \tag{2.1}$$

$$\text{Type } \mathbf{II}: \qquad T_I(\mu) = \mu^{-1} \circ \partial^{-1} \circ \mu. \tag{2.2}$$

Here χ and μ are the eigenfunction and the adjoint eigenfunction of the above integrable hierarchies, respectively. $\partial = \frac{\partial}{\partial x}$, and ∂^{-1} is the inverse of ∂. The symbol $\circ$ denotes the product of two operators. For example,

$$\partial \circ f = f\partial + \frac{\partial f}{\partial x} = f\partial + f_x, \partial^2 \circ f = f_{xx} + 2f_x\partial + f\partial^2;$$

on the other hand, the symbol $\cdot$ denotes an operator acting on something following it, $\partial \cdot f = f_x, \partial^2 \cdot f = f_{xx}$. If we introduce the conjugate operation $*$ for two operators A and B as

$$\partial^* = -\partial, \quad (\partial^{-1})^* = -\partial^{-1}, \quad (A \circ B)^* = B^* \circ A^*,$$

then

$$\big(T_D^{-1}(\chi)\big)^* = -T_I(\chi), \quad \big(T_I^{-1}(\mu)\big)^* = -T_D(\mu). \tag{2.3}$$

In particular, it follows that

$$T_D(\chi) \cdot \chi = 0, \quad \big(T_I^{-1}(\mu)\big)^* \cdot \mu = 0. \tag{2.4}$$

We omit furthermore information about the generation functions χ and μ, which can be founded in [9,13,15], and turn to the representation of the gauge transformation operator in the following.

We would like to stress that the following discussion in this section only depends on the form of the guage transformation operators, which does not depend on any concrete integrable hierarchy. Take two sets of functions $\{f_i^{(0)}, i = 1, 2, \cdots, n; f^{(0)}\}$ and $\{g_i^{(0)}, i = 1, 2, \cdots, n; g^{(0)}\}$ as the generation function of the gauge transformation operators, and suppose they satisfy following rule in the gauge transformation:

(1) The first step

$$T_D^{(1)} = T_D^{(1)}(f_1^{(0)}) = f_1^{(0)} \circ \partial \circ f_1^{(0)^{-1}}, \tag{2.5}$$

we define the rule of transformation under $T_D^{(1)}$ as

$$f^{(1)} = T_D^{(1)}(f_1^{(0)}) \cdot f^{(0)}, \quad g^{(1)} = \big(T_D^{(1)}(f_1^{(0)})\big)^{*^{-1}} \cdot g^{(0)} = -T_I(f_1^{(0)}) \cdot g^{(0)}, \tag{2.6}$$

$$f_i^{(1)} = T_D^{(1)}(f_1^{(0)}) \cdot f_i^{(0)}, \quad g_i^{(1)} = \big(T_D^{(1)}(f_1^{(0)})\big)^{*^{-1}} \cdot g_i^{(0)} = -T_I(f_1^{(0)}) \cdot g_i^{(0)}, \tag{2.7}$$

if $i \geq 2$ for $f_i^{(1)}$.

(2) The second step

$$T_D^{(2)} = T_D^{(2)}(f_2^{(1)}) = f_2^{(1)} \circ \partial \circ f_2^{(1)^{-1}}, \tag{2.8}$$

we define the rule of transformation under $T_D^{(2)}$ as

$$f^{(2)} = T_D^{(2)}(f_2^{(1)}) \cdot f^{(1)}, \quad g^{(2)} = \big(T_D^{(2)}(f_2^{(1)})\big)^{*^{-1}} \cdot g^{(1)} = -T_I(f_2^{(1)}) \cdot g^{(1)}, \tag{2.9}$$

$$f_i^{(2)} = T_D^{(2)}(f_2^{(1)}) \cdot f_i^{(1)}, \quad g_i^{(2)} = \big(T_D^{(2)}(f_2^{(1)})\big)^{*^{-1}} \cdot g_i^{(1)} = -T_I(f_2^{(1)}) \cdot g_i^{(1)}, \tag{2.10}$$

if $i \geq 3$ for $f_i^{(2)}$, under the gauge transformation of the Type I. For Type II, we may suppose the transformed rule:

(1) The first step

$$T_I^{(1)} = T_I^{(1)}(g_1^{(0)}) = g_1^{(0)^{-1}} \circ \partial^{-1} \circ g_1^{(0)}, \tag{2.11}$$

$$f^{(1)} = T_I^{(1)}(g_1^{(0)}) \cdot f^{(0)}, \quad g^{(1)} = \left(T_I^{(1)}(g_1^{(0)})\right)^{*^{-1}} \cdot g^{(0)} = -T_D(g_1^{(0)}) \cdot g^{(0)}, \tag{2.12}$$

$$f_i^{(1)} = T_I^{(1)}(g_1^{(0)}) \cdot f^{(0)}, \quad g_i^{(1)} = \left(T_I^{(1)}(g_1^{(0)})\right)^{*^{-1}} \cdot g^{(0)} = -T_D(g_1^{(0)}) \cdot g_i^{(0)} \tag{2.13}$$

for $i \geq 2$ in $g_i^{(1)}$;

(2) the second step

$$T_I^{(2)} = T_I^{(2)}(g_2^{(1)}) = g_2^{(1)^{-1}} \circ \partial^{-1} \circ g_2^{(1)}, \tag{2.14}$$

$$f^{(2)} = T_I^{(2)}(g_2^{(1)}) \cdot f^{(1)}, \quad g^{(1)} = \left(T_I^{(1)}(g_2^{(1)})\right)^{*^{-1}} \cdot g^{(1)} = -T_D(g_2^{(1)}) \cdot g^{(1)}, \tag{2.15}$$

$$f_i^{(2)} = T_I^{(2)}(g_2^{(1)}) \cdot f_i^{(1)}, \quad g_i^{(2)} = \left(T_I^{(2)}(g_2^{(1)})\right)^{*^{-1}} \cdot g_i^{(1)} = -T_D(g_2^{(1)}) \cdot g_i^{(1)} \tag{2.16}$$

for $i \geq 3$ in $g_i^{(1)}$. Obviously the gauge transformation operators can be successively applied according to the rule in (2.5)–(2.16). For the n-step gauge transformation , as shown in [13,15],

