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ON THE EMPTY CONVEX PARTITION OF A FINITE SET IN THE PLANE**

XU Changqing* DING Ren*

Abstract

The authors discuss the partition of a finite set of points in the plane into empty convex polygons, and improve some upper bound and lower bound in the related enumeration problems.

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§1. Introduction

A finite set of points in the plane is called a convex polygon if it forms the set of vertices of a convex polygon. In [5] Masatsugu Urabe studied the problem of partitioning a finite point set in the plane such that each component is a convex polygon. In this paper we improve some results in [5].

In the sequel, if not otherwise stated, P always denotes a set of 15 points in the plane, no three collinear, $CH(P)$ the convex hull of P, S the set of the vertices of $CH(P)$, and S' the set of the vertices of $CH(P-S)$. Usually points of S are denoted by x's, points of S' by y's and other points of P by z's. Polygon always refers to convex polygon.

If an m-subset of P determines a convex m-gon whose interior contains no point of P, then the convex m-gon or the m-subset itself is called an empty m-gon of P. If P is partitioned into k subsets $S_1, S_2, \ldots, S_k$ such that each $S_i, i = 1, 2, \ldots, k$, is the vertex set of a convex polygon, then the partition obtained is called a convex partition of P. A convex partition of P is called empty if each $CH(S_i)$ is an empty convex polygon of P, and disjoint if $CH(S_i) \bigcap CH(S_j) = \emptyset$ for any pair of i, j.

Let $g(P)$ be the minimum number of empty convex polygons over all empty convex partitions of P, and $f(P)$ be the minimum number of disjoint convex polygons over all disjoint convex partitions of P. Define

$G(n) =: \max\{g(P) : |P| = n\}$, $F(n) =: \max\{f(P) : |P| = n\}$.

Masatsugu Urabe in [5] proved that

$$\lceil \frac{n-1}{4} \rceil \le F(n) \le \lceil \frac{2n}{7} \rceil, \quad \lceil \frac{n-1}{4} \rceil \le G(n) \le \lceil \frac{3n}{11} \rceil.$$

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*Department of Mathematics, Hebei Normal University, Shijiazhuang 050016, China.
E-mail: chqxu@sina.com; rending@heinfo.net

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In the present paper we show that

$$\lceil\frac{n+1}{4}\rceil \le F(n) \le \lceil\frac{2n}{7}\rceil, \quad \lceil\frac{n+1}{4}\rceil \le G(n) \le \lceil\frac{4n}{15}\rceil.$$

§2. Preliminaries

We need the following lemmas.

Lemma 2.1. $F(n) \ge G(n)$.

Lemma 2.2.[5] $G(11) \le 3$.

Lemma 2.3. *Given a set of* 5 *points in the plane, no three collinear, there must exist an empty* 4*-gon in it.*

Lemma 2.4. *Let* $|P| = 15$. *If there exists a straight line which separates a convex i-gon* $(i \ge 4)$ *of P from the remaining* $15 - i$ *points, then* $g(P) \le 4$.

Proof. Since $i \ge 4$, we have $15 - i \le 11$ and then from Lemma 2.2 the result follows immediately.

The straight line extended by an edge of a convex polygon dissects the plane into two open half planes, the one not including the convex polygon is called the outer side of the line with respect to the convex polygon. For simplicity, we say outer side of an edge instead of outer side of the straight line extended by an edge, if there is no risk of ambiguity.

From Lemma 2.4 and Lemma 2.2 it is easy to obtain the following facts.

Lemma 2.5 *Let* $|P| = 15$. *If there exists an edge of* $CH(P-S)$ *such that there are at least two points of S on the outer side of the edge, then* $g(P) \le 4$.

Lemma 2.6. *Let* $|P| = 15$. *If A is an empty convex* 4*-gon of P, and the convex hull of the remaining* 11 *points contains no vertex of A, then* $g(P) \le 4$.

We call such a set of 11 points in P as in Lemma 2.6 a good 11-point set. So Lemma 2.6 says if there exists a good 11-point set in a 15-point set, then $g(P) \le 4$.

