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A HOPF INDEX THEOREM FOR A REAL VECTOR BUNDLE

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Abstract

The authors generalize the works in [5] and [6] to prove a Hopf index theorem associated to a smooth section of a real vector bundle with non-isolated zero points.

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§1. Introduction

Let X be a closed and oriented manifold of dimension $2n$. Let $E \to X$ be an oriented real vector bundle of rank $2n$. Let v be a smooth section of E. We will denote the set of zero points of v by Y.

When v is a transversal section of E, the set Y consists of isolated points. In [5], we got a purely analytic proof of a Hopf index theorem associated to v (see [4, Theorem 11.17]) by constructing a super-twisted signature operator.

In this paper, using Witten's deformation idea[8] and Bismut-Lebeau's technique (see [2, Sections 8, 9]) and also the super-twisted signature operator defined in [5], we will prove a Hopf index theorem associated to a smooth section v of E with non-isolated zero point set which is nondegenerate in the sense of Bott[3].

Let p be a zero point of a smooth section v of E. There is a linear map $\mathcal{L}_v(p)$ from the tangent space T_pX to the fibre $E|_p$ of E at p defined by

$$\mathcal{L}_v(p)(U) = \sum_{i=1}^{2n}(U\mu_i)\xi_i(p), \quad \forall U \in T_pX, \tag{1.1}$$

where $\{\xi_1, \xi_2, \cdots, \xi_{2n}\}$ is a basis for E around p and $v = \sum_{i=1}^{2n} \mu_i\xi_i$ for some smooth functions μ_i defined near p. Clearly, the definition of $\mathcal{L}_v(p)$ does not depend on the choice of the basis

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$\{\xi_1, \xi_2, \cdots, \xi_{2n}\}$. When restricted to the zero point set Y of υ, by using (1.1) pointwisely on Y, we get an intrinsic well-defined bundle homomorphism

$$\mathcal{L}_\upsilon : TX|_Y \to E|_Y. \tag{1.2}$$

Definition 1.1. *A smooth section υ of E is said to be nondegenerate in the sense of Bott*[3] *if*

(i) *the zero point set Y of υ can be expressed as a finite disjoint union*

$$Y = \bigcup_{k=1}^{m} Y_k \tag{1.3}$$

of some oriented and connected submanifolds Y_k of X;

(ii) *for each k, the kernel of the restriction $\mathcal{L}_{\upsilon,k}$ of $\mathcal{L}_\upsilon$ on Y_k is equal to TY_k.*

For each k, since $TX|_{Y_k}$ and TY_k are oriented vector bundles over Y_k, we can give an induced orientation on the normal bundle $(TX|_{Y_k})/TY_k$ of TY_k such that the orientation on TY_k and then the induced orientation on $(TX|_{Y_k})/TY_k$ together coincides with the orientation on $TX|_{Y_k}$. From Definition 1.1, the image $\mathcal{L}_{\upsilon,k}(TX|_{Y_k})$ is isomorphic to $(TX|_{Y_k})/TY_k$. So $\mathcal{L}_{\upsilon,k}(TX|_{Y_k})$ inherits an orientation from what on $(TX|_{Y_k})/TY_k$. Thus we can also give an induced orientation on the quotient bundle $(E|_{Y_k})/\mathcal{L}_{\upsilon,k}(TX|_{Y_k})$ such that the orientation on $(E|_{Y_k})/\mathcal{L}_{\upsilon,k}(TX|_{Y_k})$ and then the orientation on $\mathcal{L}_{\upsilon,k}(TX|_{Y_k})$ together coincides with the orientation on $E|_{Y_k}$.

With above data in hand, we can state the following Hopf index theorem associated to a smooth section υ of E which is nondegenerate in the sense of Bott.

Theorem 1.1.

$$\chi(E) = \sum_{k=1}^{m} \chi\left((E|_{Y_k})/\mathcal{L}_{\upsilon,k}(TX|_{Y_k})\right), \tag{1.4}$$

where $\chi(E)$ as well as $\chi\left((E|_{Y_k})/\mathcal{L}_{\upsilon,k}(TX|_{Y_k})\right)$'s are the Euler characteristics of corresponding bundles, respectively.

§2. Proof of Theorem 1.1

This section is divided in three parts. In **(a)**, we recall the definition of a super-twisted signature operator and its deformation by a section υ of E defined in [5]. Then we study the local behavior of the associated deformed operator near the zero point set Y of υ when υ is nondegenerate in the sense of Bott. In **(b)** we define for each Y_k a twisted Dirac operator and compute its index. In **(c)** we prove Theorem 1.1 by combining the technique of Bismut-Lebeau[2, Sections 8,9] and a trick of Zhang[9].

(a) A Deformed Super-Twisted Signature Operator and Its Local Behavior

Given a Riemannian metric g^{TX} on X, let ∇^{TX} denote the associated Levi-Civita connection. Given also an Euclidean inner product g^E on E, let ∇^E denote an Euclidean connection on E. Then g^{TX} and ∇^{TX} (resp. g^E and ∇^E) lift naturally to a Hermitian inner product and a Hermitian connection on the complex-valued exterior algebra bundle $\Lambda(T^*X)$ (resp. $\Lambda(E^*)$) which we will denote by $g^{\Lambda(T^*X)}$ and $\nabla^{\Lambda(T^*X)}$ (resp. $g^{\Lambda(E^*)}$ and $\nabla^{\Lambda(E^*)}$), respectively. For any $U \in TX$ and $\xi \in E$, set

$$c(U) = \varepsilon(U^*) - \iota(U), \qquad \widetilde{c}(\xi) = \varepsilon(\xi^*) - \iota(\xi), \tag{2.1}$$

where U^* and ξ^* correspond to U and ξ via g^{TX} and g^E respectively, and ε and ι are the standard notations of exterior and interior multiplications. Then for any $U, V \in TX$ and any $\xi, \eta \in E$, one has

$$\begin{aligned} c(U)c(V) + c(V)c(U) &= -2g^{TX}(U,V), \\ \widetilde{c}(\xi)\widetilde{c}(\eta) + \widetilde{c}(\eta)\widetilde{c}(\xi) &= -2g^E(\xi,\eta). \end{aligned} \tag{2.2}$$