$$T_n = T_D^{(n)}(f_n^{(n-1)}) \circ T_D^{(n-1)}(f_{n-1}^{(n-2)}) \cdots \circ T_D^{(3)}(f_3^{(2)}) \circ T_D^{(2)}(f_2^{(1)}) \circ T_D^{(1)}(f_1^{(0)}),$$

it is easy to get

$$f^{(n)} \equiv T_n \cdot f = \frac{W_{n+1}(f_1^{(0)}, f_2^{(0)}, \cdots, f_n^{(0)}, f)}{W_n(f_1^{(0)}, f_2^{(0)}, f_n^{(0)})}$$

based on the Wronskian's properties–Jacobi expansion theorem and Crum identity[13,15,18]. Here $W_n(f_1^{(0)}, f_1^{(0)}, \cdots, f_n^{(0)})$ is the Wronskian determinant. But the formula of $g^{(n)} = \left(T_n^{-1}\right)^* \cdot g^{(0)}$ can not be obtained in the the same way as [13,15]. Also, for the $(n+k)$-step gauge transformation

$$\begin{aligned} T_{n+k} = {} & T_I^{(n+k)}(g_k^{(n+k-1)}) \circ T_I^{(n+k-1)}(g_{k-1}^{(n+k-2)}) \circ \cdots \circ T_I^{(n+1)}(g_1^{(n)}) \\ & \circ T_D^{(n-1)}(f_n^{(n-1)}) \circ \cdots \circ T_D^{(2)}(f_2^{(1)}) \circ T_D^{(1)}(f_1^{(0)}), \end{aligned}$$

it is more difficult to find the final form of the

$$f^{(n+k)} = T_{n+k} \cdot f^{(0)},$$

because it will encounter the generalized Wronskian determinant. However, such two questions will be solved obviously once the determinant representation of the T_{n+k} is established.

Now let us discuss our main result. First of all, for simplicity in the following Theorem

2.1, we call

$$IW_{k,n} \equiv IW_{k,n}(g_k^{(0)}, g_{k-1}^{(0)}, \cdots, g_1^{(0)}; f_1^{(0)}, f_2^{(0)}, \cdots, f_n^{(0)})$$
$$= \begin{vmatrix} \int g_k^{(0)} \cdot f_1^{(0)} & \int g_k^{(0)} \cdot f_2^{(0)} & \int g_k^{(0)} \cdot f_3^{(0)} & \cdots & \int g_k^{(0)} \cdot f_n^{(0)} \\ \int g_{k-1}^{(0)} \cdot f_1^{(0)} & \int g_{k-1}^{(0)} \cdot f_2^{(0)} & \int g_{k-1}^{(0)} \cdot f_3^{(0)} & \cdots & \int g_{k-1}^{(0)} \cdot f_n^{(0)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \int g_1^{(0)} \cdot f_1^{(0)} & \int g_1^{(0)} \cdot f_2^{(0)} & \int g_1^{(0)} \cdot f_3^{(0)} & \cdots & \int g_1^{(0)} \cdot f_n^{(0)} \\ f_1^{(0)} & f_2^{(0)} & f_3^{(0)} & \cdots & f_n^{(0)} \\ f_{1,x}^{(0)} & f_{2,x}^{(0)} & f_{3,x}^{(0)} & \cdots & f_{n,x}^{(0)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ (f_1^{(0)})^{(n-k-1)} & (f_2^{(0)})^{(n-k-1)} & (f_3^{(0)})^{(n-k-1)} & \cdots & (f_n^{(0)})^{(n-k-1)} \end{vmatrix} \tag{2.17}$$

the generalized Wronskian determinant. In particular,

$$IW_{0,n} = W_n\big(f_1^{(0)}, f_2^{(0)}, \cdots, f_n^{(0)}\big),$$
$$(f_i^{(0)})^{(k)} = \frac{\partial^k f_i^{(0)}}{\partial x^k},$$

the notation $\int f = \int f dx$, in which the integration constant equals zero. Additionally, for the following T_{n+k}, an expansion with respect to the last column is understood, in which all sub-determinants are collected on the left of the symbols $\partial^i (i = -1, 0, 1, 2, \cdots, n-k)$; for the T_{n+k}^{-1} an expansion with respect to the first column is understood (collecting all minors on the right of $f_i^{(0)} \circ \partial^{-1}$)

Theorem 2.1. *For the $n > k$,*

$$T_{n+k} = T_I^{(n+k)}(g_k^{(n+k-1)}) \circ T_I^{(n+k-1)}(g_{k-1}^{(n+k-2)}) \cdots T_I^{(n+1)}(g_1^{(n)})$$
$$\circ T_D^{(n)}(f_n^{(n-1)}) \circ T_D^{(n-1)}(f_{n-1}^{(n-2)}) \cdots T_D^{(1)}(f_1^{(0)})$$
$$= \frac{1}{IW_{k,n}(g_k^{(0)}, g_{k-1}^{(0)}, \cdots, g_1^{(0)}; f_1^{(0)}, f_2^{(0)}, \cdots, f_n^{(0)})}$$
$$\cdot \begin{vmatrix} \int f_1^{(0)} \cdot g_k^{(0)} & \int f_2^{(0)} \cdot g_k^{(0)} & \cdots & \int f_n^{(0)} \cdot g_k^{(0)} & \partial^{-1} \circ g_k^{(0)} \\ \int f_1^{(0)} \cdot g_{k-1}^{(0)} & \int f_2^{(0)} \cdot g_{k-1}^{(0)} & \cdots & \int f_n^{(0)} \cdot g_{k-1}^{(0)} & \partial^{-1} \circ g_{k-1}^{(0)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \int f_1^{(0)} \cdot g_1^{(0)} & \int f_2^{(0)} \cdot g_1^{(0)} & \cdots & \int f_n^{(0)} \cdot g_1^{(0)} & \partial^{-1} \circ g_1^{(0)} \\ f_1^{(0)} & f_2^{(0)} & \cdots & f_n^{(0)} & 1 \\ f_{1,x}^{(0)} & f_{2,x}^{(0)} & \cdots & f_{n,x}^{(0)} & \partial \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ (f_1^{(0)})^{(n-k)} & (f_2^{(0)})^{(n-k)} & \cdots & (f_n^{(0)})^{(n-k)} & \partial^{n-k} \end{vmatrix}, \tag{2.18}$$