Let x be a point in the exterior of a convex polygon A. The weight of x with respect to A, denoted by $w_A(x)$, is defined to be the number of the edges of A whose outer sides with respect to A contains x. If $w_A(x) = 1$, x is called a single point with respect to A.

Lemma 2.7. *Let* $|P| = 15$. *If there exists a point of S which is a single point with respect to* $CH(P-S)$, *then* $g(P) \le 4$.

Proof. If $CH(P-S)$ is a segment, it is trivial that $g(P) \le 4$. Generally, due to Lemma 2.5, it suffices to prove the lemma under the following assumption:

$(*)$ There is exactly one point of S on the outer side of each edge of $CH(P-S)$.

Case 1. $|S'| > 4$. Suppose $x \in S$ is a single point about $CH(P-S)$ on the outer side of the edge (y_1, y_2). Let (y_0, y_1) and (y_2, y_3) be adjacent edges of (y_1, y_2) in the boundary of $CH(P-S)$. Let $x_1 \in S$ be on the outer side of (y_0, y_1), $x_2 \in S$ on the outer side of (y_2, y_3).

Subcase 1.1. No point of $S - \{x\}$ is on the same side of y_0y_3 as x. Choose a point z of P in the quadrilateral $y_1y_2y_3y_0$ such that y_1xy_2z is an empty convex 4-gon which is separated from the other 11 points by a line, and from Lemma 2.4 we have $g(P) \le 4$. Here y_0 or y_3 may be chosen as z.

Subcase 1.2. Exactly two points of $S - \{x\}$ are on the same side of y_0y_3 as x. By the assumption $(*)$ they must be x_1, x_2. If the interior of the quadrilateral $y_1x_1x_2y_2$ intersects P, then we can find a point z in the quadrilateral such that xy_1zy_2 is an empty convex 4-gon which is separated from the remaining 11 points. By Lemma 2.4 we obtain $g(P) \le 4$. If the interior of $y_1x_1x_2y_2$ contains no point of P, then $y_1x_1x_2y_2$ is an empty convex 4-gon of P and from the assumption $(*)$ the remaining 11-point set is good, and hence $g(P) \le 4$ by Lemma 2.6.

Subcase 1.3. Exactly one point of $S-\{x\}$ is on the same side of y_0y_3 as x. By the assumption $(*)$ this point must be x_1 or x_2, say x_1. If x and x_2 are on the different sides of x_1y_3, that is, x_1y_3 separates x, y_1, y_2 from the remaining points of $S \cup S'$, then we can choose a point z of P in the intersection of $y_1x_1y_3y_2$ and $y_0y_1y_2y_3$ (y_3 may be chosen as z) such that xy_1zy_2 is an empty convex 4-gon which is separated from the remaining points of P by a line, and by Lemma 2.4 we have $g(P) \le 4$. Otherwise, x, x_2 are on the same side of x_1y_3 and hence x_1x_2 separates x, y_1, y_2 from the remaining points of $S \cup S'$, by the same method as in Subcase 1.2 we have $g(P) \le 4$.

Case 2. $|S'| = 3, 4$. By the reasoning similar to that in Subcases 1.2 and 1.3 we obtain $g(P) \le 4$.

Lemma 2.8. *Let* $|P| = 15$. *If the outer side of each line extended by any edge of* $CH(P-S)$ *contains exactly one point of* S, *and no point of* S *is a single point with respect to* $CH(P-S)$, *then*

$$|S'| = \sum_{x \in S} w(x) \ge 2|S|.$$

Proof. Let $l_1, l_2, \cdots l_m$ denote the lines extended by the edges of $CH(P-S)$, where $m = |S'|$. For $x \in S$ let

$$w_{l_j}(x) = \begin{cases} 1, & x \in S \text{ is on the outer side of } l_j, \\ 0, & \text{otherwise,} \end{cases}$$

where $j = 1, 2, \cdots, m$. Obviously for any $x \in S$, $w(x) = \sum\limits_{j=1}^{m} w_{l_j}(x)$, and since the outer side of each edge of $CH(P-S)$ contains exactly one point of S, $\sum\limits_{x \in S} w_{l_j}(x) = 1$ for any $j = 1, 2, \cdots, m$. Hence

$$\sum_{x \in S} w(x) = \sum_{x \in S} \sum_{j=1}^{m} w_{l_j}(x) = \sum_{j=1}^{m} \sum_{x \in S} w_{l_j}(x) = m = |S'|.$$

Since each point of S is not single, we have $w(x) \ge 2$ for each $x \in S$, and therefore $|S'| = \sum\limits_{x \in S} w(x) \ge 2|S|$.