Recall that in [5] the $\mathbf{Z}_2$-grading $\Lambda(T^*X) = \Lambda_+(T^*X) \oplus \Lambda_-(T^*X)$ (resp. $\Lambda(E^*) = \Lambda_+(E^*) \oplus \Lambda_-(E^*)$) is given by the involution $\tau_{\Lambda(T^*X)}$ (resp. $\tau_{\Lambda(E^*)}$) where with respect to any oriented orthonormal basis $\{e_1, e_2, \cdots, e_{2n}\}$ (resp. $\{\xi_1, \xi_2, \cdots, \xi_{2n}\}$) for TX (resp. E), we have (see [5])

$$\begin{aligned} &\tau_{\Lambda(T^*X)} = (\sqrt{-1})^n c(e_1)c(e_2)\cdots c(e_{2n}) \\ &(\text{resp.} \quad \tau_{\Lambda(E^*)} = (\sqrt{-1})^n \widetilde{c}(\xi_1)\widetilde{c}(\xi_2)\cdots\widetilde{c}(\xi_{2n})). \end{aligned} \tag{2.3}$$

From the data above, $\Lambda(T^*X)\hat{\otimes}\Lambda(E^*)$ is a $\mathbf{Z}_2$-graded Hermitian vector bundle with the Hermitian inner product $g^{\Lambda(T^*X)} \otimes g^{\Lambda(E^*)}$ and the Hermitian connection

$$\nabla^{\Lambda(T^*X)\hat{\otimes}\Lambda(E^*)} = \nabla^{\Lambda(T^*X)}\hat{\otimes}1 + 1\hat{\otimes}\nabla^{\Lambda(E^*)} \tag{2.4}$$

and also the $\mathbf{Z}_2$-grading

$$\Lambda(T^*X)\hat{\otimes}\Lambda(E^*) = \big(\Lambda(T^*X)\hat{\otimes}\Lambda(E^*)\big)_+ \oplus \big(\Lambda(T^*X)\hat{\otimes}\Lambda(E^*)\big)_- \tag{2.5}$$

given by the involution

$$\tau_{\Lambda(T^*X)\hat{\otimes}\Lambda(E^*)} = \tau_{\Lambda(T^*X)}\hat{\otimes}\tau_{\Lambda(E^*)}, \tag{2.6}$$

where

$$\big(\Lambda(T^*X)\hat{\otimes}\Lambda(E^*)\big)_\pm = (\Lambda_+(T^*X)\otimes\Lambda_\pm(E^*)) \oplus (\Lambda_-(T^*X)\otimes\Lambda_\mp(E^*)). \tag{2.7}$$

Note that $c(U)$ and $\widetilde{c}(\xi)$ act on $\Lambda(T^*X)\hat{\otimes}\Lambda(E^*)$ obviously by $c(U)\otimes 1$ and $1\otimes\widetilde{c}(\xi)$ for any $U \in TX$ and $\xi \in E$, respectively. Moreover, $c(U)$ and $\widetilde{c}(\xi)$ anticommute with the involution $\tau_{\Lambda(T^*X)\hat{\otimes}\Lambda(E^*)}$ and satisfy

$$c(U)\widetilde{c}(\xi) + \widetilde{c}(\xi)c(U) = 0. \tag{2.8}$$

In [5] we have defined a super-twisted signature operator D^X acting on the set $\Gamma(\Lambda(T^*X)\hat{\otimes}\Lambda(E^*))$ of smooth sections of $\Lambda(T^*X)\hat{\otimes}\Lambda(E^*)$. With respect to an orthonormal basis $\{e_1, e_2, \cdots, e_{2n}\}$ for TX, we have

$$D^X = \sum_{k=1}^{2n} c(e_i)\nabla_{e_i}^{\Lambda(T^*X)\hat{\otimes}\Lambda(E^*)}. \tag{2.9}$$

Denote the restrictions of D^X on $\Gamma\left((\Lambda(T^*X)\hat{\otimes}\Lambda(E^*))_\pm\right)$ by $D^X_\pm$. By Theorem 1.1 in [5], we have

$$\operatorname{ind} D^X_+ = (-1)^n 2^{2n}\chi(E). \tag{2.10}$$

Following Witten's deformation idea (see [8]), we define the following deformation of D^X by a smooth section $v \in \Gamma(E)$:

$$D^X_T = D^X + T\sqrt{-1}\widetilde{c}(v) : \Gamma\left(\Lambda(T^*X)\hat{\otimes}\Lambda(E^*)\right) \to \Gamma\left(\Lambda(T^*X)\hat{\otimes}\Lambda(E^*)\right). \tag{2.11}$$

Note that this deformation is a little different from what in [5].

When restricting D_T^X to $\Gamma\left((\Lambda(T^*X)\hat{\otimes}\Lambda(E^*))_+\right)$, we have

$$D_{T,+}^X : \Gamma\left((\Lambda(T^*X)\hat{\otimes}\Lambda(E^*))_+\right) \to \Gamma\left((\Lambda(T^*X)\hat{\otimes}\Lambda(E^*))_-\right), \tag{2.12}$$

$$\operatorname{ind} D_{T,+}^X = \operatorname{ind} D_+^X = (-1)^n 2^{2n}\chi(E). \tag{2.13}$$

By proceeding as the proof of Lemma 1.2 and Lemma 1.3 in [5], one verifies easily that the localization principle Lemma 1.3 in [5] also holds for the deformed super-twisted operator D_T^X here. Thus we can localize our problem and need only to concentrate on the analysis near the zero point set Y of the section υ.

In the following we assume that υ is a nondegenerate section of E in the sense of Bott. Set $l_k = \dim Y_k$ in (1.3). For simplicity, we will write Y (resp. l) instead of Y_k (resp. l_k) when no confusion appears.

Let $\pi : N \to Y$ be the orthogonal bundle to TY in $TX|_Y$. Denote $\mathcal{L}_\upsilon(N)$ by E_N and denote the orthogonal bundle to E_N in $E|_Y$ by E_Y, where $E|_Y$ is the restriction of E on Y. The following isomorphisms of vector bundles are clear:

$$N \cong (TX|_Y)/TY \cong E_N = \mathcal{L}_\upsilon(TX|_Y), \quad E_Y \cong (E|_Y)/\mathcal{L}_\upsilon(TX|_Y). \tag{2.14}$$

According to the choices of the orientations in Section 1, the vector bundles TY, N, E_Y and E_N are all oriented and $\mathcal{L}_\upsilon$ is an orientation-preserving isomorphism between N and E_N.

