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THE NAGUMO EQUATION ON SELF-SIMILAR FRACTAL SETS

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Abstract

The Nagumo equation

$$u_t = \Delta u + bu(u-a)(1-u), \quad t > 0$$

is investigated with initial data and zero Neumann boundary conditions on post-critically finite (p.c.f.) self-similar fractals that have regular harmonic structures and satisfy the separation condition. Such a nonlinear diffusion equation has no travelling wave solutions because of the "pathological" property of the fractal. However, it is shown that a global Hölder continuous solution in spatial variables exists on the fractal considered. The Sobolev-type inequality plays a crucial role, which holds on such a class of p.c.f self-similar fractals. The heat kernel has an eigenfunction expansion and is well-defined due to a Weyl's formula. The large time asymptotic behavior of the solution is discussed, and the solution tends exponentially to the equilibrium state of the Nagumo equation as time tends to infinity if b is small.

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§1. Introduction

The Nagumo equation in $\mathbb{R}^N (N \geq 1)$ reads

$$u_t = \Delta u + bu(u-a)(1-u), \quad t > 0, \tag{1.1}$$

where $b > 0$ and $0 < a < 1$. The equation (1.1) is used as a model to describe the spread of genetic traits or the propagatoin of nerve pulse in a nerve axon[1,15,29]. There has been an extensive study of (1.1) on the real line $\mathbb{R}$ (see for example [1,5,6,15,17,29]).

Our concern here is different. We work with (1.1) on a certain class of self-similar fractal sets in $\mathbb{R}^N$, of which the Sierpiński gasket is most typical. The Sierpiński gasket in $\mathbb{R}^2$ is defined as follows. Let $F_i : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$F_i(x) = \frac{1}{2}(x - p_i) + p_i, \quad x \in \mathbb{R}^2, \quad i = 1, 2, 3,$$

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where $\{p_1, p_2, p_3\} \equiv V_0$ is the set of vertices of a triangle in $\mathbb{R}^2$ with each side of length 1. The Sierpiński gasket in $\mathbb{R}^2$ is the closure of $V_* = \bigcup_{m=0}^{\infty} V_m$ with $V_m = \bigcup_{i=1}^{3} F_i(V_{m-1}), m \geq 1$ (see Fig. 1).

Fig.1

The reader may think of the Sierpiński gasket in $\mathbb{R}^2$ as an example when encountering a p.c.f. self-similar fractal I mentioned in this paper.

Laplacians have been defined on a certain class of self-similar fractals (see for example [3,4, 8, 12, 14, 21, 23]). In particular, Kigami[21] defined a standard Laplacian on post-critically finite (p.c.f.) self-similar fractals that have regular harmonic structures. Note that the Laplacian defined on a fractal degenerates to the usual second derivative if the fractal set degenerates to an interval on the real line $\mathbb{R}$.

A remarkable difference for (1.1) on fractals from on classical domains is that (1.1) has no travelling wave solutions on fractals, that is, (1.1) has no solutions of the form $u(t, x) = u(x - ct)$ with wave speed $c > 0$.

In this paper we show that (1.1) with initial data and zero Neumann boundary conditions has a unique solution on p.c.f. self-similar fractals if the Sobolev-type inequality (2.8) holds on fractals (see below); in particular, on p.c.f. self-similar fractals which possess regular harmonic structures and satisfy the separation condition. We also demonstrate that the solution decays exponentially to its spatial average over the fractal V, that is, to

$$\bar{u}(t) = \int_V u(t, x)\, d\mu(x), \quad t > 0,$$

if b is small, where μ is a regular Borel measure supported on V so that $\mu(V) = 1$ and $0 < \mu(U) < \infty$ for any open subset U of V. Such a measure exists for most basic self-similar fractals. The key is to employ the Sobolev-type inequality[14,18,23] (see also [24,25]).

The arrangement of this paper is as follows. In Section 2 we give the heat kernel by using the eigenvalues and eigenfunctions. This is reminiscent of Mercer's theorem on classical domains[30]. A Weyl's formula[22] gives rise to the uniform convergence of eigenfunction expansion of the heat kernel for $t \geq \eta > 0$. In Section 3 we prove the existence and uniqueness of global non-negative solution of (1.1) with non-negative initial data and zero Neumann boundary condition. The equation (1.1) admits an invariant interval $[0, 1]$ on p.c.f. self-similar fractals as on classical domains, that is, solutions of (1.1) lie in $[0, 1]$ if initial values lie in $[0, 1]$. This comes from a maximum principle on self-similar fractals. Finally, in Section 4 we discuss the large time behavoir of the solution of (1.1), and show that the solution of (1.1) decays exponentially to that of the associated ordinary differential equation.