Given any edge of $CH(P)$, there exists a point $y \in S'$ which is nearest to the edge. We call the point y a near point to the edge or a near point of $CH(P)$. The point in S' which is not a near point to any edge of $CH(P)$ is called a non-near point of $CH(P)$.

Lemma 2.9. *If* $(*)$ *there is exactly one point of* S *on the outer side of each edge of* $CH(P-S)$, *and* $(**)$ *no point of* S *is single about* $CH(P-S)$, *then*

(1) *each edge of* $CH(P)$ *has exactly one near point*;

(2) *the near points of two consecutive edges of* $CH(P)$ *are distinct*;

(3) *between any two consecutive near points of* $CH(P)$ *there is at least one non-near point*;

(4) *near points and non-near points of* $CH(P)$ *appear alternatively if, moreover,* $|S'| = 2|S|$.

Proof. (1) Suppose the contrary. If the edge (x_1, x_2) of $CH(P)$ had two near points y_1, y_2 in S', then the outer side of y_1y_2 would have two points of S, i.e. x_1, x_2, which contradicts $(*)$.

(2) If two consecutive edges (x_1, x_2) and (x_2, x_3) of $CH(P)$ shared a common near point y_2, it would be easy to get a contradiction to $(*)$.

(3) If there were two consecutive near points y_1, y_2 of $CH(P)$ without any non-near point between them, then the point $x \in S$ on the outer side of y_1y_2 would be single about $CH(P-S)$, a contradiction to $(**)$.

(4) If there were two non-near points, say y_1', y_2', between the near point y_1 to (x_1, x_2) and the near point y_2 to (x_2, x_3), then $w(x_2) \geq 3$ and

$$|S'| = \sum_{x \in S} w(x) = \sum_{x \in S - \{x_2\}} w(x) + w(x_2) > 2|S|,$$

a contradiction. Therefore when $|S'| = 2|S|$, near points and non-near points of $CH(P)$ appear alternatively.

§3. Main Theorems

Theorem 3.1. $G(15) \leq 4$.

Proof. Since $G(15) =: \max\{g(P) : |P| = 15\}$, we need only to prove that for any finite set P of 15 points in the plane we have $g(P) \leq 4$.

By Lemma 2.5 and Lemma 2.7, it suffices to prove our claim under the previous assumptions $(*)$ and $(**)$.

By $(*)$, $(**)$ and Lemma 2.8 it is easy to see that $|S'| \geq 2|S|$, $|S'| + |S| \leq 15$, hence $|S| \leq 5$ and so we have the following cases to consider.

Case 1. $|S| = 5$.

From $(*)$ and Lemma 2.8, $|S'| \geq 2|S| = 10$, but $|S'| \leq 15 - |S| = 10$, so $|S'| = 10$. Then the 10 points of S' form an empty 10-gon, while the 5 points of S can be partitioned into three empty convex polygons, and hence $g(P) \leq 4$.

Case 2. $|S| = 4$. By an argument similar to that in Case 1 we have $|S'| = 8, 9, 10, 11$. It is easy to obtain $g(P) \leq 4$ when $|S'| = 9, 10, 11$. Now consider $|S'| = 8$. Notice that here $|S'| = 2|S|$. Let $P - S \cup S' = \{z_1, z_2, z_3\}$. Consider the number of points of S' on the outer side of z_1z_2, z_2z_3 and z_3z_1. Without loss of generality we suppose that the number of points of S' on the outer side of z_1z_2 is the maximum and let it be t. Obviously $t = 3, 4, 5, 6, 7$.