Since Theorem 1.1 is purely topological and does not depend on the metrics and connections on the bundles involved, we can and will choose g^{TX} such that Y is a totally geodesic submanifold of X. Thus the restriction $\nabla^{TX|_Y}$ of ∇^{TX} on $TX|_Y$ preserves $\Gamma(TY)$ and $\Gamma(N)$ respectively. When restricting $\nabla^{TX|_Y}$ to TY (resp. N), we get a connection ∇^{TY} (resp. ∇^N) on TY (resp. N). Clearly, ∇^{TY} is the Levi-Civita connection on Y associated to the restricted metric $g^{TY} = g^{TX}|_Y$. Denote the connections on $\Lambda(T^*Y)$ and $\Lambda(N^*)$ lifted from ∇^{TY} and ∇^N by $\nabla^{\Lambda(T^*Y)}$ and $\nabla^{\Lambda(N^*)}$, respectively. On the other hand, since $\mathcal{L}_\upsilon$ is an isomorphism between N and E_N, we can and will choose a Euclidean inner product g^E on E such that when restricted to Y, $\mathcal{L}_\upsilon$ is an isometry from N onto E_N. We can also choose a Euclidean connection ∇^E such that its restriction $\nabla^{E|_Y}$ on $E|_Y$ preserves $\Gamma(E_Y)$ and $\Gamma(E_N)$, respectively. Similarly, we can define the connections ∇^{E_Y}, ∇^{E_N}, $\nabla^{\Lambda(E_Y^*)}$ and $\nabla^{\Lambda(E_N^*)}$ on the bundles E_Y, E_N, $\Lambda(E_Y^*)$ and $\Lambda(E_N^*)$, respectively.

Let $\epsilon_0 > 0$ be such that for any $\epsilon \in (0, \epsilon_0)$, the set $B_\epsilon = \{Z \in N \mid |Z| < \epsilon\}$ can be identified with a tubular neighborhood $\mathcal{U}_\epsilon$ of Y in X by the exponential map $(y, Z) \to \exp_y^X(Z) \in X$, where $y \in Y$ and $Z \in N_y \cap B_\epsilon$. Let $\mathbf{W}$ (resp. W) be the set of smooth sections of $\pi^*\left((\Lambda^*(T^*X)\hat{\otimes}\Lambda(E^*))|_Y\right)$ (resp. $\Lambda^*(T^*X)\hat{\otimes}\Lambda(E^*)$). For any s_1, s_2 in $\mathbf{W}$ with compact support in B_ϵ, define

$$\langle s_1, s_2\rangle = \int_Y \Big(\int_{N_y} \langle s_1, s_2\rangle(y, Z)d\sigma_{N_y}(Z)\Big)d\sigma_Y(y), \tag{2.15}$$

where $d\sigma_Y$ and $d\sigma_{N_y}$ denote the volume elements of Y and fiber N_y at $y \in Y$, respectively. Denote the volume element of X by $d\sigma_X$. Let $k(y, Z)$ be such that

$$d\sigma_X(y, Z) = k(y, Z)d\sigma_Y(y)d\sigma_{N_y}(Z). \tag{2.16}$$

Then $k(y, Z)$ is a positive smooth function on $\mathcal{U}_\epsilon$ and $k(y, 0) = 1$.

Let W_ϵ (resp. $\mathbf{W}_\epsilon$) be the set of elements in W (resp. $\mathbf{W}$) with compact support in $\mathcal{U}_{\epsilon_0}$ (resp. B_ϵ). By the trivialization of $\Lambda^*(T^*X)$ and $\Lambda(E^*)$ on $\mathcal{U}_\epsilon$ along the geodesic in X perpendicular to Y, an element $s \in \mathbf{W}$ can be considered as an element in W. One verifies easily that $k^{\frac{1}{2}}D_T^X k^{-\frac{1}{2}}$ acts as a formal self-adjoint operator on $\mathbf{W}_\epsilon$ with respect to the L^2 inner product (2.15).

For any $U \in TY$, let U^H denote the horizontal lifting of U with respect to the connection ∇^N. Then for any orthonormal basis $\{e_1, \cdots, e_l, f_{l+1}, \cdots, f_{2n}\}$ for $TX|_Y$ with $\{e_1, \cdots, e_l\}$ (resp. $\{f_{l+1}, \cdots, f_{2n}\}$) an orthonormal basis for TY (resp. N), set

$$D^H = \sum_{i=1}^{l} c(e_i)\pi^*\nabla^{\Lambda(T^*X)\hat{\otimes}\Lambda(E^*)|_Y}_{e_i^H} : \mathbf{W} \to \mathbf{W}, \tag{2.17}$$

$$D^N = \sum_{\alpha=l+1}^{2n} c(f_\alpha)\pi^*\nabla^{\Lambda(T^*X)\hat{\otimes}\Lambda(E^*)|_Y}_{f_\alpha} : \mathbf{W} \to \mathbf{W}. \tag{2.18}$$

Clearly, the definitions of D^H and D^N do not depend on the choice of the basis $\{e_1, \cdots, e_l, f_{l+1}, \cdots, f_{2n}\}$. Thus D^H and D^N are two well-defined operators acting on $\mathbf{W}$.

Consider the splitting

$$E|_{\mathcal{U}_\epsilon} = E_Y^\tau \oplus E_N^\tau, \tag{2.19}$$

where E_Y^τ and E_N^τ are the parallel transports of E_Y and E_N along the geodesic in X perpendicular to Y. With respect to (2.19), υ has a decomposition

$$\upsilon = \upsilon_{E_Y} + \upsilon_{E_N} \tag{2.20}$$

on $\mathcal{U}_\epsilon$ with $\upsilon_{E_Y} \in \Gamma(E_Y^\tau)$ and $\upsilon_{E_N} \in \Gamma(E_N^\tau)$.

