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ON ASYMPTOTIC NORMALITY OF PARAMETERS IN LINEAR EV MODEL***

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Abstract

This paper studies the parameter estimation of one dimensional linear errors-in-variables (EV) models in the case that replicated observations are available in some experimental points. Asymptotic normality is established under mild conditions, and the parameters entering the asymptotic variance are consistently estimated to render the result useable in construction of large-sample confidence regions.

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§1. Introduction

EV (Errors-in-Variables) model is just the regression model with both dependent and independent variables subject to error (see, for example, [1, p.403], [2] and the literature cited there). It is well known that in such models the parameters in the regression function cannot be consistently estimated without some restrictive conditions imposed upon the error variances. A way out is to take replicated observations. Consider that in many practical applications, making artificial conditions upon error variances is not practical, but taking replicated observations presents no essential difficulties. This procedure was studied in [3], in which estimators of α and β are introduced, and their weak and strong consistency are proved under mild conditions. Their asymptotic normality are established respectively in [4], but with a severe restriction that the errors are assumed normality distributed. Recently we succeed in getting rid of this restriction, thus place the result on a broader base. This constitutes the main result of this paper.

We write the model studied in this paper as

$$\xi_{ij} = x_i + \delta_{ij}, \eta_{ij} = y_i + \varepsilon_{ij} = \alpha + \beta x_i + \varepsilon_{ij}, j = 1, 2, \cdots n_i; i = 1, 2, \cdots k \cdots$$

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with the following conditions imposed:

$$(A)\begin{cases}(\delta_{ij},\varepsilon_{ij}) : j=1,2,\cdots,n_i;\ i=1,2,\cdots, \quad \text{are} \quad \text{i.i.d,}\\ E\delta_{11}=E\varepsilon_{11}=0, 0<E\delta_{11}^2=\sigma_1^2<\infty, 0<E\varepsilon_{11}^2=\sigma_2^2<\infty,\\ \text{There are infinitely many integers in } \{n_i\} \text{ which are greater than one.}\end{cases}$$

(B) : $\delta_{11},\varepsilon_{11}$ are independent and δ_{11} is symmetric .

Here (ξ_{ij},η_{ij}) are observable, $x_1,x_2,\cdots,\sigma_1^2,\sigma_2^2$ and α,β are not.

In the following we adhere to the following notations:

$$\xi_i=\sum_{j=1}^{n_i}\xi_{ij}/n_i, \quad \text{similarly} \quad \eta_i,\delta_i,\varepsilon_i;$$

$$N_k=\sum_{i=1}^{k}n_i, \bar{\xi}=\sum_{i=1}^{k}n_i\xi_i/N_k, \text{ similarly } \bar{\eta},\bar{\delta},\bar{\varepsilon};$$

$$\bar{x}=\sum_{i=1}^{k}n_ix_i/N_k, \quad S=\sum_{i=1}^{k}n_i(x_i-\bar{x})^2.$$

Note that $\bar{x},S$ and $\bar{\xi},\bar{\eta},\cdots$ all depend upon k. To simplify the writing, we put

$$\sum_{i=1}^{k}\sum_{j=1}^{n_i}a_{ij}=\sum a_{ij}, \quad \sum_{i=1}^{k}a_i=\sum a_i$$

for any quantity a_{ij} depending upon i,j and any quantity a_i depending upon i. Other summations will be written in its full detail.

As in [3], we introduce the consistent estimates for σ_1^2,σ_2^2 and β,α as follows:

$$\hat{\sigma}_1^2=\sum(\xi_{ij}-\xi_i)^2/(N_k-k), \tag{1.1}$$

$$\hat{\sigma}_2^2=\sum(\eta_{ij}-\eta_i)^2/(N_k-k), \tag{1.2}$$

$$\hat{\beta}=\sum(\xi_{ij}-\bar{\xi})(\eta_{ij}-\bar{\eta})\Big/\Big(\sum(\xi_{ij}-\bar{\xi})^2-N_k\hat{\sigma}_1^2\Big), \tag{1.3}$$

$$\hat{\alpha}=\bar{\eta}-\hat{\beta}\bar{\xi}. \tag{1.4}$$

The purpose of this paper is to prove the asymptotic normality of $\hat{\alpha}$ and $\hat{\beta}$ under condition (A) and supplementary condition (B). This paper is organized as follow: in Section 2 and Section 3, asymptotic normality of $\hat{\beta}$ and $\hat{\alpha}$ are established respectively, in Section 4 we introduce some estimators to make interval estimation and hypothesis testing of $\hat{\alpha}$ and $\hat{\beta}$, in the large-sample sense.

§2. Asymptotic Normality of $\hat{\beta}$

The following lemmas will be needed in the following.

Lemma 2.1. *Let w_i be a sequence of independent random variables with zero means and bounded variances. $\{a_i\}$ and $\{c_i>0\}$ are constant sequences such that $\sum_{i=1}^{n}(a_i-\bar{a}_n)^2/c_n$ is bounded and*

$$\lim_{n\to\infty}\sum_{i=1}^{n}(a_i-\bar{a}_n)^2=\infty.$$

Then

$$\sum_{i=1}^{n}(a_i-\bar{a}_n)w_i/c_n \xrightarrow{\text{a.s.}} 0, \quad \text{as } n\to\infty.$$