and

$$T_{n+k}^{-1}$$
$$=\begin{vmatrix} f_1^{(0)}\circ\partial^{-1} & \int g_k^{(0)}\cdot f_1^{(0)} & \cdots & \int g_1^{(0)}\cdot f_1^{(0)} & f_1^{(0)} & f_{1,x}^{(0)} & \cdots & (f_1^{(0)})^{(n-k-2)} \\ f_2^{(0)}\circ\partial^{-1} & \int g_k^{(0)}\cdot f_2^{(0)} & \cdots & \int g_1^{(0)}\cdot f_2^{(0)} & f_2^{(0)} & f_{2,x}^{(0)} & \cdots & (f_2^{(0)})^{(n-k-2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{n-1}^{(0)}\circ\partial^{-1} & \int g_k^{(0)}\cdot f_{n-1}^{(0)} & \cdots & \int g_1^{(0)}\cdot f_{n-1}^{(0)} & f_{n-1}^{(0)} & f_{n-1,x}^{(0)} & \cdots & (f_{n-1}^{(0)})^{(n-k-2)} \\ f_n^{(0)}\circ\partial^{-1} & \int g_k^{(0)}\cdot f_n^{(0)} & \cdots & \int g_1^{(0)}\cdot f_n^{(0)} & f_n^{(0)} & f_{n,x}^{(0)} & \cdots & (f_n^{(0)})^{(n-k-2)} \end{vmatrix}$$
$$\cdot\frac{(-1)^{n-1}}{IW_{k,n}(g_k^{(0)},g_{k-1}^{(0)},\cdots,g_1^{(0)};f_1^{(0)},f_2^{(0)},\cdots,f_n^{(0)})}. \tag{2.19}$$

Proof. The process of the recursive construction of T_{n+k} shows that T_{n+k} is an $(n+k)$-th pseudo-differential operator with a normalized leading coefficient, which annihilates all generation functions $f_1^{(0)},f_2^{(0)},\cdots,f_n^{(0)}$, and its conjugate operator $(T_{n+k}^{-1})^*$ annihilates $g_1^{(0)},g_2^{(0)},\cdots,g_k^{(0)}$ from (2.4). We may assume

$$T_{n+k}=\sum_{p=-k}^{-1}a_p\circ\partial^{-1}\circ g_{|p|}^{(0)}+\sum_{p=0}^{n-k}a_p\circ\partial^p, \tag{2.20}$$

with $a_{n-k}=1$, and

$$T_{n+k}^{-1}=\sum_{j=1}^{n}f_j^{(0)}\circ\partial^{-1}\circ b_j. \tag{2.21}$$

It follows that $T_{n+k}\cdot f_i^{(0)}=0$ $(i=1,2,\cdots,n)$ and $\left(T_{n+k}^{-1}\right)^*\cdot g_i^{(0)}=0$; explicitly,

$$\begin{aligned} & a_{-k}\int g_k^{(0)}f_1^{(0)}+\cdots+a_{-1}\int g_1^{(0)}f_1^{(0)}+a_0f_1^{(0)}+a_1f_{1,x}^{(0)}+\cdots+a_{n-k-1}(f_1^{(0)})^{(n-k-1)} \\ &=-(f_1^{(0)})^{(n-k)}, \\ & a_{-k}\int g_k^{(0)}f_2^{(0)}+\cdots+a_{-1}\int g_1^{(0)}f_2^{(0)}+a_0f_1^{(0)}+a_1f_{2,x}^{(0)}+\cdots+a_{n-k-1}(f_2^{(0)})^{(n-k-1)} \\ &=-(f_2^{(0)})^{(n-k)}, \\ &\qquad\qquad\vdots \\ & a_{-k}\int g_k^{(0)}f_n^{(0)}+\cdots+a_{-1}\int g_1^{(0)}f_n^{(0)}+a_0f_1^{(0)}+a_1f_{n,x}^{(0)}+\cdots+a_{n-k-1}(f_n^{(0)})^{(n-k-1)} \\ &=-(f_n^{(0)})^{(n-k)}, \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} b_1\cdot\Big(\int g_k^{(0)}\cdot f_1^{(0)}\Big)+b_2\cdot\Big(\int g_k^{(0)}\cdot f_2^{(0)}\Big)+\cdots+b_n\cdot\Big(\int g_k^{(0)}\cdot f_n^{(0)}\Big)&=0, \\ b_1\cdot\Big(\int g_{k-1}^{(0)}\cdot f_1^{(0)}\Big)+b_2\cdot\Big(\int g_{k-1}^{(0)}\cdot f_2^{(0)}\Big)+\cdots+b_n\cdot\Big(\int g_{k-1}^{(0)}\cdot f_n^{(0)}\Big)&=0, \\ \vdots\qquad\qquad & \\ b_1\cdot\Big(\int g_1^{(0)}\cdot f_1^{(0)}\Big)+b_2\cdot\Big(\int g_1^{(0)}\cdot f_2^{(0)}\Big)+\cdots+b_n\cdot\Big(\int g_1^{(0)}\cdot f_n^{(0)}\Big)&=0. \end{aligned} \tag{2.23}$$

Equations (2.22) can be solved via Cramer's rule and result in

$$a_{-i} = \frac{(-1)^{n+1+k-i+1}}{IW_{k,n}(g_k^{(0)},\cdots,g_1^{(0)};f_1^{(0)},f_2^{(0)},\cdots,f_n^{(0)})} \cdot \begin{vmatrix} \int f_1^{(0)}\cdot g_k^{(0)} & \int f_2^{(0)}\cdot g_k^{(0)} & \int f_3^{(0)}\cdot g_k^{(0)} & \cdots & \int f_n^{(0)}\cdot g_k^{(0)} \\ \int f_1^{(0)}\cdot g_{k-1}^{(0)} & \int f_2^{(0)}\cdot g_{k-1}^{(0)} & \int f_3^{(0)}\cdot g_{k-1}^{(0)} & \cdots & \int f_n^{(0)}\cdot g_{k-1}^{(0)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \hat{i} & \hat{i} & \hat{i} & \cdots & \hat{i} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \int f_1^{(0)}\cdot g_1^{(0)} & \int f_2^{(0)}\cdot g_1^{(0)} & \int f_3^{(0)}\cdot g_1^{(0)} & \cdots & \int f_n^{(0)}\cdot g_1^{(0)} \\ f_1^{(0)} & f_2^{(0)} & f_3^{(0)} & \cdots & f_n^{(0)} \\ f_{1,x}^{(0)} & f_{2,x}^{(0)} & f_{3,x}^{(0)} & \cdots & f_{n,x}^{(0)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ (f_1^{(0)})^{(n-k)} & (f_2^{(0)})^{(n-k)} & (f_3^{(0)})^{(n-k)} & \cdots & (f_n^{(0)})^{(n-k)} \end{vmatrix}, \tag{2.24}$$