(1) t=7. z_1, z_2 and the 7 points of S' form an empty convex 9-gon, z_3 and the remaining point of S' form a segment, i.e. an empty 2-gon, the 4 points in S form two segments and hence we obtain an empty partition of P into 4 empty convex polygons, therefore $g(P) \leq 4$.

(2) t=6. By the argument similar to that in the case t=7 we get $g(P) \leq 4$.

(3) t=5. z_1, z_2 and the 5 points of S' form an empty convex 7-gon. Since $|S'| = 2|S|$, by Lemma 2.9, among the remaining 3 points of S' there must be a near point to some edge of $CH(P)$ from which we obtain an empty 3-gon, the other 4 points in S and S' form two segments and so $g(P) \leq 4$.

(4) t=4. z_1, z_2 and the 4 points of S' form an empty convex 6-gon. Consider z_3 and the remaining 4 points y_1, y_2, y_3, y_4 clockwise in S'. By Lemma 2.9 we may suppose y_1, y_3 are two near points to $CH(P)$. Without loss of generality, let z_3 be on the same side of y_2y_4 as y_1. We obtain an empty convex 4-gon $z_3y_2y_3y_4$, the near point y_1 and the edge to which y_1 is closest form an empty 3-gon, and the last two points of P in S form a segment, therefore $g(P) \leq 4$.

(5) t=3. By Lemma 2.9, we may suppose that the outer side of z_1z_2 contains $y_6, y_7, y_8 \in S'$ among which y_7 is the only near point to $CH(P)$. Denote the remaining 5 points of S' by y_1, y_2, y_3, y_4, y_5 clockwise. By Lemma 2.9, y_1, y_3, y_5 are near points to $CH(P)$. Notice that $z_1y_6y_7y_8z_2$ is an empty convex 5-gon. Since P is a set of points with no three collinear, z_3 must be on one side of y_2y_4 and so either $y_1y_2z_3y_4y_5$ is an empty 5-gon or $y_2y_3y_4z_3$ is an

empty convex 4-gon. By the definition of near point, the remaining 5 points of $S \cup S'$ form an empty convex 3-gons and a segment or the remaining 6 points of $S \cup S'$ form two empty 3-gons. And hence $g(P) \leq 4$.

Case 3. $|S| = 3$. From $(*)$ and the equality $2|S| \leq |S'| \leq 15 - |S|$ we get $6 \leq |S'| \leq 12$. When $|S'| = 10, 11, 12$, we get $g(P) \leq 4$ immediately by the simple argument as before. Now we discuss the remaining cases.

Subcase 3.1. $|S'| = 9$.

Now that $|S'| > 2|S|$, by $(*)$ and Lemma 2.9 there exists a non-near point between any two near points of $CH(P)$. Denote $P - (S \cup S') = \{z_1, z_2, z_3\}$. Consider the maximum number of points of S' on the outer sides extended by the edges of 3-gon $z_1z_2z_3$, and without loss of generality let the maximum number t be attained on the outer side of z_1z_2. Clearly t=$3, 4, 5, 6, 7, 8$.

(1) $t \geq 5$. The conclusion $g(P) \leq 4$ is obvious.

(2) t=4. z_1, z_2 and the 4 points of S' on the outer side of z_1z_2 form an empty convex 6-gon. $CH(P)$ has three near points. Since $|S'| = 9$, and $|S| = 3$, we see that $|S'| > 2|S|$. By Lemma 2.9 there exists a non-near point between any two near points of $CH(P)$. So among the 4 points of S' on the outer side of z_1z_2 there are at most 2 near points and hence there exists a near point of $CH(P)$ in the remaining 5 points of S'. This near point and its corresponding edge in $CH(P)$ form an empty 3-gon. Now by Lemma 2.3, we can get an empty convex 4-gon from the 5 points left in S' and z_3. The remaining two points in P form a segment, and we obtain $g(P) \leq 4$.