§2. Preliminaries

Let $D \geq 2$ be an integer. An iterated function system (IFS) is a family of contraction mappings $\{F_1, F_2, \cdots, F_D\}$ on $\mathbb{R}^N$, that is, $F_i : \mathbb{R}^N \to \mathbb{R}^N$ $(1 \le i \le D)$ and

$$|F_i(x) - F_i(y)| \le \alpha_i |x - y| \quad \text{for all } x, y \in \mathbb{R}^N, \tag{2.1}$$

where $0 < \alpha_i < 1$ and $|\,.\,|$ is the Euclidean metric. If (2.1) is replaced by equality then F_i is a similitude. For an IFS $\{F_1, F_2, \cdots, F_D\}$ on $\mathbb{R}^N$ there exists a unique, non-empty compact set $V \subset \mathbb{R}^N$ satisfying $V = \bigcup_{i=1}^{D} F_i(V)$ (see [11,p.30]). Such a set V is called the attractor of the IFS $\{F_1, F_2, \cdots, F_D\}$; if the F_i are all similitudes, V is called a self-similar fractal. An IFS $\{F_1, F_2, \cdots, F_D\}$ satisfies the open set condition if there exists a non-empty bounded open set $U \subset \mathbb{R}^N$ such that $\bigcup_{i=1}^{D} F_i(U) \subset U$ with this union disjoint. For an IFS $\{F_1, F_2, \cdots, F_D\}$ of similitudes satisfying the open set condition there is a unique number $d_f > 0$ such that $\sum_{i=1}^{D} \alpha_i^{d_f} = 1$. Such a d_f is the Hausdorff dimension of the self-similar fractal V of the IFS $\{F_1, F_2, \cdots, F_D\}$ (see [10, pp.118–120]).

A certain class of self-similar fractals, termed post-critically finite (p.c.f.) self-similar fractals, was introduced in [9,12]. Let $\sum$ be a shift space based on $S = \{1, 2, \cdots, D\}$, that is, $\sum = \{\omega : \omega = i_1 i_2 \cdots \ \text{ with } i_k \in S \ \text{ for all } k \in \mathbb{N}\}$, where $\mathbb{N}$ is the collection of all positive integers. For the self-similar fractal V of an IFS $\{F_1, F_2, \cdots, F_D\}$ we define a mapping $\pi : \sum \to V$ by

$$\pi(\omega) = \bigcap_{m=1}^{\infty} F_{i_1 i_2 \cdots i_m}(V),$$

for all $\omega = i_1 i_2 \cdots \in \sum$, where $F_{i_1 i_2 \cdots i_m} = F_{i_1} \circ F_{i_2} \circ \cdots \circ F_{i_m}$. It is easy to see that π is well defined since $\bigcap_{m=1}^{\infty} F_{i_1 i_2 \cdots i_m}(V)$ is a singleton, due to the contraction property of F_i $(1 \le i \le D)$. Let

$$\Lambda = \bigcup_{\substack{i,j \in S \\ i \ne j}} F_i(V) \cap F_j(V), \quad \Pi = \pi^{-1}(\Lambda), \quad P = \bigcup_{k=1}^{\infty} \sigma^k(\Pi),$$

where σ is a shift map, that is, $\sigma(i_1 i_2 i_3 \cdots) = i_2 i_3 \cdots$. We call V a post-critically finite self-similar fractal if the post-critical set P is finite. Let $V_0 = \pi(P)$, termed the boundary of V. Then V is the closure of V_* under the Euclidean metric, where $V_* = \bigcup_{m=0}^{\infty} V_m$ and $V_{m+1} = \bigcup_{i \in S} F_i(V_m)$ with $V_m \subset V_{m+1}, m \ge 0$ (see [19, Lemma 1.3.10]). Let $L^2(V)$ be the usual space of square integrable functions on V with respect to μ, with the norm $\|\,.\,\|_2$, and $\mathcal{D}(W)$ a dense subspace of $L^2(V)$. A Dirichlet form W on V is a non-negative, closed, Markovian and symmetric bilinear form on $\mathcal{D}(W) \times \mathcal{D}(W)$ (see [16, pp.3–5]). The Dirichlet form on p.c.f. self-similar fractals may be obtained in the following way. Let V be a p.c.f. self-similar fractal in $\mathbb{R}^N$, with the boundary $V_0 = \{p_1, p_2, \cdots, p_{n_0}\}$ $(n_0 \ge 2)$. Define a quadratic form W_0 on V_0 by

$$W_0(u, v) = \frac{1}{2} \sum_{1 \le i,j \le n_0} c_{ij}(u(p_i) - u(p_j))(v(p_i) - v(p_j))$$

for $u, v : V_0 \to \mathbb{R}$, where $c_{ij} = c_{ji} \ge 0$. We suppose that W_0 is irreducible, that is,

$$W_0(u, u) = 0 \quad \text{if and only if } u \text{ is constant on } V_0. \tag{2.2}$$