Note that we can always deform υ near Y so that $\upsilon_{E_Y} = 0$, leaving Y and thus N, $E|_Y$, E_N and E_Y in the text unchanged. For the simplicity, we assume below that

$$\upsilon \in \Gamma(E_N^\tau) \quad \text{on} \quad \mathcal{U}_\epsilon. \tag{2.21}$$

For any $y \in Y$ and any orthonormal basis $\{f_{l+1}, \cdots, f_{2n}\}$ of N_y, set

$$\eta_\alpha = \mathcal{L}_\upsilon(f_\alpha) \quad \text{for} \quad l+1 \le \alpha \le 2n. \tag{2.22}$$

Then $\{\eta_{l+1}, \cdots, \eta_{2n}\}$ is an orthonormal basis of E_{N_y}. Let $(z_{l+1}, z_{l+2}, \cdots, z_{2n})$ denote the Euclidean coordinate system on N_y corresponding to $\{f_{l+1}, \cdots, f_{2n}\}$. For any $U \in TX|_Y$ (resp. $\xi \in E|_Y$), denote by $\widetilde{U}$ (resp. $\widetilde{\xi}$) the parallel transport of U (resp. ξ) along the geodesic in X perpendicular to Y. For any $Z = (z_{l+1}, z_{l+2}, \cdots, z_{2n}) \in N_y$ with $|Z| < \epsilon$, let

$$\upsilon(y, Z) = \sum_{\alpha=l+1}^{2n} \mu_\alpha(y, Z)\widetilde{\eta}_\alpha(y, Z). \tag{2.23}$$

Set

$$\upsilon_1(y, Z) = \sum_{\alpha=l+1}^{2n}\sum_{\beta=l+1}^{2n} \frac{\partial \mu_\alpha}{\partial z_\beta}(y) z_\beta \widetilde{\eta}_\alpha(y, Z), \tag{2.24}$$

$$\upsilon_2(y, Z) = \frac{1}{2}\sum_{\alpha=l+1}^{2n}\sum_{\beta,\gamma=l+1}^{2n} \frac{\partial^2 \mu_\alpha}{\partial z_\beta \partial z_\gamma}(y) z_\beta z_\gamma \widetilde{\eta}_\alpha(y, Z). \tag{2.25}$$

The definitions (2.24) and (2.25) are obviously independent of the choice of the basis $\{f_{l+1}, \cdots, f_{2n}\}$. Clearly,

$$\upsilon(y, Z) = \upsilon_1(y, Z) + \upsilon_2(y, Z) + O(|Z|^3). \tag{2.26}$$

Moreover, from (2.22) and the definition of $\mathcal{L}_\upsilon$, one verifies easily that

$$\upsilon_1(y,Z)=\sum_{\alpha=l+1}^{2n} z_\alpha \widetilde{\eta}_\alpha(y,Z). \tag{2.27}$$

Similar to Theorem 8.18 in [2], one verifies easily the following proposition which describes the local behavior of D_T^X as $T\to\infty$.

Proposition 2.1. *As $T\to+\infty$, we have the following formula on $\mathbf{W}_\epsilon$:*

$$k^{1/2}D_T^X k^{-1/2}=D^H+D^N+T\sqrt{-1}\widetilde{c}(\upsilon_1)+T\sqrt{-1}\widetilde{c}(\upsilon_2)+R_T, \tag{2.28}$$

where

$$R_T=O(|Z|\partial^H+|Z|^2\partial^N+|Z|+T|Z|^3), \tag{2.29}$$

and ∂^H, ∂^N represent horizontal and vertical differential operators, respectively.

Set

$$D_T^N=D^N+T\sqrt{-1}\widetilde{c}(\upsilon_1). \tag{2.30}$$

Note that D_T^N is a self-adjoint elliptic operator acting fibrewisely on $\Gamma(\pi^*(\Lambda^*(N^*)\otimes\Lambda(E_N^*)))$. Given an orthonormal frame $\{f_{l+1},\cdots,f_{2n}\}$ for N, let $\{\eta_{l+1},\cdots,\eta_{2n}\}$ be determined by (2.22). Then from (2.18), (2.27), one verifies easily that

$$D_T^N=\sum_{\alpha=l+1}^{2n} c(f_\alpha)\pi^*\nabla_{f_\alpha}^{\Lambda(T^*X)\hat{\otimes}\Lambda(E^*)|_Y}+T\sqrt{-1}\sum_{\alpha=l+1}^{2n} z_\alpha\widetilde{c}(\eta_\alpha), \tag{2.31}$$

$$(D_T^N)^2=\sum_{\alpha=l+1}^{2n}\left(-\frac{\partial^2}{\partial z_\alpha^2}+T^2z_\alpha^2-T\right)+T\sum_{\alpha=l+1}^{2n}\big(1+\sqrt{-1}c(f_\alpha)\widetilde{c}(\eta_\alpha)\big). \tag{2.32}$$

Set

$$\widehat{\mathcal{L}}_\upsilon=\sum_{\alpha=l+1}^{n}\big(1+\sqrt{-1}c(f_\alpha)\widetilde{c}(\eta_\alpha)\big):\Lambda(N^*)\hat{\otimes}\Lambda(E_N^*)\to\Lambda(N^*)\hat{\otimes}\Lambda(E_N^*). \tag{2.33}$$

Clearly, the definition of $\widehat{\mathcal{L}}_\upsilon$ does not depend on the choice of $\{f_{l+1},\cdots,f_{2n}\}$ and thus $\widehat{\mathcal{L}}_\upsilon$ is a well-defined bundle map on $\Lambda(N^*)\hat{\otimes}\Lambda(E_N^*)$.

Similar to (2.3), the involution

$$\tau_{\Lambda(N^*)\hat{\otimes}\Lambda(E_N^*)}=(\sqrt{-1})^{2n-l}c(f_{l+1})\cdots c(f_{2n})\widetilde{c}(\eta_{l+1})\cdots\widetilde{c}(\eta_{2n}) \tag{2.34}$$

gives a $\mathbf{Z}_2$-grading in $\Lambda(N^*\oplus E_N^*)=\Lambda(N^*)\hat{\otimes}\Lambda(E_N^*)$,

$$\Lambda(N^*)\hat{\otimes}\Lambda(E_N^*)=\big(\Lambda(N^*)\hat{\otimes}\Lambda(E_N^*)\big)_+\oplus\big(\Lambda(N^*)\hat{\otimes}\Lambda(E_N^*)\big)_-. \tag{2.35}$$

Set

$$o_Y(\upsilon)=\ker\widehat{\mathcal{L}}_\upsilon. \tag{2.36}$$

Lemma 2.1. (i) rk $o_Y(\upsilon)=2^{2n-l}$.