(3) t=3. The outer side of each edge of 3-gon $z_1z_2z_3$ contains three points of S'. z_1, z_2 and the 3 points of S' on the outer side of z_1z_2 form an empty convex 5-gon. From the remaining 6 points in S' we can always choose 4 points to form an empty convex 5-gon with z_3 while the remaining two points in S' are adjacent in the boundary of $CH(P - S)$. These two points and a point in S form an empty 3-gon and the last two points in S form a segment, and thus $g(P) \leq 4$.

Subcase 3.2. $|S'| = 8$.

In this case there are 4 points of P in the interior of $CH(P-S)$, that is, $|P-(S\cup S')| = 4$.

Fig.1 Three open regions R_1, R_2, R_3

(1) $CH(P - (S \cup S'))$ is a triangle, say, $z_1z_2z_3$ whose interior contains exactly one point z_0. The rays z_0z_1, z_0z_2, z_0z_3 divide the plane into three open regions R_1, R_2, R_3. Suppose $t =: \max\limits_{1\leq i\leq 3} |R_i \cap S'| = |R_1 \cap S'|$. Clearly t=$3, 4, 5, 6, 7$. If $t \geq 4$, immediately we have $g(P) \leq 4$. Now suppose $t = 3$. Considering the definition of t, we may suppose $|R_1 \cap S'| = |R_2 \cap S'| = 3, |R_3 \cap S'| = 2$. It is easy to form a partition of P into one empty convex 6-gon, one empty convex 4-gon, one empty 3-gon and a segment, which leads to $g(P) \leq 4$.

(2) $CH(P - (S \cup S'))$ is an empty 4-gon, denoted by $z_1z_2z_3z_4$. z_1z_2 and z_3z_4 determine

three open regions in the plane: R_1, R_2, R_3 (see Fig.1). Let $t =: \max_{1\le i\le 3} |R_i \cap S'|$. Clearly t= 3, 4, 5, 6, 7. From $t \ge 4$ it is easy to deduce $g(P) \le 4$. We focus on the case t=3. When t=3, there must be two regions each of which contains 3 points of S'.

(a) $|R_1 \cap S'| = 3, |R_2 \cap S'| = 3, |R_3 \cap S'| = 2$. In each of the closures of R_1 and R_2 we may construct an empty convex 5-gon of P; then we can select one point of S to form an empty triangle together with the two points of S' in R_3; the last two points of S form a segment. Hence we have $g(P) \le 4$.

(b) $|R_1 \cap S'| = 2, |R_2 \cap S'| = 3, |R_3 \cap S'| = 3$. In exactly the same way as in the previous case we get $g(P) \le 4$.

(c) $|R_1 \cap S'| = 3, |R_2 \cap S'| = 2, |R_3 \cap S'| = 3$.

(i) The two points of S' in R_2 are adjacent vertices of $CH(P - S)$. In each of the closures of R_1, R_3, we may construct one empty convex 5-gons, and then we can select one point from S to form an empty 3-gon together with the two points of S' in R_2, and the last two points of S form a segment, and thus we get $g(P) \le 4$.

(ii) The two points of S' in R_2 are not adjacent vertices of $CH(P-S)$. Then we have a configuration of the 15 points of P as in Fig.2. Let $Z = \{z_1, z_2, z_3, z_4\}$. Noticing the weights of points in S' with respect to Z, we assume $w_Z(y_1) = w_Z(y_3) = w_Z(y_5) = w_Z(y_7) = 1$, and $w_Z(y_2) = w_Z(y_4) = w_Z(y_6) = w_Z(y_8) = 2$. If there is a near point of $CH(P)$ in $\{y_1, y_3, y_5, y_7\}$, it is easy to get a partition of P into four empty convex polygons and hence $g(P) \le 4$. Otherwise, no point in $\{y_1, y_3, y_5, y_7\}$ is a near point of $CH(P)$. Without loss of generality we assume the three near points of $CH(P)$ are y_2, y_4, y_6. Also we assume that $w_{S'}(x_1) = 4$ and $w_{S'}(x_2) = w_{S'}(x_3) = 2$. Considering the positions of points of Z we have two more subcases:

(A) There exists at least one point of Z between x_1y_1 and x_1y_7. Then z_4 must be such point.