This is a special case of a result of [5]. Let

$$Q_k=\sum(\xi_{ij}-\bar{\xi})^2-N_k\hat{\sigma}_1^2. \tag{2.1}$$

Lemma 2.2.[3] *Assume that*

$$\liminf_{k\to\infty}(S/N_k)>0. \tag{2.2}$$

If condition (A) is satisfied, then

$$\lim_{k\to\infty}Q_k/S=1 \quad \text{a.s.} \tag{2.3}$$

Hence also

$$\liminf_{k\to\infty}Q_k/N_k>0 \quad \text{a.s.} \tag{2.4}$$

Lemma 2.3.[6] *Let $\{x_i\}_{i=1}^n$ be independent random variables with zero means and finite absolute moments of order $p\ge 2$. Then $E\Big|\sum\limits_{i=1}^n x_i\Big|^p \le c_p n^{p/2-1}\sum\limits_{i=1}^n E\,|x_i|^p$, where c_p is constant.*

Lemma 2.4. *Let $\{x_i\}$ be an i.i.d sequence with zero means. If $E\,|x_1|^r<\infty$ for some $r>0$, then $\max\limits_{1\le i\le k}|x_i|\,/k^{1/r}\xrightarrow{\text{a.s.}}0$.*

Proof. Since $E|x_1|^r<\infty$, it follows that $\sum\limits_{k=1}^{\infty}P\left(|x_1|>k^{1/r}\right)<\infty$. Hence

$$\sum_{k=1}^{\infty}P\left(|x_1|>\varepsilon k^{1/r}\right)<\infty \text{ for each } \varepsilon>0, \tag{2.5}$$

$$\begin{aligned}
P\Big(\bigcup_{m=k}^{\infty}\Big\{\max_{1\le i\le m}|x_i|/m^{1/r}>\varepsilon\Big\}\Big) &= P\Big(\bigcup_{m=k}^{\infty}\bigcup_{i=1}^{m}\{|x_i|>\varepsilon m^{1/r}\}\Big)\\
&= P\Big(\bigcup_{i=1}^{k}\{|x_i|>\varepsilon k^{1/r}\}\bigcup\bigcup_{m=k+1}^{\infty}\{|x_m|>\varepsilon m^{1/r}\}\Big)\\
&\le \sum_{i=1}^{k}P(|x_i|>\varepsilon k^{1/r})+\sum_{m=k+1}^{\infty}P(|x_m|>\varepsilon m^{1/r})\\
&= kP(|x_1|>\varepsilon k^{1/r})+\sum_{m=k+1}^{\infty}P(|x_1|>\varepsilon m^{1/r}).
\end{aligned}$$

Since $E|x_1|^r<\infty$, it follows that $kP(|x_1|>\varepsilon k^{1/r})\to 0$ as $k\to\infty$. By (2.5)

$$\sum_{m=k+1}^{\infty}P(|x_1|>\varepsilon m^{1/r})\to 0 \quad \text{as } \ k\to\infty,$$

therefore we obtain for each $\varepsilon>0$,

$$P\Big(\bigcup_{m=k}^{\infty}\{\max_{1\le i\le m}|x_i|/m^{1/r}>\varepsilon\}\Big)\to 0 \quad \text{as } k\to\infty. \tag{2.6}$$

Thus we prove Lemma 2.4.

Theorem 2.1. *If, in addition to* $(A), (B)$ *and* (2.2), *the following conditions are satisfied*:
(1) $\exists r > 0, E|\delta_{11}|^{4+r} < \infty$,
(2) $\max\limits_{1\le i\le k} n_i/N_k \to 0$, $\max\limits_{1\le i\le k} n_i\,(x_i-\bar{x})^2/S \to 0$ *as* $k\to\infty$,
(3) $\liminf\limits_{k\to\infty} N_k/k > 1$;
then

$$\frac{S(\hat{\beta}-\beta)}{\Big\{N_k\sigma_1^2\sigma_2^2+(\beta^2\sigma_1^2+\sigma_2^2)S+\beta^2\Big[\frac{N_k}{(N_k-k)^2}\Big(N_k\sum\limits_{i=1}^{k}1/n_i-k^2\Big)(E\delta_{11}^4-3\sigma_1^4)+\frac{2N_kk}{N_k-k}\sigma_1^4\Big]\Big\}^{1/2}}$$

$$\xrightarrow{\text{d.f.}} N(0,1). \tag{2.7}$$

Proof. Write

$$\hat{\beta}-\beta = W_k/Q_k, \tag{2.8}$$

where

$$W_k=\beta\Big(N_k\hat{\sigma}_1^2-\sum(\delta_{ij}-\bar{\delta})^2\Big)-\beta\sum(x_i-\bar{x})\delta_{ij}+\sum(\xi_{ij}-\bar{\xi})\varepsilon_{ij}. \tag{2.9}$$

Q_k is defined in (2.1).

Let

$$m_k=\beta\Big(N_k\hat{\sigma}_1^2-\sum(\delta_{ij}-\bar{\delta})^2\Big)-\beta\sum(x_i-\bar{x})\delta_{ij}, \tag{2.10}$$

$$W_k-m_k=\sum(\xi_{ij}-\bar{\xi})\varepsilon_{ij}. \tag{2.11}$$

Since

$$\sum(\xi_{ij}-\bar{\xi})^2=\sum(\delta_{ij}-\bar{\delta})^2+2\sum(x_i-\bar{x})\delta_{ij}+S,$$

by Kolmogorov's SLLN, Lemma 2.1 and Lemma 2.2, it follows that

$$\sum(\xi_{ij}-\bar{\xi})^2/(N_k\sigma_1^2+S)\xrightarrow{\text{a.s.}}1. \tag{2.12}$$

Since

$$\max_{i,j}|\xi_{ij}-\bar{\xi}|=\max_{i,j}|\delta_{ij}-\bar{\delta}+x_i-\bar{x}|\le 2\max_{i,j}|\delta_{ij}|+\max_i|\sqrt{n_i}(x_i-\bar{x})|,$$

from the conditions (1), (2) and Lemma 2.4, we have

$$\max_{i,j}|\xi_{ij}-\bar{\xi}|/\sqrt{N_k\sigma_1^2+S}\xrightarrow{\text{a.s.}}0. \tag{2.13}$$