(ii)

$$o_Y(\upsilon)\subset\begin{cases}(\Lambda(N^*)\hat{\otimes}\Lambda(E_N^*))_+, & \text{if } n+\dfrac{l(l-1)}{2} \text{ is even,}\\[2mm] (\Lambda(N^*)\hat{\otimes}\Lambda(E_N^*))_-, & \text{if } n+\dfrac{l(l-1)}{2} \text{ is odd.}\end{cases}$$

Proof. One can prove the lemma by applying Theorem 2.1 and Theorem 2.2 in [5], which are related close to [7]. Since the case here is much simpler, we will give the lemma a direct proof. Clearly, the linear map $\sqrt{-1}c(f)\widetilde{c}(\eta)$ acting on the complex vector space

$$\Lambda(\{f^*,\eta^*\}) = \mathbf{C}\{1, f^*, \eta^*, f^*\wedge\eta^*\} \tag{2.37}$$

is an involution. A direct computation shows that the -1 eigenspace of $\sqrt{-1}c(f)\widetilde{c}(\eta)$ is

$$\mathbf{C}\{f^* - \sqrt{-1}\eta^*, 1-\sqrt{-1}f^*\wedge\eta^*\}, \tag{2.38}$$

which is also the kernel of the map $1+\sqrt{-1}c(f)\widetilde{c}(\eta)$. From (2.33), (2.38) and $\Lambda(N^*)\hat{\otimes}\Lambda(E_N^*) = \bigwedge_{\alpha=l+1}^{2n}(\Lambda(\{f_\alpha^*,\eta_\alpha^*\}))$, we get

$$o_Y(v) = \bigotimes_{\alpha=l+1}^{2n}\left(\mathbf{C}\{f_\alpha^* - \sqrt{-1}\eta_\alpha^*, 1-\sqrt{-1}f_\alpha^*\wedge\eta_\alpha^*\}\right). \tag{2.39}$$

Thus $\dim o_Y(v) = 2^{2n-l}$.

On the other hand, from (2.34) one verifies easily the following two equalities:

$$\tau_{\Lambda(N^*)\hat{\otimes}\Lambda(E_N^*)} = (-1)^{n+\frac{l(l+1)}{2}}\prod_{\alpha=l+1}^{2n}(\sqrt{-1}c(f_\alpha))\widetilde{c}(\eta_\alpha)), \tag{2.40}$$

$$\tau_{\Lambda(N^*)\hat{\otimes}\Lambda(E_N^*)}|_{o_Y(v)} = (-1)^{n+\frac{l(l-1)}{2}}. \tag{2.41}$$

From (2.41) we complete the proof of Lemma 2.1.

From (2.32), Lemma 2.1 and the spectral theory of harmonic ocillators (see [7, Lemma 2.1]), one verifies the following lemma easily.

Lemma 2.2. *Take $T>0$. Then for any $y\in Y$, the operator $(D_T^N)^2$ acting on $\Gamma(\Lambda^*(N_y^*))$ over N_y is nonnegative with the 2^{2n-l} dimensional kernel*:

$$\exp\Big(-\frac{T|Z|^2}{2}\Big)\otimes o_Y(v)|_y. \tag{2.42}$$

Furthermore, the nonzero eigenvalues of $(D_T^N)^2$ are all $\geq 2(2n-l)T$.

(b) A Twisted Dirac Operator on Y and Its Index

Note that $\Lambda(T^*Y)\hat{\otimes}\Lambda(E_Y^*)$ is a $\mathbf{Z}_2$-graded Hermitian vector bundle over Y with the Hermitian connection

$$\nabla^{\Lambda(T^*Y)\hat{\otimes}\Lambda(E_Y^*)} = \nabla^{\Lambda(T^*Y)}\hat{\otimes}1 + 1\hat{\otimes}\nabla^{\Lambda(E_Y^*)} \tag{2.43}$$

and the $\mathbf{Z}_2$-grading

$$\Lambda(T^*Y)\hat{\otimes}\Lambda(E_Y^*) = (\Lambda(T^*Y)\hat{\otimes}\Lambda(E_Y^*))_+ \oplus (\Lambda(T^*Y)\hat{\otimes}\Lambda(E_Y^*))_- \tag{2.44}$$

given by the involution

$$\tau_{\Lambda(T^*Y)\hat{\otimes}\Lambda(E_Y^*)} = (\sqrt{-1})^l c(e_1)\cdots c(e_l)\widetilde{c}(\xi_1)\cdots\widetilde{c}(\xi_l), \tag{2.45}$$

where $\{e_1,e_2,\cdots,e_l\}$ (resp. $\{\xi_1,\xi_2,\cdots,\xi_l\}$) is an oriented orthonormal basis for TY (resp. E_Y). On the other hand, let

$$P^{o_Y(v)}: \Lambda(N^*)\hat{\otimes}\Lambda(E_N^*) \to o_Y(v) \tag{2.46}$$

denote the orthogonal projection of $\Lambda(N^*)\hat{\otimes}\Lambda(E_N^*)$ to $o_Y(v)$. Then $o_Y(v)$ is a Hermitian vector bundle over Y with the Hermitian connection

$$\nabla^{o_Y(v)} = P^{o_Y(v)}\nabla^{\Lambda(N^*)\hat{\otimes}\Lambda(E_N^*)}P^{o_Y(v)}. \tag{2.47}$$

Set

$$\widetilde{\nabla}^{\Lambda(T^*Y)\hat{\otimes}\Lambda(E_Y^*)} = \nabla^{\Lambda(N^*)\hat{\otimes}\Lambda(E_N^*)} \otimes 1 + 1 \otimes \nabla^{o_Y(\upsilon)}. \tag{2.48}$$

For any orthonormal basis $\{e_1, e_2, \cdots, e_l\}$ for TY, set

$$\widetilde{D}^Y = \sum_{i=1}^{l} c(e_i)\widetilde{\nabla}_{e_i}^{\Lambda(T^*Y)\hat{\otimes}\Lambda(E_Y^*)}. \tag{2.49}$$

Clearly, (2.49) defines a twisted Dirac operator

$$\widetilde{D}^Y : \Gamma\left((\Lambda(T^*Y)\hat{\otimes}\Lambda(E_Y^*)) \otimes o_Y(\upsilon)\right) \to \Gamma\left((\Lambda(T^*Y)\hat{\otimes}\Lambda(E_Y^*)) \otimes o_Y(\upsilon)\right). \tag{2.50}$$

Denote the restriction of $\widetilde{D}^Y$ on $\Gamma\left((\Lambda(T^*Y)\hat{\otimes}\Lambda(E_Y^*))_\pm \otimes o_Y(\upsilon)\right)$ by $\widetilde{D}^Y_\pm$.