• z_4 is between x_1y_7 and x_1y_8. If z_4 is not in the interior of $y_1y_6y_7y_8$, then no point of Z is in the interior of $y_1y_6y_7y_8$ and in Z z_4 is nearest to y_1y_6. Therefore we can partition P into 4 empty convex polygons as follows: 5-gon $y_1z_4y_6y_7y_8$, 3-gon $y_2x_1x_2$, 4-gon $y_3z_1z_2z_3$, and 3-gon $y_4y_5x_3$. If z_4 is in the interior of $y_1y_6y_7y_8$, then the empty convex 4-gon $x_1y_8z_4y_7$ is separated from the remaining 11 points of P by y_1y_6 or a line parallel to y_1y_6. Therefore we reach the conclusion $g(P) \le 4$.

• z_4 is between x_1y_1 and x_1y_8. Similar to the above reasoning we consider the relative position of z_4 with respect to the interior of $y_2y_7y_8y_1$ and easily we obtain $g(P) \le 4$.

(B) No point of Z is between x_1y_1 and x_1y_7. Then either both z_4, z_3 are between x_1y_7 and x_1y_6 or both z_4, z_1 are between x_1y_1 and x_1y_2. In the former case we have an empty convex 4-partition of P: 5-gon $x_1y_7z_4z_3y_6$, 3-gon $z_2y_3y_5$, 4-gon $y_8y_1y_2z_1$, and 3-gon $y_4x_2x_3$. Similarly we have an empty convex 4-partition of P in the latter case. All these facts lead to the conclusion that $g(P) \le 4$.

Subcase 3.3. $|S'| = 6, 7$. By similar argument we obtain the conclusion $g(P) \le 4$.

Theorem 3.2. $G(n) \le \lceil \frac{4n}{15} \rceil$.

Proof. It suffices to prove that for any set P of n points, no three collinear, in the plane, $g(P) \le \lceil \frac{4n}{15} \rceil$, that is, P has a $\lceil \frac{4n}{15} \rceil$-empty convex partition. Take a line l not parallel to any line determined by any two points of P. Move l parallel to itself and divide the plane into $\lceil \frac{n}{15} \rceil$ open strip regions such that each region contains 15 points of P except probably the last region R which contains r points of P, $1 \le r \le 15$. If n is divisible by 15, then each aforesaid region contains 15 points of P which has a 4-empty convex partition and therefore P has a $\frac{4n}{15}$-empty convex partition. Now suppose n is not divisible by 15. Then in the $\lceil \frac{4n}{15} \rceil$

regions each of the first $\lfloor \frac{n}{15} \rfloor$ contains 15 points of P and the last region contains r points of P with $1 \le r \le 14$. For the points of P in the first $\lfloor \frac{n}{15} \rfloor$ regions there exists a $4\lfloor \frac{n}{15} \rfloor$-empty convex partition. When r =1, 2, 3, the r points of P in R has a 1-empty convex partition, $4\lfloor \frac{n}{15} \rfloor + 1 = \lceil \frac{4n}{15} \rceil$, and so P has a $\lceil \frac{4n}{15} \rceil$-empty convex partition. When r =4, 5, 6, 7, it is easy to see that the r points of P in R has a 2-empty convex partition, $4\lfloor \frac{n}{15} \rfloor + 2 = \lceil \frac{4n}{15} \rceil$. When r= 8, 9, 10, 11, by Lemma 2.2, the r points of P in R has a 3-empty convex partition, $4\lfloor \frac{n}{15} \rfloor + 3 = \lceil \frac{4n}{15} \rceil$. When r = 12, 13, 14, by Theorem 3.1, the r points of P in R has a 4-empty convex partition, $4\lfloor \frac{n}{15} \rfloor + 4 = \lceil \frac{4n}{15} \rceil$, so in each case P has a $\lceil \frac{4n}{15} \rceil$-empty convex partition.