By (2.12) and (2.13), it follows that

$$\max_{i,j}|\xi_{ij}-\bar{\xi}|/\sqrt{\sum(\xi_{ij}-\bar{\xi})^2}\xrightarrow{\text{a.s.}}0. \tag{2.14}$$

Denote by $A|B$ the conditional distribution of A given B. From the condition (B), (2.11), (2.12), (2.14) and central limit theorem, it follows that the following assertion holds true with probability one: As $k\to\infty$, the conditional distribution

$$\frac{W_k-m_k}{\sqrt{(N_k\sigma_1^2+S)\sigma_2^2}}\bigg|\{\delta_{ij}\}\xrightarrow{\text{d.f.}}N(0,1). \tag{2.15}$$

Now turn to m_k. By (2.10), write $m_k = \beta(T_k + N_k\bar{\delta}^2)$, where

$$\begin{aligned} T_k &= \Big(N_k\hat{\sigma}_1^2 - \sum \delta_{ij}^2\Big) - \sum (x_i - \bar{x})\delta_{ij} \\ &= \frac{k}{N_k - k}\sum \delta_{ij}^2 - \frac{N_k}{N_k - k}\sum n_i\delta_i^2 - \sum n_i (x_i - \bar{x})\,\delta_i \\ &= \frac{k}{N_k - k}\sum \left(\delta_{ij}^2 - \sigma_1^2\right) - \frac{N_k}{N_k - k}\sum \left(n_i\delta_i^2 - \sigma_1^2\right) - \sum n_i (x_i - \bar{x})\,\delta_i. \end{aligned} \tag{2.16}$$

Let

$$Y_{ki} = \frac{k}{N_k - k}\sum_{j=1}^{n_i} \left(\delta_{ij}^2 - \sigma_1^2\right) - \frac{N_k}{N_k - k}\left(n_i\delta_i^2 - \sigma_1^2\right) - n_i (x_i - \bar{x})\,\delta_i, \quad EY_{ki} = 0,$$

$$\begin{aligned} EY_{ki}^2 = &\Big(\frac{k}{N_k - k}\Big)^2 n_i \left(E\delta_{11}^4 - \sigma_1^4\right) + \Big(\frac{N_k}{N_k - k}\Big)^2 \left[\left(E\delta_{11}^4 - 3\sigma_1^4\right)/n_i + 2\sigma_1^4\right] \\ &+ n_i (x_i - \bar{x})^2 \sigma_1^2 - \frac{2kN_k}{(N_k - k)^2}\left(E\delta_{11}^4 - \sigma_1^4\right). \end{aligned}$$

Let

$$B_k^2 = \sum EY_{ki}^2 = \frac{N_k\left(N_k\sum n_i^{-1} - k^2\right)}{(N_k - k)^2}\left(E\delta_{11}^4 - \sigma_1^4\right) + \frac{2N_k^2\left(k - \sum n_i^{-1}\right)}{(N_k - k)^2}\sigma_1^4 + S\sigma_1^2. \tag{2.17}$$

From Lemma 2.3 and $E\,|\delta_{11}|^{4+r} < \infty$, simple calculations show that

$$E\Big|\sum_{j=1}^{n_i} \left(\delta_{ij}^2 - \sigma_1^2\right)\Big|^{2+r/2} \le Cn_i^{1+r/4}, \tag{2.18}$$

$$E|n_i\delta_i^2 - \sigma_1^2|^{2+r/2} \le C, \tag{2.19}$$

$$E|n_i (x_i - \bar{x})\,\delta_i|^{2+r/2} \le C[n_i (x_i - \bar{x})^2]^{1+r/4}. \tag{2.20}$$

By the condition (3), it follows that

$$k/(N_k - k) = O(1), \quad N_k/(N_k - k) = O(1).$$

Thus together with (2.18), (2.19) and (2.20) we obtain

$$\sum E|Y_{ki}|^{2+r/2} = O\Big(\sum \Big(n_i^{1+r/4} + [n_i (x_i - \bar{x})^2]^{1+r/4}\Big)\Big). \tag{2.21}$$

By the condition (2), (2.2) and (2.21), it follows that

$$\frac{1}{B_k^{2+r/2}}\sum E|Y_{ki}|^{2+r/2} = o\,(1)\,. \tag{2.22}$$

Since $T_k = \sum Y_{ki}$, from (2.17), (2.22) and central limit theorem, we have

$$T_k/B_k \xrightarrow{\text{d.f.}} N(0,1). \tag{2.23}$$

Since $EN_k\bar{\delta}^2/B_k = \sigma_1^2/B_k \to 0$, together with (2.23) we obtain

$$m_k/(\beta B_k) \xrightarrow{\text{d.f.}} N(0,1). \tag{2.24}$$

Summarizing above it follows that the following assertion holds true with probability one: As $k \to \infty$, the conditional distribution $\frac{W_k}{\sqrt{(N_k\sigma_1^2 + S)\sigma_2^2}}\Big|\{\delta_{ij}\}$ tends to the distribution of $Y_1 + Y_2$, where Y_1, Y_2 are independent, and

$$Y_1 \sim N\left(0, \beta^2 B_k^2/[\left(N_k\sigma_1^2 + S\right)\sigma_2^2]\right), Y_2 \sim N(0,1).$$

Hence

$$\frac{W_k}{\sqrt{(N_k\sigma_1^2+S)\,\sigma_2^2+\beta^2B_k^2}} \xrightarrow{\text{d.f.}} N(0,1). \tag{2.25}$$

Returning to (2.8), and noticing (2.2), we obtain

$$\frac{S\left(\hat{\beta}-\beta\right)}{\sqrt{(N_k\sigma_1^2+S)\,\sigma_2^2+\beta^2B_k^2}} \xrightarrow{\text{d.f.}} N(0,1).$$

Therefore we prove Theorem 2.1.