Theorem 2.1. *The following equalities hold*:

$$\operatorname{ind} \widetilde{D}^Y_+ = \begin{cases} (-1)^{\frac{l}{2}} 2^{2n}\chi(E_Y), & if \quad l = even; \\ 0, & if \quad l = odd. \end{cases}$$

Proof. The case for odd l is trivial. When l is even, the involution

$$\tau_{\Lambda(T^*Y)} = (\sqrt{-1})^{l/2} c(e_1)c(e_2)\cdots c(e_l)$$
$$(resp. \quad \tau_{\Lambda(E_Y^*)} = (\sqrt{-1})^{l/2}\widetilde{c}(\xi_1)\widetilde{c}(\xi_2)\cdots\widetilde{c}(\xi_l)) \tag{2.51}$$

gives the signature $\mathbf{Z}_2$-grading in $\Lambda(T^*Y)$ (resp. $\Lambda(E_Y^*)$). Moreover, we have

$$\tau_{\Lambda(T^*Y)\hat{\otimes}\Lambda(E_Y^*)} = \tau_{\Lambda(T^*Y)}\hat{\otimes}\tau_{\Lambda(E_Y^*)}, \tag{2.52}$$

$$(\Lambda(T^*Y)\hat{\otimes}\Lambda(E_Y^*))_\pm = (\Lambda_+(T^*Y)\otimes\Lambda_\pm(E_Y^*)) \oplus (\Lambda_-(T^*Y)\otimes\Lambda_\mp(E_Y^*)). \tag{2.53}$$

Compared with the definition of the super-twisted signature operator in [5], $\widetilde{D}^Y$ can be viewed as a twisted super-twisted signature operator on Y. From the proof of Theorem 1.1 in [5], we have

$$\begin{aligned} \operatorname{ch}\left((\Lambda_+(E_Y^*) - \Lambda_-(E_Y^*)) \otimes o_Y(\upsilon)\right) &= \left(\operatorname{ch}(\Lambda_+(E_Y^*)) - \operatorname{ch}(\Lambda_-(E_Y^*))\right)\operatorname{ch}(o_Y(\upsilon)) \\ &= 2^{l/2}(\sqrt{-1})^{-l/2}\operatorname{Pf}(-R^{E_Y}) \cdot 2^{2n-l} \\ &= 2^{2n-l/2}(\sqrt{-1})^{-l/2}\operatorname{Pf}(-R^{E_Y}), \end{aligned}$$

where R^{E_Y} is the curvature of ∇^{E_Y}. From the local index theorem for twisted Dirac operator (see [1, Theorem 4.3]) and Theorem 1.1 in [5] we get

$$\operatorname{ind} \widetilde{D}^Y_+ = (-1)^{l/2} 2^{2n}\chi(E_Y). \tag{2.54}$$

(c) Proof of Theorem 1.2.

For any $\mu \geq 0$, let W^μ (resp. $\mathbf{F}^\mu$) be the set of sections of $\Lambda^*(T^*X)$ on X (resp. of $\Lambda^*(T^*Y)\otimes o_Y(\upsilon)$ on Y) which lie in the μ-th Sobolev space. Let $|| \quad ||_{\mathrm{W}^\mu}$ (resp. $|| \quad ||_{\mathbf{F}^\mu}$) be the Sobolev norm on W^μ (resp. $\mathbf{F}^\mu$).

Let $\gamma : \mathbf{R} \to [0,1]$ be a smooth even function with $\gamma(a) = 1$ if $|a| \leq \frac{1}{2}$ and $\gamma(a) = 0$ if $|a| \geq 1$. For any $T > 0$ and $y \in Y$, set

$$\alpha_T(y) = \int_{N_y} \gamma\Big(\frac{|Z|}{2}\Big)^2 \exp(-T|Z|^2) d\sigma_{N_y}(Z), \tag{2.55}$$

$$G_T(y, Z) = \alpha_T^{-\frac{1}{2}}(y)\gamma\Big(\frac{|Z|}{2}\Big)\exp\Big(-\frac{T|Z|^2}{2}\Big), \tag{2.56}$$

where $\epsilon \in (0, \epsilon_0)$ and ϵ_0 is defined in Section 2 (a). Clearly, the values of functions $\alpha_T(y)$ and $G_T(y, Z)$ do not depend on $y \in Y$.

For $\mu \geq 0, T > 0$, let $J_T : \mathbf{F}^\mu \to \mathrm{W}^\mu$ be a linear map defined by

$$J_T u = k^{-1/2} G_T \pi^* u, \qquad \forall u \in \mathbf{F}^\mu. \tag{2.57}$$

Let W_T^μ be the image of J_T in W^μ and let $\mathrm{W}_T^{0,\perp}$ be the orthogonal complement of W_T^0 in W^0. Set

$$\mathrm{W}^{1,\perp} = \mathrm{W}^1 \cap \mathrm{W}_T^{0,\perp}. \tag{2.58}$$

Let p_T and $p_T^\perp$ be the orthogonal projection operators from W^0 onto W_T^0 and $\mathrm{W}_T^{0,\perp}$, respectively. Set

$$D_{T,1} = p_T D_T^X p_T, \quad D_{T,2} = p_T D_T^X p_T^\perp, \quad D_{T,3} = p_T^\perp D_T^X p_T, \quad D_{T,4} = p_T^\perp D_T^X p_T^\perp. \tag{2.59}$$

We have

$$D_T^X = D_{T,1} + D_{T,2} + D_{T,3} + D_{T,4}. \tag{2.60}$$

Now by using Proposition 2.1 and proceeding as in [2, Section 9], we can prove the following lemma.

Lemma 2.3. (i) *The following formula holds on* $\Gamma(\Lambda^*(T^*Y) \otimes o_Y(\upsilon))$ *as* $T \to +\infty$,

$$J_T^{-1} D_{T,1} J_T = \widetilde{D}^Y + O\Big(\frac{1}{\sqrt{T}}\Big), \tag{2.61}$$

where $O(\frac{1}{\sqrt{T}})$ *is a first order differential operator with smooth coefficients dominated by* $C/\sqrt{T}$.