where $i = 1,\ 2,\ \cdots,k, \hat{i}$ represents that the row containing g_i is deleted;

$$a_{i} = \frac{(-1)^{n+1+i+k+1}}{IW_{k,n}(g_k^{(0)},\cdots,g_1^{(0)};f_1^{(0)},f_2^{(0)},\cdots,f_n^{(0)})} \cdot \begin{vmatrix} \int f_1^{(0)}\cdot g_k^{(0)} & \int f_2^{(0)}\cdot g_k^{(0)} & \int f_3^{(0)}\cdot g_k^{(0)} & \cdots & \int f_n^{(0)}\cdot g_k^{(0)} \\ \int f_1^{(0)}\cdot g_{k-1}^{(0)} & \int f_2^{(0)}\cdot g_{k-1}^{(0)} & \int f_3^{(0)}\cdot g_{k-1}^{(0)} & \cdots & \int f_n^{(0)}\cdot g_{k-1}^{(0)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \int f_1^{(0)}\cdot g_1^{(0)} & \int f_2^{(0)}\cdot g_1^{(0)} & \int f_3^{(0)}\cdot g_1^{(0)} & \cdots & \int f_n^{(0)}\cdot g_1^{(0)} \\ f_1^{(0)} & f_2^{(0)} & f_3^{(0)} & \cdots & f_n^{(0)} \\ f_{1,x}^{(0)} & f_{2,x}^{(0)} & f_{3,x}^{(0)} & \cdots & f_{n,x}^{(0)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \hat{i} & \hat{i} & \hat{i} & \cdots & \hat{i} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ (f_1^{(0)})^{(n-k)} & (f_2^{(0)})^{(n-k)} & (f_3^{(0)})^{(n-k)} & \cdots & (f_n^{(0)})^{(n-k)} \end{vmatrix}, \tag{2.25}$$

in which $i = 0,1,2,\cdots,n-k-1$, and $\hat{i}$ indicates that the ith derivative is omitted. Hence, taking (2.24) and (2.25) back into (2.20) leads to (2.18)

Let us turn to the determination of the coefficients in T_{n+k}^{-1}. The T_{n+k}^{-1} is the inverse operator of T_{n+k}, so it must satisfy the following relation

$$\left(T_{n+k}\circ T_{n+k}^{-1}\right)_- = 0. \tag{2.26}$$

Here for a pseudo-differential operator

$$A = \sum a_i\partial^i, \quad A_+ = \sum_{i\geq 0} a_i\partial^i,$$
$$A_- = A_{<0} = \sum_{i<0} a_i\partial^i.$$

By (2.20) and (2.21), we have

$$\Big(T_{n+k}\circ T_{n+k}^{-1}\Big)_{<0} = \Big((\sum_{p=0}^{n-k} a_p\circ\partial^p + a_{-k}\circ\partial^{-1}\circ g_k^{(0)} + \cdots + a_{-1}\circ\partial^{-1}\circ g_1^{(0)}) \circ (\sum_{j=1}^{n} f_j^{(0)}\circ\partial^{-1}\circ b_j)\Big)_{<0}$$
$$= \sum_{p=0}^{n-k}\sum_{j=1}^{n} a_p\circ(f_j^{(0)})^{(p)}\circ\partial^{-1}\circ b_j + \sum_{j=1}^{n} a_{-k}\circ\partial^{-1}\circ(f_j^{(0)}\cdot g_k^{(0)})\circ\partial^{-1}\circ b_j$$
$$+ \sum_{j=1}^{n} a_{-(k-1)}\circ\partial^{-1}\circ(f_j^{(0)}\cdot g_{k-1}^{(0)})\circ\partial^{-1}\circ b_j$$
$$\vdots$$
$$+ \sum_{j=1}^{n} a_{-1}\circ\partial^{-1}\circ(f_j^{(0)}\cdot g_1^{(0)})\circ\partial^{-1}\circ b_j. \tag{2.27}$$

Using $\partial^{-1}\circ f\circ\partial^{-1} = (\int f)\circ\partial^{-1} - \partial^{-1}\circ(\int f)$, the above equation can be written as

$$\Big(T_{n+k}\circ T_{n+k}^{-1}\Big)_{<0}$$
$$= \sum_{j=1}^{n}\sum_{p=0}^{n-k} a_p\circ(f_j^{(0)})^{(p)}\circ\partial^{-1}\circ b_j$$
$$+ \sum_{j=1}^{n} a_{-k}\Big(\int g_k^{(0)}\cdot f_j^{(0)}\Big)\circ\partial^{-1}\circ b_j - \sum_{j=1}^{n} a_{-k}\circ\partial^{-1}\circ\Big(\int g_k^{(0)}\cdot f_j^{(0)}\Big)\circ b_j$$
$$+ \sum_{j=1}^{n} a_{-(k-1)}\Big(\int g_{k-1}^{(0)}\cdot f_j^{(0)}\Big)\circ\partial^{-1}\circ b_j - \sum_{j=1}^{n} a_{-(k-1)}\circ\partial^{-1}\circ\Big(\int g_{k-1}^{(0)}\cdot f_j^{(0)}\Big)\circ b_j$$
$$\vdots$$
$$+ \sum_{j=1}^{n} a_{-1}\Big(\int g_1^{(0)}\cdot f_j^{(0)}\Big)\circ\partial^{-1}\circ b_j - \sum_{j=1}^{n} a_{-1}\circ\partial^{-1}\circ\Big(\int g_1^{(0)}\cdot f_j^{(0)}\Big)\circ b_j. \tag{2.28}$$