From the discussion above, we can see that if the distribution of δ_{11} is normality, by using the same method as in [4], the conditions can be simplified. We have the following

Theorem 2.2. *If, in addition to* $(A), (B)$ *and* (2.2), *the following conditions are satisfied*:

(1) $\delta_{11}\sim N\left(0,\sigma_1^2\right)$,

(2) $\max\limits_{1\le i\le k}|x_i-\bar{x}|/\sqrt{S}\to 0$ *as* $k\to\infty$,

then

$$\frac{S\left(\hat{\beta}-\beta\right)}{\sqrt{(\beta^2\sigma_1^2+\sigma_2^2)\,S+N_k\sigma_1^2\sigma_2^2+2kN_k/\left(N_k-k\right)\beta^2\sigma_1^4}} \xrightarrow{\text{d.f.}} N(0,1). \tag{2.26}$$

§3. Asymptotic Normality of $\hat{\alpha}$

Theorem 3.1. *If, in addition to* $(A), (B)$ *and* (2.2), *the following conditions are satisfied*:

(1) $\exists r>0, E\left|\delta_{11}\right|^{4+r}<\infty$,

(2) $\max\limits_{1\le i\le k} n_i/N_k\to 0$, $\max\limits_{1\le i\le k} n_i\left(x_i-\bar{x}\right)^2/S\to 0$ *as* $k\to\infty$,

(3) $\liminf\limits_{k\to\infty} N_k/k>1$;

then

$$S(\hat{\alpha}-\alpha)\Big/\Big\{N_k\bar{x}^2\sigma_1^2\sigma_2^2+(\beta^2\sigma_1^2+\sigma_2^2)\Big(S\bar{x}^2+\frac{S^2}{N_k}\Big)+\beta^2\bar{x}^2\Big[\frac{N_k}{\left(N_k-k\right)^2}$$
$$\cdot\Big(N_k\sum_{i=1}^k n_i^{-1}-k^2\Big)\left(E\delta_{11}^4-3\sigma_1^4\right)+\frac{2N_kk}{N_k-k}\sigma_1^4\Big]\Big\}^{1/2} \xrightarrow{\text{d.f.}} N(0,1). \tag{3.1}$$

Proof. By (1.4), it follows that

$$\hat{\alpha}-\alpha=\bar{\xi}\left(\beta-\hat{\beta}\right)+\bar{\varepsilon}-\beta\bar{\delta}\equiv W_k'/Q_k, \tag{3.2}$$

where $W_k'=-\bar{\xi}W_k-Q_k\beta\bar{\delta}+Q_k\bar{\varepsilon}$, W_k is defined in (2.9). Let

$$m_k'=-\bar{\xi}m_k-Q_k\beta\bar{\delta}, \tag{3.3}$$

where m_k is defined in (2.10). We obtain

$$W_k'-m_k'=-\bar{\xi}\sum\left(\xi_{ij}-\bar{\xi}\right)\varepsilon_{ij}+Q_k\bar{\varepsilon}=\sum\left[Q_k/N_k-\bar{\xi}\left(\xi_{ij}-\bar{\xi}\right)\right]\varepsilon_{ij}. \tag{3.4}$$

Since

$$\sum\left[Q_k/N_k-\bar{\xi}\left(\xi_{ij}-\bar{\xi}\right)\right]^2=Q_k^2/N_k+\bar{\xi}^2\sum\left(\xi_{ij}-\bar{\xi}\right)^2,\quad \bar{\delta}=\bar{\xi}-\bar{x}\xrightarrow{\text{a.s.}}0,$$

by Lemma 2.2 and (2.12) we have

$$\sum\left[Q_k/N_k-\bar{\xi}\left(\xi_{ij}-\bar{\xi}\right)\right]^2/\left[S^2/N_k+\bar{x}^2\left(N_k\sigma_1^2+S\right)\right]\xrightarrow{\text{a.s.}}1. \tag{3.5}$$

Since

$$\max_{i,j}|Q_k/N_k-\bar{\xi}(\xi_{ij}-\bar{\xi})|\le Q_k/N_k+|\bar{\xi}|[2\max_{i,j}|\delta_{ij}|+\max_i|\sqrt{n_i}(x_i-\bar{x})|],$$

from the conditions (1), (2), Lemma 2.2 and Lemma 2.4, it follows that

$$\max_{i,j}|Q_k/N_k-\bar{\xi}\left(\xi_{ij}-\bar{\xi}\right)|/\sqrt{S^2/N_k+\bar{x}^2\left(N_k\sigma_1^2+S\right)}\xrightarrow{\text{a.s.}}0. \tag{3.6}$$

By (3.5) and (3.6), it follows that

$$\max_{i,j}|Q_k/N_k-\bar{\xi}\left(\xi_{ij}-\bar{\xi}\right)|/\sqrt{\sum\left[Q_k/N_k-\bar{\xi}\left(\xi_{ij}-\bar{\xi}\right)\right]^2}\xrightarrow{\text{a.s.}}0. \tag{3.7}$$

From the condition (B), (3.4), (3.5), (3.7) and central limit theorem, it follows that the following assertion holds true with probability one: As $k\to\infty$, the conditional distribution

$$\frac{W_k'-m_k'}{\sqrt{\left[S^2/N_k+\bar{x}^2\left(N_k\sigma_1^2+S\right)\right]\sigma_2^2}}\Big|\{\delta_{ij}\}\xrightarrow{\text{d.f.}}N(0,1). \tag{3.8}$$