(ii) *There exist* $C_1 > 0$, $C_2 > 0$ *and* $T_0 > 0$ *such that for any* $T \geq T_0$, $s \in \mathrm{W}_T^{1,\perp}$ *and* $s' \in \mathrm{W}_T^1$, *we have*

$$||D_{T,2}s||_{\mathrm{W}^0} \leq C_1\Big(\frac{||s||_{\mathrm{W}^1}}{\sqrt{T}} + ||s||_{\mathrm{W}^0}\Big), \tag{2.62}$$

$$||D_{T,3}s'||_{\mathrm{W}^0} \leq C_1\Big(\frac{||s'||_{\mathrm{W}^1}}{\sqrt{T}} + ||s'||_{\mathrm{W}^0}\Big), \tag{2.63}$$

$$||D_{T,4}s||_{\mathrm{W}^0} \geq C_2(||s||_{\mathrm{W}^1} + \sqrt{T}||s||_{\mathrm{W}^0}). \tag{2.64}$$

Proof. One verifies the inequalities in (ii) in the lemma easily by following the proofs of Theorem 9.10, Theorem 9.11 and Theorem 9.14 in [2, Section 9]. Note that our case is much simpler than what in [2, Section 9]. So we need only to prove the first part in the lemma.

For any $u \in \Gamma(\Lambda^*(T^*Y) \otimes o_Y(\upsilon))$, we have

$$\begin{aligned} J_T^{-1} D_{T,1} J_T u &= J_T^{-1} p_T D_T^X p_T J_T u \\ &= J_T^{-1} p_T k^{-\frac{1}{2}} \left(D^H + D^N + T\sqrt{-1}\widetilde{c}(\upsilon_1) + T\sqrt{-1}\widetilde{c}(\upsilon_2) + R_T\right) G_T \pi^* u \\ &= J_T^{-1} p_T k^{-\frac{1}{2}} D^H G_T \pi^* u + J_T^{-1} p_T k^{-\frac{1}{2}} (D^N + T\sqrt{-1}\widetilde{c}(\upsilon_1)) G_T \pi^* u \\ &\quad + T\sqrt{-1} J_T^{-1} p_T k^{-\frac{1}{2}} \widetilde{c}(\upsilon_2) G_T \pi^* u + J_T^{-1} p_T k^{-\frac{1}{2}} R_T G_T \pi^* u. \end{aligned}$$

From Lemma 2.2 and (2.56) and the definition of p_T, one verifies easily that

$$J_T^{-1} p_T k^{-\frac{1}{2}} (D^N + T\sqrt{-1}\widetilde{c}(\upsilon_1)) G_T \pi^* u = O\Big(\frac{1}{\sqrt{T}}\Big) u. \tag{2.65}$$

Since $\widetilde{c}(\upsilon_2)$ interchanges $(\Lambda(T^*Y)\hat{\otimes}\Lambda(E_Y^*))_+$ and $(\Lambda(T^*Y)\hat{\otimes}\Lambda(E_Y^*))_-$, from the second part in Lemma 2.1 and again the definition of p_T, one gets

$$p_T k^{-\frac{1}{2}}\widetilde{c}(\upsilon_2)G_T\pi^*u = 0. \tag{2.66}$$

From (2.29) and Proposition 9.3 in [2, Section 9], we have

$$J_T^{-1}p_T k^{-\frac{1}{2}}R_T G_T\pi^*u = O\Big(\frac{1}{\sqrt{T}}\Big)u. \tag{2.67}$$

On the other hand, from (2.17), (2.56) and the choices of connections ∇^{TM} and ∇^E in Section 2**a)**, we have

$$\begin{aligned}J_T^{-1}p_T k^{-\frac{1}{2}}D^H G_T\pi^*u &= J_T^{-1}p_T k^{-\frac{1}{2}}\sum_{i=1}^{l}c(e_i)\left(\pi^*\nabla_{e_i^H}^{\Lambda(T^*X)\hat{\otimes}\Lambda(E^*)|_Y}\right)G_T\pi^*u\\ &= J_T^{-1}p_T k^{-\frac{1}{2}}G_T\pi^*\Big(\sum_{i=1}^{l}c(e_i)\widetilde{\nabla}_{e_i}^{\Lambda(N^*)\hat{\otimes}\Lambda(E_N^*)}u\Big)\\ &= J_T^{-1}k^{-\frac{1}{2}}G_T\pi^*D_Y u = \widetilde{D}_Y u.\end{aligned}$$

So the lemma follows.

From (2.34) and (2.45) we have

$$\tau_{\Lambda(T^*X)\hat{\otimes}\Lambda(E^*)}|_Y = (-1)^l\tau_{\Lambda(T^*Y)\hat{\otimes}\Lambda(E_Y^*)}\hat{\otimes}\tau_{\Lambda(N^*)\hat{\otimes}\Lambda(E_N^*)}. \tag{2.68}$$

So for even l,

$$\begin{aligned}(\Lambda(T^*X)\hat{\otimes}\Lambda(E^*))_\pm = &\big((\Lambda(T^*Y)\hat{\otimes}\Lambda(E_Y^*))_+\otimes(\Lambda(N^*)\hat{\otimes}\Lambda(E_N^*))_\pm\big)\\ &\oplus\big((\Lambda(T^*Y)\hat{\otimes}\Lambda(E_Y^*))_-\otimes(\Lambda(N^*)\hat{\otimes}\Lambda(E_N^*))_\mp\big),\end{aligned} \tag{2.69}$$

and for odd l,

$$\begin{aligned}(\Lambda(T^*X)\hat{\otimes}\Lambda(E^*))_\pm = &\big((\Lambda(T^*Y)\hat{\otimes}\Lambda(E_Y^*))_+\otimes(\Lambda(N^*)\hat{\otimes}\Lambda(E_N^*))_\mp\big)\\ &\oplus\big((\Lambda(T^*Y)\hat{\otimes}\Lambda(E_Y^*))_-\otimes(\Lambda(N^*)\hat{\otimes}\Lambda(E_N^*))_\pm\big).\end{aligned} \tag{2.70}$$

Let

$$\mathrm{W}^\mu = \mathrm{W}_+^\mu\oplus\mathrm{W}_-^\mu \tag{2.71}$$

be the decomposition with respect to the natural extension of the $\mathbf{Z}_2$-grading in $\Gamma(\Lambda(T^*X)\hat{\otimes}\Lambda(E^*))$.