It is easy to see that the right hand side of (2.28) equals zero, because of (2.22) and (2.23). On the other hand, according to the identity

$$f\circ\partial^{-1}\circ g = \sum_{i\geq 0}\partial^{-1-i}\circ f^{(i)}\circ g$$

with $f^{(i)} = \dfrac{\partial^i f}{\partial x^i}$, the T_{n+k}^{-1} can be rewritten as

$$T_{n+k}^{-1} = \sum_{n-k-2\geq i\geq 0}\sum_{j=1}^{n}\partial^{-1-i}\circ(f_j^{(0)})^{(i)}\circ b_j + \partial^{-n+k}\sum_{j=1}^{n}(f_j^{(0)})^{(n-k-1)}\circ b_j$$
$$+ \sum_{i\geq n-k}\sum_{j=1}^{n}\partial^{-1-i}\circ(f_j^{(0)})^{(i)}\circ b_j. \tag{2.29}$$

However, in order to satisfy $T_{n+k}\circ T_{n+k}^{-1}=1$, T_{n+k}^{-1} should be taken the form as

$$T_{n+k}^{-1}=\partial^{-n+k}+\text{the lower order terms.} \tag{2.30}$$

So, comparing the (2.29) and (2.30) gives the following system of equations,

$$\begin{aligned}
&f_1^{(0)}b_1+f_2^{(0)}b_2+f_3^{(0)}b_3+\cdots+f_n^{(0)}b_n=0,\\
&f_{1,x}^{(0)}b_1+f_{2,x}^{(0)}b_2+f_{3,x}^{(0)}b_3+\cdots+f_{n,x}^{(0)}b_n=0,\\
&\qquad\qquad\vdots\\
&(f_1^{(0)})^{(n-k-2)}b_1+(f_2^{(0)})^{(n-k-2)}b_2+(f_3^{(0)})^{(n-k-2)}b_3+\cdots+(f_n^{(0)})^{(n-k-2)}b_n=0,\\
&(f_1^{(0)})^{(n-k-1)}b_1+(f_2^{(0)})^{(n-k-1)}b_2+(f_3^{(0)})^{(n-k-1)}b_3+\cdots+(f_n^{(0)})^{(n-k-1)}b_n=1.
\end{aligned} \tag{2.31}$$

From (2.23) and (2.31), the b_i $(i=1,2,\cdots,n)$ is obtained as follows:

$$b_i=\frac{(-1)^{n+i}}{IW_{k,n}}\cdot\begin{vmatrix}
\int f_1^{(0)}\cdot g_k^{(0)} & \int f_2^{(0)}\cdot g_k^{(0)} & \cdots & \hat{i} & \cdots & \int f_n^{(0)}\cdot g_k^{(0)}\\
\int f_1^{(0)}\cdot g_{k-1}^{(0)} & \int f_2^{(0)}\cdot g_{k-1}^{(0)} & \cdots & \hat{i} & \cdots & \int f_n^{(0)}\cdot g_{k-1}^{(0)}\\
\vdots & \vdots & \cdots & \hat{i} & \cdots & \vdots\\
f_1^{(0)} & f_2^{(0)} & \cdots & \hat{i} & \cdots & f_n^{(0)}\\
f_{1,x}^{(0)} & f_{2,x}^{(0)} & \cdots & \hat{i} & \cdots & f_{n,x}^{(0)}\\
\vdots & \vdots & \cdots & \hat{i} & \cdots & \vdots\\
(f_1^{(0)})^{(n-k-2)} & (f_2^{(0)})^{(n-k-2)} & \cdots & \hat{i} & \cdots & (f_n^{(0)})^{(n-k-2)}
\end{vmatrix}, \tag{2.32}$$

where $i=1,2,\cdots,n$. The symbol $\hat{i}$ denotes the column with $f_i^{(0)}$ is deleted. The determinant representation of T_{n+k}^{-1} is obtained by taking b_i back into (2.21).

Under the case of $n=k$, we need some modification of the formula about the T_{n+k}^{-1}. We may assume

$$T_{n+k}=1+\sum_{p=-1}^{-k}a_p\circ\partial^{-1}\circ g_{|p|}^{(0)}, \tag{2.33}$$

which is consistent with (2.20), but T_{n+k}^{-1} must be deformed as

$$T_{n+k}^{-1}=1+\sum_{j=1}^{n}f_j^{(0)}\circ\partial^{-1}\circ b_j. \tag{2.34}$$

By similar argument about Theorem 2.1, we can show

Theorem 2.2. *Under the case of $n = k$, the T_{n+k} is also given by* (2.18), *i.e.,*

$$\begin{aligned}T_{n+k} &= T_I^{(n+k)}(g_k^{(n+k-1)}) \circ T_I^{(n+k-1)}(g_{k-1}^{(n+k-2)}) \cdots T_I^{(n+1)}(g_1^{(n)}) \\ &\quad \circ T_D^{(n)}(f_n^{(n-1)}) \circ T_D^{(n-1)}(f_{n-1}^{(n-2)}) \cdots T_D^{(1)}(f_1^{(0)}) \\ &= \frac{1}{IW_{k,n}(g_k^{(0)}, g_{k-1}^{(0)}, \cdots, g_1^{(0)}; f_1^{(0)}, f_2^{(0)}, \cdots, f_n^{(0)})} \\ &\quad \cdot \begin{vmatrix} \int f_1^{(0)} \cdot g_k^{(0)} & \int f_2^{(0)} \cdot g_k^{(0)} & \cdots & \int f_n^{(0)} \cdot g_k^{(0)} & \partial^{-1} \circ g_k^{(0)} \\ \int f_1^{(0)} \cdot g_{k-1}^{(0)} & \int f_2^{(0)} \cdot g_{k-1}^{(0)} & \cdots & \int f_n^{(0)} \cdot g_{k-1}^{(0)} & \partial^{-1} \circ g_{k-1}^{(0)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \int f_1^{(0)} \cdot g_1^{(0)} & \int f_2^{(0)} \cdot g_1^{(0)} & \cdots & \int f_n^{(0)} \cdot g_1^{(0)} & \partial^{-1} \circ g_1^{(0)} \\ f_1^{(0)} & f_2^{(0)} & \cdots & f_n^{(0)} & \partial^0 \end{vmatrix},\end{aligned} \tag{2.35}$$