Now turn to m_k'. By (3.3), (2.1) and (2.10), it follows that

$$\begin{aligned}m_k'&=-\bar{\xi}\Big[\beta\Big(N_k\hat{\sigma}_1^2-\sum\left(\delta_{ij}-\bar{\delta}\right)^2\Big)-\beta\sum\left(x_i-\bar{x}\right)\delta_{ij}\Big]-\Big[\sum\left(\xi_{ij}-\bar{\xi}\right)^2-N_k\hat{\sigma}_1^2\Big]\beta\bar{\delta}\\&=-\bar{x}m_k-\beta S\bar{\delta}-\beta\bar{\delta}\sum\left(x_i-\bar{x}\right)\delta_{ij}\\&=-\beta\Big(\bar{x}T_k+\bar{x}N_k\bar{\delta}^2+S\bar{\delta}+\bar{\delta}\sum\left(x_i-\bar{x}\right)\delta_{ij}\Big),\end{aligned} \tag{3.9}$$

where T_k is defined in (2.16). Write $m_k'=-\beta\left(T_k'+\bar{x}N_k\bar{\delta}^2+\bar{\delta}\sum\left(x_i-\bar{x}\right)\delta_{ij}\right)$, where

$$\begin{aligned}T_k'=\bar{x}T_k+S\bar{\delta}&=\frac{k\bar{x}}{N_k-k}\sum\left(\delta_{ij}^2-\sigma_1^2\right)-\frac{N_k\bar{x}}{N_k-k}\sum\left(n_i\delta_i^2-\sigma_1^2\right)\\&\quad-\sum n_i\left[\left(x_i-\bar{x}\right)\bar{x}-S/N_k\right]\delta_i.\end{aligned} \tag{3.10}$$

Let

$$Y_{ki}'=\frac{k\bar{x}}{N_k-k}\sum_{j=1}^{n_i}\left(\delta_{ij}^2-\sigma_1^2\right)-\frac{N_k\bar{x}}{N_k-k}\left(n_i\delta_i^2-\sigma_1^2\right)-n_i\left[\left(x_i-\bar{x}\right)\bar{x}-S/N_k\right]\delta_i,\ EY_{ki}'=0,$$

$$\begin{aligned}EY_{ki}'^2&=\Big(\frac{k\bar{x}}{N_k-k}\Big)^2n_i\left(E\delta_{11}^4-\sigma_1^4\right)+\Big(\frac{N_k\bar{x}}{N_k-k}\Big)^2\left[\left(E\delta_{11}^4-3\sigma_1^4\right)/n_i+2\sigma_1^4\right]\\&\quad+n_i\left[\left(x_i-\bar{x}\right)\bar{x}-S/N_k\right]^2\sigma_1^2-\frac{2kN_k\bar{x}^2}{\left(N_k-k\right)^2}\left(E\delta_{11}^4-\sigma_1^4\right).\end{aligned}$$

Let

$$\begin{aligned}B_k'^2=\sum EY_{ki}'^2&=\frac{\bar{x}^2N_k\left(N_k\sum n_i^{-1}-k^2\right)}{\left(N_k-k\right)^2}\left(E\delta_{11}^4-\sigma_1^4\right)+\frac{2\bar{x}^2N_k^2\left(k-\sum n_i^{-1}\right)}{\left(N_k-k\right)^2}\sigma_1^4\\&\quad+\left(S\bar{x}^2+S^2/N_k\right)\sigma_1^2.\end{aligned} \tag{3.11}$$

By (2.18), (2.19), (2.20) and the condition (3), we have

$$\begin{aligned}\sum E|Y_{ki}'|^{2+r/2}=O\Big(&\sum(n_i^{1+r/4}+[n_i(x_i-\bar{x})^2]^{1+r/4})|\bar{x}|^{2+r/2}\\&+(S/N_k)^{2+r/2}\sum n_i^{1+r/4}\Big).\end{aligned} \tag{3.12}$$

By the condition (2), (2.2) and (3.12) we obtain

$$\frac{1}{B_k'^{2+r/2}}\sum E\left|Y_{ki}'\right|^{2+r/2}=o(1). \tag{3.13}$$

Since $T'_k = \sum Y'_{ki}$, from (3.11), (3.13) and central limit theorem, it follows that

$$T'_k / B'_k \xrightarrow{\text{d.f.}} N(0,1). \tag{3.14}$$

Since

$$E\left|\bar{\delta}\sum\left(x_i-\bar{x}\right)\delta_{ij}/B'_k\right|^2 \le B_k'^{-2}E\bar{\delta}^2E\left|\sum\left(x_i-\bar{x}\right)\delta_{ij}\right|^2 = \sigma_1^4 S/\left(N_k B_k'^2\right) \to 0,$$

$E\left|\bar{x}N_k\bar{\delta}^2/B'_k\right| = |\bar{x}|\,\sigma_1^2/B'_k \to 0$, together with (3.14) we obtain

$$m'_k/\left(\beta B'_k\right) \xrightarrow{\text{d.f.}} N(0,1). \tag{3.15}$$

Summarizing above it follows that the following assertion holds true with probability one: As $k\to\infty$, the conditional distribution $\frac{W'_k}{\sqrt{[S^2/N_k+\bar{x}^2\left(N_k\sigma_1^2+S\right)]\sigma_2^2}}\Big|\{\delta_{ij}\}$ tends to the distribution of $Y'_1+Y'_2$, where Y'_1, Y'_2 are independent, and

$$Y'_1 \sim N\left(0, \beta^2 B_k'^2/\{[S^2/N_k+\bar{x}^2\left(N_k\sigma_1^2+S\right)]\sigma_2^2\}\right), Y'_2 \sim N(0,1).$$

Hence

$$\frac{W'_k}{\sqrt{[S^2/N_k+\bar{x}^2\left(N_k\sigma_1^2+S\right)]\,\sigma_2^2+\beta^2 B_k'^2}} \xrightarrow{\text{d.f.}} N(0,1). \tag{3.16}$$

Returning to (3.2), and noticing (2.2), we obtain

$$\frac{S\left(\hat{\alpha}-\alpha\right)}{\sqrt{[S^2/N_k+\bar{x}^2\left(N_k\sigma_1^2+S\right)]\,\sigma_2^2+\beta^2 B_k'^2}} \xrightarrow{\text{d.f.}} N(0,1).$$

Therefore we prove Theorem 3.1.