Fig.2 A configuration of the 15 points of P

§4. The Lower Bound of $F(n), G(n)$

Lemma 4.1.[5] *If $n \ge 10$, then $F(n) \ge G(n) \ge \lceil \frac{n-1}{4} \rceil$.*

Lemma 4.2. *If $n = 4k + 1$ $(k = 0, 1, 2, \cdots)$, then $F(n) \ge G(n) \ge \lceil \frac{n+1}{4} \rceil$.*

Proof. By Lemma 2.1 it suffices to prove $G(n) \ge \lceil \frac{n+1}{4} \rceil$. When $k = 0, 1, 2$, obviously $G(n) = \lceil \frac{n+1}{4} \rceil$. Consider $k \ge 3$. We need only to construct a configuration P of $n = 4k + 1$ points, no three collinear, such that $g(P) \ge \lceil \frac{n+1}{4} \rceil$, i.e. $g(P) \ge k + 1$. Take a regular $(2k + 1)$-gon $u_1 u_2 \cdots u_{2k+1}$ with center at point o. Then choose $v_1, v_2, \cdots, v_{2k+1}$ such that $P = \{u_1, u_2, \cdots, u_{2k}, v_1, v_2, \cdots, v_{2k+1}\}$ meets the following requirements (see Fig.3):

(1) no three points from $\{u_1, u_2, \cdots, u_{2k}\}$ are in a vertex set of an empty convex polygon of P;

(2) if u_i, u_j (except u_1, u_{2k}) are in a vertex set of an empty convex polygon of P, then v_{2k+1} is not in that vertex set.

Then any empty convex partition of P contains at least $k+1$ empty convex polygons and we reach the conclusion that $g(P) \ge k + 1 = \lceil \frac{n+1}{4} \rceil$.

Fig.3 A regular $(2k+1)$-gon with center at o

Lemma 4.3. *If $n = 4k$ (k is a positive integer), then $F(n) \geq G(n) \geq \lceil \frac{n+1}{4} \rceil$.*

Proof. We prove $G(n) \geq \lceil \frac{n+1}{4} \rceil$. For $n = 4, 8$, obviously $G(n) = \lceil \frac{n+1}{4} \rceil$. For $n > 10$ we construct the following point set P:

Let $P_1 = \{u_1, u_2, \cdots, u_{2k+1}\}$ be the vertices of a regular $2k+1$-gon. $u_{2k-3}u_{2k-1}$ and $u_{2k-2}u_{2k}$ meet at o_1; $u_{2k-1}u_{2k+1}$ and $u_{2k}u_1$ meet at o_2. In the interior of $u_{2k-2}u_{2k-1}o_1$ choose a point ν_{2k-2} close to o_1. In the interior of $u_{2k}u_{2k+1}o_2$ choose a point ν_{2k-1} close to o_2. In the interior of $u_{2k-1}u_{2k}u_i$ and $u_{i-1}u_iu_{i+1}$ choose a point ν_i close to $u_{i-1}u_{i+1}$ $(i = 1, 2, \cdots, 2k-3)$. In this way we obtain the $4k$-point set

$$P = \{u_1, u_2, \cdots, u_{2k+1}, \nu_1, \cdots, \nu_{2k-1}\}$$

with no three collinear. It is easy to see that each vertex set of an empty convex polygon in P contains at most 2 points of P_1. Therefore any empty convex partition of P contains at least $k+1$ empty convex polygon and we have $g(P) \geq k+1$.

Theorem 4.1. *$F(n) \geq G(n) \geq \lceil \frac{n+1}{4} \rceil$ for any positive integer n.*

Proof. By Lemma 4.2 and Lemma 4.3 it suffices to prove only for $4k+2, 4k+3$ $(k = 0, 1, 2, \cdots)$. When $n = 4k+2, 4k+3$ we have $\lceil \frac{n-1}{4} \rceil = \lceil \frac{n+1}{4} \rceil$. From Lemma 4.1 we obtain $F(n) \geq G(n) \geq \lceil \frac{n+1}{4} \rceil$ for $n > 10$. $G(n) \geq \lceil \frac{n+1}{4} \rceil$ when $n = 2, 3, 6, 7$, and the proof is complete.

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