but the T_{n+k}^{-1} becomes

$$T_{n+k}^{-1} = - \begin{vmatrix} -\partial^0 & g_k^{(0)} & g_{k-1}^{(0)} & \cdots & g_1^{(0)} \\ f_1^{(0)} \circ \partial^{-1} & \int g_k^{(0)} f_1^{(0)} & \int g_{k-1}^{(0)} f_1^{(0)} & \cdots & \int g_1^{(0)} f_1^{(0)} \\ f_2^{(0)} \circ \partial^{-1} & \int g_k^{(0)} f_2^{(0)} & \int g_{k-1}^{(0)} f_2^{(0)} & \cdots & \int g_1^{(0)} f_2^{(0)} \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ f_n^{(0)} \circ \partial^{-1} & \int g_k^{(0)} f_n^{(0)} & \int g_{k-1}^{(0)} f_n^{(0)} & \cdots & \int g_1^{(0)} f_n^{(0)} \end{vmatrix} \frac{1}{IW_{k,n}}. \tag{2.36}$$

Proof. The proof is omitted because it is similar to Theorem 2.1.

Particularly, the sufficient condition about the existence of the determinant representation of the gauge transformation operator is $IW_{k,n} \neq 0$.

We would like to point out that the T_{n+k} and T_{n+k}^{-1} in Theorem 2.1 and Theorem 2.2 reduces to the determinant expression of T_n and $(T_n^{-1})^*$ in [11] under the case of $k = 0$.

§3. Example

In this section, in order to show the usage of the determinant expression of the T_{n+k} and T_{n+k}^{-1}, we will apply it to the KP hierarchy. This well-known integrable hierarchy can be described as[9]

$$\frac{\partial B_m}{\partial x_n} - \frac{\partial B_n}{\partial x_m} + [B_m, B_n] = 0 \quad (m, n = 2, 3, \cdots,), \tag{3.1}$$

where $B_n = (L^n)_+$ denotes the part of the differential operators, and

$$L = \partial + u_2 \partial^{-1} + u_3 \partial^{-2} + \cdots. \tag{3.2}$$

An important character for the KP hierarchy is that there exists a sigle function $\tau(x)$, the so-called tau function, such that

$$u_2 = \frac{\partial^2}{\partial x_1^2} \log \tau, \quad u_3 = \frac{1}{2}\Big[\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{\partial^3}{\partial x_1^3}\Big] \log \tau. \tag{3.3}$$

Thus the central task is to establish the final form of the tau function after the $n + k$ step gauge transformations. Following [9], we need to introduce linear system

$$\frac{\partial}{\partial x_n} \phi(x_1, x_2, \cdots,) = B_n \cdot \phi(x_1, x_2, \cdots,) \tag{3.4}$$

and its conjugation systems

$$\frac{\partial}{\partial x_n} \psi(x_1, x_2, \cdots,) = -B_n^* \cdot \psi(x_1, x_2, \cdots,) \tag{3.5}$$

to construct two kinds of the gauge transformation , i.e.,

$$T_D = \phi \circ \partial \circ \phi^{-1}, \tag{3.6}$$

$$T_I = \psi^{-1} \circ \partial^{-1} \circ \psi, \tag{3.7}$$

as mentioned in (2.1) and (2.2).

Let us discuss the transformed result for inintial KP hierarchy $B_n^{(0)}$ with initial wave functions $\{\phi^{(0)}, \phi_i^{(0)}, i = 1, 2, \cdots, n\}$ and $\{\psi^{(0)}, \psi_i^{(0)}\}$. We will use them to generate gauge transformation and its repeated iteration. Considering the results in [9] and previous (2.18) and (2.19), the following Lemma 3.1 and Theorem 3.1 can be reduced easily, so we omit their proof. For the n-step gauge transformation chain,

$$B_n^{(0)} \overset{T_D^{(1)}(\phi_1^{(0)})}{\rightarrow} B_n^{(1)} \overset{T_D^{(2)}(\phi_2^{(1)})}{\rightarrow} B_n^{(2)} \overset{T_D^{(3)}(\phi_3^{(2)})}{\rightarrow} B_n^{(3)} \overset{T_D^{(4)}(\phi_4^{(3)})}{\rightarrow} \cdots \rightarrow \cdots B_n^{(n-1)} \overset{T_D^{(n)}(\phi_n^{(n-1)})}{\rightarrow} B_n^{(n)},$$

we have

Lemma 3.1.

$$T_n = T_D^{(n)}(\phi_n^{(n-1)}) \circ T_D^{(n-1)}(\phi_{n-1}^{(n-2)}) \cdots \circ T_D^{(2)}(\phi_2^{(1)}) \circ T_D^{(1)}(\phi_1^{(0)})$$

$$= \frac{1}{W_n} \cdot \begin{vmatrix} \phi_1^{(0)} & \phi_2^{(0)} & \phi_3^{(0)} & \cdots & \phi_n^{(0)} & 1 \\ \phi_{1,x}^{(0)} & \phi_{2,x}^{(0)} & \phi_{3,x}^{(0)} & \cdots & \phi_{n,x}^{(0)} & \partial \\ \phi_{1,xx}^{(0)} & \phi_{2,xx}^{(0)} & \phi_{3,xx}^{(0)} & \cdots & \phi_{n,xx}^{(0)} & \partial^2 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ (\phi_1^{(0)})^{(n)} & (\phi_2^{(0)})^{(n)} & (\phi_3^{(0)})^{(n)} & \cdots & (\phi_n^{(0)})^{(n)} & \partial^n \end{vmatrix}, \tag{3.8}$$

and

$$\tau^{(n)} = W_n\big(\phi_1^{(0)}, \phi_2^{(0)}, \cdots, \phi_n^{(0)}\big) \cdot \tau^{(0)}$$

in which $i = 1, 2, \cdots, n$.