From the discussion above, we can see that if the distribution of δ_{11} is normality, by using the same method as in [4], the conditions can be simplified. We have the following

Theorem 3.2. *If, in addition to* $(A), (B)$ *and* (2.2)*, the following conditions are satisfied*:

(1) $\delta_{11} \sim N\left(0, \sigma_1^2\right)$,

(2) $\max\limits_{1\le i\le k}|x_i-\bar{x}|/\sqrt{S} \to 0$ *as* $k\to\infty$;

then

$$\frac{S\left(\hat{\alpha}-\alpha\right)}{\left\{N_k\bar{x}^2\sigma_1^2\sigma_2^2+\left(\beta^2\sigma_1^2+\sigma_2^2\right)\left(S\bar{x}^2+S^2/N_k\right)+\beta^2\bar{x}^2\frac{2N_k k}{N_k-k}\sigma_1^4\right\}^{1/2}} \xrightarrow{\text{d.f.}} N(0,1). \tag{3.17}$$

§4. Estimation of $E\delta_{11}^4$

Since $\hat{\sigma}_1^2, \hat{\sigma}_2^2, Q_k, \bar{\xi}$ and $\hat{\beta}, \hat{\alpha}$ are consistent estimates of $\sigma_1^2, \sigma_2^2, S, \bar{x}$ and β, α respectively, we can replace the latter by the former in the denominator of (2.7) and (3.1) respectively without invalidating the asymptotic normality. In order that the modified form can be used to make interval estimation and hypothesis testing of β and α, in the large-sample sense, we need to estimate $E\delta_{11}^4$. For estimating $E\delta_{11}^4$, we use

$$\hat{\mu}_4 = \left[\sum\left(\xi_{ij}-\xi_i\right)^4 - \sum\left(6-15/n_i+9/n_i^2\right)\hat{\sigma}_1^4\right]/\sum\left(n_i-4+6/n_i-3/n_i^2\right). \tag{4.1}$$

Remark 4.1. From (4.1) we can see that those $\{\xi_{ij}, 1\le j\le n_i\}$ with n_i equaling 1 do not contribute in estimating $\hat{\mu}_4$. When we discuss the consistency of $\hat{\mu}_4$, we can without loss of generality assume that all $n_i \ge 2$ in this section.

Lemma 4.1.[7] *Let* $\{x_n\}$ *be a sequence of independent random variables with zero means. If* $a_n \uparrow \infty$ *and* $\sum\limits_{n=1}^{\infty} E|x_n|^p/a_n^p < \infty$ *for some* $p, 1\le p\le 2$*, letting* $S_n = \sum\limits_{i=1}^{n} x_i$*, then* $S_n/a_n \xrightarrow{\text{a.s}} 0$.

Lemma 4.2.[8] *Let $\{x_i\}_{i=1}^n$ be independent random variables with zero means and finite absolute moments of order $p, 1 \le p < 2$. Then*

$$E\left|\sum_{i=1}^n x_i\right|^p \le 2\sum_{i=1}^n E\left|x_i\right|^p.$$

Lemma 4.3. *Let $\{\delta_{ij}\}$ be an i.i.d sequence with zero means. Write*

$$\mu_4 = \left[\sum\left(\delta_{ij}-\delta_i\right)^4 - \sum\left(6-15/n_i+9/n_i^2\right)\sigma_1^4\right] / \sum\left(n_i-4+6/n_i-3/n_i^2\right). \tag{4.2}$$

(1) *If $E\left|\delta_{11}\right|^{4+r} < \infty$ for some $r>0$, then $\mu_4 \xrightarrow{\text{pr.}} E\delta_{11}^4$.*

(2) *If $n_i \le N$ for all i and $E\left|\delta_{11}\right|^{4+r} < \infty$ for some $r>0$, then $\mu_4 \xrightarrow{\text{a.s.}} E\delta_{11}^4$.*

(3) *If δ_{11} is symmetric distribution and $E\delta_{11}^6 < \infty$, then $\mu_4 \xrightarrow{\text{a.s.}} E\delta_{11}^4$.*

Proof. Since $n_i \ge 2$ for all i, this shows that $n_i - 4 + 6/n_i - 3/n_i^2 \ge n_i/8$. Therefore

$$N_k / \sum\left(n_i-4+6/n_i-3/n_i^2\right) \le 8. \tag{4.3}$$

Let

$$Y_k = \sum\left(\delta_{ij}-\delta_i\right)^4 - \sum\left(6-15/n_i+9/n_i^2\right)\sigma_1^4 - \sum\left(n_i-4+6/n_i-3/n_i^2\right)E\delta_{11}^4.$$