Following [9, Section 2**(c)**], for any $t\in\mathbf{R}$, set

$$D_{T,+}^X(t) = D_{T,1}+D_{T,2}+t(D_{T,3}+D_{T,4}):\mathrm{W}_+^1\to\mathrm{W}_-^0. \tag{2.72}$$

From Lemma 2.3 and proceeding similarly as the proof of Lemma 2.2 in [9], we get

Lemma 2.4. *There exists $T_1>0$ such that for any $T\ge T_1$, $D_{T,+}^X(t)$, $t\in[0,1]$, is a continuous curve of Fredholm operators.*

From Lemma 2.4, we have

$$\operatorname{ind} D_{T,+}^X = \operatorname{ind} D_{T,+}^X(1) = \operatorname{ind} D_{T,+}^X(0) = \operatorname{ind} D_{T,1} + \operatorname{ind} D_{T,4}, \tag{2.73}$$

where in the last line, $D_{T,1}$ (resp. $D_{T,4}$) is now regarded as a Fredholm operator mapping from $\mathrm{W}_{T,+}^1$ (resp. $\mathrm{W}_{T,+}^{0,\perp}$) to $\mathrm{W}_{T,-}^0$ (resp. $\mathrm{W}_{T,-}^{0,\perp}$). By the similar reason as (2.13) in [9, Section 2(**b**)], for sufficiently large $T > 0$, we have

$$\operatorname{ind} D_{T,4} = 0. \tag{2.74}$$

Thus for sufficiently large $T > 0$, we have

$$\operatorname{ind} D_{T,+}^{X} = \operatorname{ind} D_{T,1} = \operatorname{ind} J_T^{-1} D_{T,1} J_T. \tag{2.75}$$

Now from the first part of Lemma 2.3 and (2.69), (2.70) and (ii) in Lemma 2.1, we get for sufficiently large $T > 0$,

$$\begin{aligned}\operatorname{ind} D_{T,+}^{X} &= \sum_{l_k=even} (-1)^{n+l_k(l_k-1)/2} \operatorname{ind} \widetilde{D}_{+}^{Y_k} - \sum_{l_k=odd} (-1)^{n+l_k(l_k-1)/2} \operatorname{ind} \widetilde{D}_{+}^{Y_k} \\ &= \sum_{l_k=even} (-1)^{n+l_k/2} \operatorname{ind} \widetilde{D}_{+}^{Y_k}.\end{aligned}$$

Thus by Theorem 2.1 we have

$$\begin{aligned}\operatorname{ind} D_{T,+}^{X} &= \sum_{l_k=even} (-1)^{n+l_k/2} (-1)^{l_k/2} 2^{2n} \chi(E_{Y_k}) \\ &= (-1)^n 2^{2n} \sum_{l_k=even} \chi(E_{Y_k}) \\ &= (-1)^n 2^{2n} \sum_{k=1}^{m} \chi(E_{Y_k}),\end{aligned}$$

where the last equality is from the fact that an odd real vector bundle has the vanishing Euler characteristic. Now from (2.13) we get

$$\chi(E) = \sum_{k=1}^{m} \chi(E_{Y_k}). \tag{2.76}$$

Finally, by the orientation-preserving isomorphisms in (2.14), we get Theorem 1.1.

Remark 2.1. (i) When υ is a transversal section of E, Y consists of isolated points $Y_k \in X$. Comparing the definition of ind $(\upsilon; Y_k)$ in [5, Section 2] with the orientation on E_{Y_k}, one gets easily that

$$\chi(E_{Y_k}) = \operatorname{ind} (\upsilon; Y_k), \tag{2.77}$$

and thus Theorem 11.17 in [4, p.125], which was proved in [5] in a purely analytic way.

(ii) When $E = TX$, one compares the definition of ind (υ, Y_k) in [6, Section. 2] with the orientation on E_{Y_k}, one gets easily that

$$\chi(E_{Y_k}) = \operatorname{ind} (\upsilon, Y_k) \chi(Y_k), \tag{2.78}$$

and thus Theorem 4.2 in [6].

As a simple application, we consider a rank $2q$ oriented real vector bundle with $q < n$. Let υ be a transversal section of E. Let Y still denote the zero point set of υ. In this case, each connected component Y_k of Y is a $2n - 2q$ dimensional orientable submanifold of X. Moreover, the normal bundle $(TX|_{Y_k})/TY_k$ of TY_k in $TX|_{Y_k}$ can be identified with $E|_{Y_k}$. We will choose the orientation on each Y_k such that the identification $TY_k \oplus E|_{Y_k}$ with $TX|_{Y_k}$ preserves the orientations on them.

From Theorem 1.1, we get easily the following corollary.

Corollary 2.1. *For any oriented real vector bundle $F \to X$ of rank $2n-2q$, the following equality holds:*

$$\langle e(F)e(E), [X]\rangle = \sum_{k=1}^{m} \chi(F|_{Y_k}), \tag{2.79}$$

where $e(E)$ and $e(F)$ denote the integral Euler classes of E and F respectively, and $[X]$ denotes the homology class determined by X.

Proof. Clearly, the transversal section v of E can be naturally considered as a nondegenerate section of $F \oplus E$ in the sense of Bott. Then by Theorem 1.2 we have

$$\sum_{k=1}^{m} \chi(F|_{Y_k}) = \chi(F \oplus E) = \langle e(F \oplus E), [X]\rangle = \langle e(F)e(E), [X]\rangle.$$

Remark 2.2. Let E be an oriented even dimensional subbundle of TX. In [10], by constructing a sub-signature operator $\widetilde{D}_{E,+}$ associated to E, Zhang computed directly the local index of $\widetilde{D}_{E,+}$, by which he got the following index formula in [10, (6)]:

$$\operatorname{ind} \widetilde{D}_{E,+} = \langle \mathcal{L}(E)e(TX/E), [X]\rangle, \tag{2.80}$$

where $\mathcal{L}(E)$ is the Hirzebruch $\mathcal{L}$-characteristic class. His formula provides an index theorem interpretation for the Euler class of E. In general, for any rank $2q$ oriented real vector bundle E over X with $q < n$, one can get an analogue index theorem interpretation by extending the definition of the super-twisted signature operator in [5] to this case naturally.

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