For the $(n+k)$-step gauge transformation, $n > k$,

$$B_n^{(0)} \overset{T_D^{(1)}(\phi_1^{(0)})}{\rightarrow} B_n^{(1)} \overset{T_D^{(2)}(\phi_2^{(1)})}{\rightarrow} B_n^{(2)} \overset{T_D^{(3)}(\phi_3^{(2)})}{\rightarrow} B_n^{(3)} \cdots \rightarrow B_n^{(n-1)} \overset{T_D^{(n)}(\phi_n^{(n-1)})}{\rightarrow} B_n^{(n)}$$

$$\overset{T_I^{(n+1)}(\psi_1^{(n)})}{\rightarrow} B_n^{(n+1)} \overset{T_I^{(n+2)}(\psi_2^{(n+1)})}{\rightarrow} B_n^{(n+2)} \cdots \rightarrow B_n^{(n+k-1)} \overset{T_I^{(n+k)}(\psi_k^{(n+k-1)})}{\rightarrow} B_n^{(n+k)},$$

we have

Theorem 3.1.

$$T_{n+k} = T_I^{(n+k)}(\psi_k^{(n+k-1)}) \circ T_I^{(n+k-1)}(\psi_{k-1}^{(n+k-2)}) \cdots T_I^{(n+1)}(\psi_1^{(n)})$$
$$\circ T_D^{(n)}(\phi_n^{(n-1)}) \circ T_D^{(n-1)}(\phi_{n-1}^{(n-2)}) \cdots T_D^{(1)}(\phi_1^{(0)})$$
$$= \frac{1}{IW_{k,n}(\psi_k^{(0)}, \psi_{k-1}^{(0)}, \cdots, \psi_1^{(0)}; \phi_1^{(0)}, \phi_2^{(0)}, \cdots, \phi_n^{(0)})}$$
$$\cdot \begin{vmatrix} \int \phi_1^{(0)} \cdot \psi_k^{(0)} & \int \phi_2^{(0)} \cdot \psi_k^{(0)} & \cdots & \int \phi_n^{(0)} \cdot \psi_k^{(0)} & \partial^{-1} \circ \psi_k^{(0)} \\ \int \phi_1^{(0)} \cdot \psi_{k-1}^{(0)} & \int \phi_2^{(0)} \cdot \psi_{k-1}^{(0)} & \cdots & \int \phi_n^{(0)} \cdot \psi_{k-1}^{(0)} & \partial^{-1} \circ \psi_{k-1}^{(0)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \int \phi_1^{(0)} \cdot \psi_1^{(0)} & \int \phi_2^{(0)} \cdot \psi_1^{(0)} & \cdots & \int \phi_n^{(0)} \cdot \psi_1^{(0)} & \partial^{-1} \circ \psi_1^{(0)} \\ \phi_1^{(0)} & \phi_2^{(0)} & \cdots & \phi_n^{(0)} & 1 \\ \phi_{1,x}^{(0)} & \phi_{2,x}^{(0)} & \cdots & \phi_{n,x}^{(0)} & \partial \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ (\phi_1^{(0)})^{(n-k)} & (\phi_2^{(0)})^{(n-k)} & \cdots & (\phi_n^{(0)})^{(n-k)} & \partial^{n-k} \end{vmatrix}, \tag{3.10}$$

$$\tau^{(n+k)} = \psi_k^{(n+k-1)} \cdot \psi_{k-1}^{(n+k-2)} \cdots \psi_1^{(n)} \cdot \tau^{(n)} = IW_{k,n}(\psi_k^{(0)}, \psi_{k-1}^{(0)}, \cdots, \psi_1^{(0)}; \phi_1^{(0)}, \phi_2^{(0)}, \cdots, \phi_n^{(0)}) \cdot \tau^{(0)}. \tag{3.11}$$

The above Lemma 3.1 and Theorem 3.1 show that the determinant representation of gauge transformation is convenient for us to get the transformed wave function and the conjugation wave function. This fact is very important for us to solve the components of cKP hierarchy, which will be discussed in future. Moreover, one can see that it is still necessary to present the above discussion, although $\tau^{(n+k)}$ is given in [9] without proof.

As we stated in the introduction, the Jacobi expansion theorem and the Crum identity (see [13, 15, 18]) are important properties when we consider the repeated iteration of the gauge transformations. By using the determinant expression of the gauge transformation operator, we also provide alternative proof of the Jacobi expansion theorem of the Wronskian. Comparing $\phi^{(n)} = T_n \cdot \phi^{(0)}$ with $\phi^{(n)} = T_D^{(n)}(\phi_n^{(n-1)})\phi^{(n-1)} = \phi_n^{(n-1)} \cdot \left(\frac{\phi^{(n-1)}}{\phi_n^{(n-1)}}\right)_x$, we re-obtain the well-known

Corollary 3.1 (Jacobi Expansion Theorem).[13,15,18]

$$\left(\frac{W_n\big(\phi_1^{(0)}, \phi_2^{(0)}, \cdots, \phi_{n-1}^{(0)}, \phi^{(0)}\big)}{W_n\big(\phi_1^{(0)}, \phi_2^{(0)}, \cdots, \phi_{n-1}^{(0)}, \phi_n^{(0)}\big)}\right)_x = \frac{W_{n+1}\big(\phi_1^{(0)}, \phi_2^{(0)}, \cdots, \phi_{n-1}^{(0)}, \phi_n^{(0)}, \phi^{(0)}\big) W_{n-1}\big(\phi_1^{(0)}, \phi_2^{(0)}, \cdots, \phi_{n-1}^{(0)}\big)}{W_n^2\big(\phi_1^{(0)}, \phi_2^{(0)}, \cdots, \phi_{n-1}^{(0)}, \phi_n^{(0)}\big)}. \tag{3.12}$$

Now we can generalize this theorem to the case of the generalized Wronskian determinant.