From $Y_k = \sum\left(n_i-4+6/n_i-3/n_i^2\right)\left(\mu_4 - E\delta_{11}^4\right)$ and (4.3), we only need to verify that

$$Y_k/N_k \to 0 \quad \text{a.s. or pr.} \tag{4.4}$$

Simple calculations show that

$$\begin{aligned} Y_k = & \sum\left(\delta_{ij}^4 - E\delta_{11}^4\right) - 4\sum\left[\sum_{j=1}^{n_i}\delta_{ij}^3\delta_i - E\delta_{11}^4\right] \\ & + 6\sum\left[\sum_{j=1}^{n_i}\delta_{ij}^2\delta_i^2 - E\delta_{11}^4/n_i - \sigma_1^4\left(n_i-1\right)/n_i\right] \\ & - 3\sum\left[n_i\delta_i^4 - E\delta_{11}^4/n_i^2 - 3\sigma_1^4\left(n_i-1\right)/n_i^2\right] \\ \equiv & L_{1k} - 4L_{2k} + 6L_{3k} - 3L_{4k}. \end{aligned} \tag{4.5}$$

By Lemma 2.3, it follows that

$$E\left|\delta_i\right|^{4+r} = n_i^{-(4+r)}E\left|\sum_{j=1}^{n_i}\delta_{ij}\right|^{4+r} \le C_{4+r}n_i^{-(2+r/2)}E\left|\delta_{11}\right|^{4+r}. \tag{4.6}$$

Hence

$$\begin{aligned} & E\left|n_i\delta_i^4 - E\delta_{11}^4/n_i^2 - 3\sigma_1^4\left(n_i-1\right)/n_i^2\right|^{1+r/4} \\ \le & 3^{r/4}\left[E\left|n_i\delta_i^4\right|^{1+r/4} + \left(E\delta_{11}^4\right)^{1+r/4} + 3^{1+r/4}\sigma_1^{4+r}\right] \le C. \end{aligned}$$

In the following C will denote a constant, although not necessarily the same constant each time. Thus

$$\sum_{i=1}^{\infty} E\left|n_i\delta_i^4 - E\delta_{11}^4/n_i^2 - 3\sigma_1^4\left(n_i-1\right)/n_i^2\right|^{1+r/4} / N_i^{1+r/4} < \infty. \tag{4.7}$$

Combining $En_i\delta_i^4 = E\delta_{11}^4/n_i^2 + 3\sigma_1^4\left(n_i-1\right)/n_i^2$ and Lemma 4.1, it can be shown that

$$L_{4k}/N_k \equiv \sum\left[n_i\delta_i^4 - E\delta_{11}^4/n_i^2 - 3\sigma_1^4\left(n_i-1\right)/n_i^2\right]/N_k \xrightarrow{\text{a.s.}} 0. \tag{4.8}$$

From (4.6) it follows that

$$\begin{aligned} E\Big|\sum_{j=1}^{n_i}\delta_{ij}^2\delta_i^2\Big|^{1+r/4} &\le n_i^{r/4}\sum_{j=1}^{n_i}E\left|\delta_{ij}^{2+r/2}\delta_i^{2+r/2}\right| \\ &\le n_i^{1+r/4}\left(E\left|\delta_{11}^{4+r}\right|\cdot E\left|\delta_i^{4+r}\right|\right)^{1/2} \\ &\le \sqrt{C_{4+r}}E\left|\delta_{11}\right|^{4+r}. \end{aligned} \tag{4.9}$$

Hence

$$\begin{aligned} &E\Big|\sum_{j=1}^{n_i}\delta_{ij}^2\delta_i^2 - E\delta_{11}^4/n_i - \sigma_1^4\left(n_i-1\right)/n_i\Big|^{1+r/4} \\ &\le 3^{r/4}\Big[E\Big|\sum_{j=1}^{n_i}\delta_{ij}^2\delta_i^2\Big|^{1+r/4} + \left(E\delta_{11}^4\right)^{1+r/4} + \sigma_1^{4+r}\Big] \le C. \end{aligned}$$

Thus

$$\sum_{i=1}^{\infty}E\Big|\sum_{j=1}^{n_i}\delta_{ij}^2\delta_i^2 - E\delta_{11}^4/n_i - \sigma_1^4\left(n_i-1\right)/n_i\Big|^{1+r/4}\Big/N_i^{1+r/4} < \infty. \tag{4.10}$$

Combining

$$E\sum_{j=1}^{n_i}\delta_{ij}^2\delta_i^2 = E\delta_{11}^4/n_i + \sigma_1^4\left(n_i-1\right)/n_i$$

and Lemma 4.1, it follows that

$$L_{3k}/N_k \equiv \sum\Big[\sum_{j=1}^{n_i}\delta_{ij}^2\delta_i^2 - E\delta_{11}^4/n_i - \sigma_1^4\left(n_i-1\right)/n_i\Big]\Big/N_k \xrightarrow{\text{a.s.}} 0. \tag{4.11}$$

By Kolmogorov's SLLN it follows that

$$L_{1k}/N_k \equiv \sum\left(\delta_{ij}^4 - E\delta_{11}^4\right)/N_k \xrightarrow{\text{a.s.}} 0. \tag{4.12}$$

Combining (4.5), (4.8), (4.11) and (4.12), in order to prove (4.4) we only need to verify

$$L_{2k}/N_k \equiv \sum\Big[\sum_{j=1}^{n_i}\delta_{ij}^3\delta_i - E\delta_{11}^4\Big]/N_k \to 0 \quad \text{a.s. or pr.} \tag{4.13}$$

Since

$$E\Big|\sum_{j=1}^{n_i}\delta_{ij}^3\Big|^{(4+r)/3} \le n_i^{(4+r)/3}E\left|\delta_{11}\right|^{4+r},$$

by the Holder inequality and (4.6), we obtain

$$\begin{aligned} &E\Big|\sum_{j=1}^{n_i}\delta_{ij}^3\delta_i\Big|^{1+r/4} \le \Big(E\Big|\sum_{j=1}^{n_i}\delta_{ij}^3\Big|^{(4+r)/3}\Big)^{3/4}\Big(E\left|\delta_i\right|^{4+r}\Big)^{1/4} \\ &\le n_i^{(4+r)/8}\cdot\left(C_{4+r}\right)^{1/4}E\left|\delta_{11}\right|^{4+r}. \end{aligned} \tag{4.14}$$