Corollary 3.2.

$$\left(\frac{IW_{k,n+1}(\psi_k^{(0)}, \psi_{k-1}^{(0)}, \cdots, \psi_1^{(0)}; \phi_1^{(0)}, \phi_2^{(0)}, \cdots, \phi_n^{(0)}, \phi^{(0)})}{IW_{k-1,n}(\psi_{k-1}^{(0)}, \psi_{k-2}^{(0)}, \cdots, \psi_1^{(0)}; \phi_1^{(0)}, \phi_2^{(0)}, \cdots, \phi_{n-1}^{(0)})}\right)_x = \frac{IW_{k,n} IW_{k-1,n+1}}{IW_{k-1,n}^2}. \tag{3.13}$$

Corollary 3.3. *For $k \geq 1$,*

$$\left(\frac{IW_{k,n}(\psi^{(0)}, \psi_{k-1}^{(0)}, \psi_{k-2}^{(0)}, \cdots, \psi_1^{(0)}; \phi_1^{(0)}, \cdots, \phi_n^{(0)})}{IW_{k,n}(\psi_k^{(0)}, \psi_{k-1}^{(0)}, \psi_{k-2}^{(0)}, \cdots, \psi_1^{(0)}; \phi_1^{(0)}, \cdots, \phi_n^{(0)})}\right)_x = -\frac{IW_{k-1,n}(\psi_{k-1}^{(0)}, \psi_{k-2}^{(0)}, \cdots, \psi_1^{(0)}; \phi_1^{(0)}, \cdots, \phi_n^{(0)})}{IW_{k,n}^2(\psi_k^{(0)}, \psi_{k-1}^{(0)}, \cdots, \psi_1^{(0)}; \phi_1^{(0)}, \cdots, \phi_n^{(0)})} \cdot IW_{k+1,n}(\psi^{(0)}, \psi_k^{(0)}, \psi_{k-1}^{(0)}, \cdots, \psi_1^{(0)}; \phi_1^{(0)}, \cdots, \phi_n^{(0)}). \tag{3.14}$$

§4. Conclusion

Our main results, Theorem 2.1 and Theorem 2.2, are the determinant representations of the gauge transformation operator. Applying Theorem 2.1 to the KP hierarchy, we have provided an exact proof (Theorem 3.1) of the transformed τ function, which involves the generalized Wronskian determinant. Furthermore, the Jacobi expansion theorem and the Crum identity of the Wronskian determinant have been re-obtained by using Theorem 2.1. Particularly, Corollary 3.2 and Corollary 3.3 mean the generalization of the Jacobi expansion theorem for the generalized Wronskian determinant.

The advantage of the determinant representation of the gauge transformation operator is that it not only brings out the transformed wave function but also leads to the transformed conjugation wave function (see the details in Theorem 3.1). This fact inspires us to plan to reduce the generalized Wronskian τ-function of the KP to the k-constrained sub-hierarchy, which is similar to the work of Oevel and Strampp[19] for the Wronskian τ-function. Using Theorem 2.1, the exact proof the binary-type τ-function introduced by [15] can be obtained.

This new τ-function also involves the generalized Wronskian determinant, which does not hold for the cKP hierarchy with the one component. Hence, based on Theorem 2.1, we will consider to solve the 1-constrained KP hierarchy with one component: $L = \partial + \phi \circ \partial^{-1} \circ \psi$ in a forthcoming paper.

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References

[1] Date, E., Jimbo, M., Kashiwara, M. & Miwa, T., Transformation groups for soliton equation [A], in Nonlinear Integrable Systems—Classical and Quantum Theory [M], edited by M. Jimbo and T. Miwa, World Scientific, Singapore, 1983, 39–119.

[2] Ohta, Y., Satsuma, J., Takahashi, D. & Tokihiro, T., An elementary introduction to Sato theory [J], *Theor. Prog. Phys. Suppl.,* **94**(1988), 210–241.

[3] Dickey, L. A., Soliton equations and Hamiltonian systems [M], World Scintific, Singapore, 1991.

[4] Konopelchenko, B. G., Sidorenko, J. & Strampp, W., (1+1)-dimensional integrable systems as symmetry constraints of $(2+1)$-dimensional systems [J], *Phys. Lett.,* **A157**(1991), 17–21.

[5] Cheng, Y. & Li, Y. S., The constraint of the Kadomtsev-Petviashvili equation and its special solutions [J], *Phys. Lett.,* **A157**(1991), 22–26.

[6] Cheng, Y., Constraints of the Kadomtsev-Petviashvili hierarchy [J], *J. Math. Phys.,* **33**(1992), 3774–3782.

[7] Cheng, Y., Modifying the KP, the nth constrained KP hierarchies and their Hamiltonian structures [J], *Commun. Math. Phys.,* **171**(1995), 661–682.

[8] Loris, I. & Willox, R., KP symmetry reductions and a generalized constraint [J], *J. Phys.,* **A30**(1997), 6925-6938.

[9] Chau, L. L., Shaw, J. C. & Yen, H. C., Solving the KP hierarchy by gauge transformations [J], *Commun. Math. Phys.,* **149**(1992), 263–278.

[10] Oevel, W. & Schief, W., Darobux theorem and the KP hierarchy [A], in Application of Nonlinear Differential Equations [M], edited by P. A. Clarkson, Dordrecht, Kluwer Academic Publisher, 1993, 193–206.

[11] Oevel, W., Darboux theorems and Wronskian formulas for integrable systems I [J], *Constrained KP flows, Physica,* **A195**(1993), 533–576.

[12] Nimmo, J. J., Darboux transformation from reduction of the KP hierarchy [A], in Nonlinear Evolution Equation and Dynamical Systems [M], edited by V. G. Makhankov et al., World Scientific, Singapore, 1995, 168–177.

[13] Aratyn, H., Nissimov, E. & Pacheva, S., Darboux-Backlund solutions of SL(p,q) KP-KdV hierarchies, constrained generalized Toda lattices, and two-matrix string model [J], *Phys. Lett.,* **A201**(1995), 293–305.

[14] Aratyn, H., Nissimov, H. & Pacheva, S., Constrained KP Hierarchies: Darboux-Bäcklund solutions and additional symmetries [R], Preprint (http://xxx.lanl.gov/abs/solv-int/9512008).

[15] Chau, L. L., Shaw, J. C. & Tu, M. H., Solving the constrained KP hierarchy by gauge transformations [J], *J. Math. Phys.,* **38**(1997), 4128–4137.

[16] Willox, R., Loris, I. & Gilson, C. R., Binary Darboux transformations for constrained KP hierarchies [J], *Inverse Problems,* **13**(1997), 849–865.

[17] Loris, I., On reduced CKP equations [J], *Inverse Problems,* **15**(1999), 1099–1109.

[18] Adler, M. & van Moerbeke, P., Birkhoff strata, Backlund transformations, and regularization of isospectral operators [J], *Adv. Math.,* **108**(1994), 140–204.

[19] Oevel, W. & Strampp, W., Wronskian solutions of the constrained Kadomtsev-Petviashvili hierarchy [J], *J. Math. Phys.,* **37**(1996), 6213–6219.