Hence

$$\begin{aligned} E\Big|\sum_{j=1}^{n_i}\delta_{ij}^3\delta_i - E\delta_{11}^4\Big|^{1+r/4} &\le 2^{r/4}\Big[n_i^{(4+r)/8}\cdot\left(C_{4+r}\right)^{1/4}E\left|\delta_{11}\right|^{4+r} + \left(E\delta_{11}^4\right)^{1+r/4}\Big] \\ &\le C\cdot n_i^{(4+r)/8}. \end{aligned} \tag{4.15}$$

Note that

$$E\sum_{j=1}^{n_i}\delta_{ij}^3\delta_i = E\delta_{11}^4.$$

By Lemma 4.3 it follows that

$$\begin{aligned} E\left|L_{2k}/N_k\right|^{1+r/4} &\equiv E\Big|\sum\Big[\sum_{j=1}^{n_i}\delta_{ij}^3\delta_i - E\delta_{11}^4\Big]/N_k\Big|^{1+r/4} \\ &\le 2C\sum n_i^{(4+r)/8}/N_k^{1+r/4} \to 0. \end{aligned} \tag{4.16}$$

Thus

$$L_{2k}/N_k \equiv \sum\Big[\sum_{j=1}^{n_i}\delta_{ij}^3\delta_i - E\delta_{11}^4\Big]/N_k \xrightarrow{pr.} 0,$$

we prove Lemma 4.3 (1).

If $n_i \le N$ for all i, by (4.15) it follows that

$$\sum_{i=1}^{\infty}E\Big|\sum_{j=1}^{n_i}\delta_{ij}^3\delta_i - E\delta_{11}^4\Big|^{1+r/4}/N_i^{1+r/4} < \infty. \tag{4.17}$$

By Lemma 4.1,

$$L_{2k}/N_k \equiv \sum\Big[\sum_{j=1}^{n_i}\delta_{ij}^3\delta_i - E\delta_{11}^4\Big]/N_k \xrightarrow{\text{a.s.}} 0,$$

we prove Lemma 4.3 (2).

If δ_{11} is symmetric distribution and $E\delta_{11}^6 < \infty$, by Lemma 2.3 it follows that

$$\begin{aligned} E\Big|\sum_{j=1}^{n_i}\delta_{ij}^3\delta_i\Big|^{3/2} &\le \Big(E\Big|\sum_{j=1}^{n_i}\delta_{ij}^3\Big|^2\Big)^{3/4}\Big(E\left|\delta_i\right|^6\Big)^{1/4} \\ &\le \Big(n_iE\left|\delta_{11}\right|^6\Big)^{3/4}\Big(C_6n_i^{-3}E\left|\delta_{11}\right|^6\Big)^{1/4} \\ &= (C_6)^{1/4}E\left|\delta_{11}\right|^6. \end{aligned} \tag{4.18}$$

Hence

$$E\Big|\sum_{j=1}^{n_i}\delta_{ij}^3\delta_i - E\delta_{11}^4\Big|^{3/2} \le C.$$

We obtain

$$\sum_{i=1}^{\infty}E\Big|\sum_{j=1}^{n_i}\delta_{ij}^3\delta_i - E\delta_{11}^4\Big|^{3/2}/N_i^{3/2} < \infty. \tag{4.19}$$

By Lemma 4.1,

$$L_{2k}/N_k \equiv \sum\Big[\sum_{j=1}^{n_i}\delta_{ij}^3\delta_i - E\delta_{11}^4\Big]/N_k \xrightarrow{\text{a.s.}} 0,$$

we prove Lemma 4.3 (3).

Theorem 4.1. *Let $\{\delta_{ij}\}$ be an i.i.d sequence with zero means.*

(1) *If $E\left|\delta_{11}\right|^{4+r} < \infty$ for some $r > 0$, then $\hat{\mu}_4 \xrightarrow{\text{pr.}} E\delta_{11}^4$.*

(2) *If $n_i \le N$ for all i and $E\left|\delta_{11}\right|^{4+r} < \infty$ for some $r > 0$, then $\hat{\mu}_4 \xrightarrow{\text{a.s.}} E\delta_{11}^4$.*

(3) *If δ_{11} is symmetric distribution and $E\delta_{11}^6 < \infty$, then $\hat{\mu}_4 \xrightarrow{\text{a.s.}} E\delta_{11}^4$.*

Proof. By Lemma 4.3, we only need to verify

$$\hat{\mu}_4 - \mu_4 \xrightarrow{\text{a.s.}} 0. \tag{4.20}$$

From (4.1), (4.2) and

$$\xi_{ij} - \xi_i = \delta_{ij} - \delta_i,$$

it follows that

$$\hat{\mu}_4 - \mu_4 = \Big[\sum \left(6 - 15/n_i + 9/n_i^2\right) \Big/ \sum \left(n_i - 4 + 6/n_i - 3/n_i^2\right)\Big] \left(\sigma_1^4 - \hat{\sigma}_1^4\right). \tag{4.21}$$

Since $6 - 15/n_i + 9/n_i^2 < 3n_i$ and (4.3), this shows that

$$\sum \left(6 - 15/n_i + 9/n_i^2\right) \Big/ \sum \left(n_i - 4 + 6/n_i - 3/n_i^2\right) < 24 \tag{4.22}$$

By (4.21), (4.22) and $\hat{\sigma}_1^2 \xrightarrow{\text{a.s.}} \sigma_1^2$, we prove (4.20). Therefore Theorem 4.1 is proved.

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