We inductively define a quadratic form W_{m+1} on V_{m+1} by

$$W_{m+1}(u,v) = \sum_{i=1}^{D} r_i^{-1}\, W_m(u \circ F_i, v \circ F_i) \tag{2.3}$$

for $m \geq 0$ and $u, v : V_{m+1} \to \mathbb{R}$, where $r_i > 0$ for all $i \in S$. For $u : V_0 \to \mathbb{R}$, we define

$$\mathbb{W}_1(u,u) = \min\{W_1(v,v) |\ v : V_1 \to \mathbb{R} \text{ and } v|_{V_0} = u\}. \tag{2.4}$$

A p.c.f. self-similar fractal V is said to possess a harmonic structure, denoted by (J, r), if there exist an $n_0 \times n_0$ matrix $J = -(c_{ij})$ and a vector $r = (r_1, r_2, \cdots, r_D)$ such that

$$\mathbb{W}_1(u,u) = W_0(u,u), \tag{2.5}$$

for all $u : V_0 \to \mathbb{R}$. The harmonic structure (J, r) is said to be regular if $r_i < 1$ for all $i \in S$ (see [21,22]). It is an open question whether or not a general p.c.f. self-similar fractal possesses a regular harmonic structure although a positive answer was obtained for nested fractals in [26, 28]. For a p.c.f. self-similar fractal V with a harmonic structure, we see that W_m is increasing in m. Thus we may define

$$W(u,u) = \lim_{m\to\infty} W_m(u,u), \tag{2.6}$$

for all $u : V_* \to \mathbb{R}$ (possibly $W(u,u) = \infty$). The W in (2.6) is only defined on V_*. By a continuous extension such a W may be viewed as the Dirichlet form on V with the domain $\mathcal{D}(W)$ dense in $C(V)$, the space of all continuous functions on V. Note that this construction of Dirichlet form does not depend on the measure μ on V.

Let V be a p.c.f. self-similar fractal in $\mathbb{R}^N$ which possesses a regular harmonic structure. The V satisfies the separation condition if for all $m \geq 1$ and $\omega_1, \omega_2 \in S^m$, there exist some $\delta_0 > 0$ and some $d \in (0, 1)$ such that

$$\text{dist}\,(F_{\omega_1}(V), F_{\omega_2}(V)) \equiv \min_{\substack{x\in F_{\omega_1}(V)\\ y\in F_{\omega_2}(V)}} \{|x-y|\} \geq \delta_0\, d^m \tag{2.7}$$

whenever $F_{\omega_1}(V) \cap F_{\omega_2}(V) = \emptyset$. If V is a p.c.f. self-similar fractal having a regular harmonic structure and satisfying the separation condition (2.7), then the Sobolev-type inequality

$$|u(x) - u(y)| \leq c|x-y|^{\beta} W(u,u)^{\frac{1}{2}} \tag{2.8}$$

holds for all $x, y \in V$ and all $u \in C(V)$, where $c, \beta > 0$ and $W(u,u)$ is the Dirichlet form on V given by (2.6) (see [18]). For the Sierpiński gasket in $\mathbb{R}^{N-1} (N \geq 3)$, we have (2.8) with $c = 2N + 3$ and $\beta = \log((N + 2)/N)/\log 4$, and

$$W(u,u) = \lim_{m\to\infty} \left(\frac{N+2}{N}\right)^m \sum_{\substack{x,y\in V_m\\ |x-y|=2^{-m}}} |u(x) - u(y)|^2,$$

(see [14,23]). Define

$$H(V) = \{u \in C(V) |\ W(u,u) < \infty\}. \tag{2.9}$$

Then $H(V)$ is a Hilbert space with the norm $\|u\| = \sqrt{W(u,u) + \|u\|_2^2}$, where $\|u\|_2$ is the L^2-norm given by $\|u\|_2 = \left(\int_V u(x)^2\, d\mu(x)\right)^{\frac{1}{2}}$.

By the Arzela-Ascoli Theorem, we see from (2.8) that

$$H(V) \hookrightarrow C(V) \quad \text{compactly}. \tag{2.10}$$

Let V_0 be the boundary of V. The Neumann derivative $(du) : V_0 \to \mathbb{R}$ of u was defined in

[19,21]. For the Sierpiński gasket in $\mathbb{R}^{N-1}(N \geq 3)$, we have

$$(du)(p) = -\lim_{m\to\infty}\left(\frac{N+2}{N}\right)^m \sum_{\substack{x\in V_m \\ |x-p|=2^{-m}}} (u(x)-u(p)), \quad p \in V_0$$

(see [20,21]).

Let $H^1(V) = \{u \in H(V) | \ (du) \text{ exists on } V_0 \text{ and } (du)|_{V_0} = 0\}$. We say $u \in H^1(V)$ admits a weak Laplacian Δu if there exists $\Delta u \in L^2(V)$ such that

$$W(u,v) = -\int_V \Delta u(x)v(x)\, d\mu(x) \quad \text{for all} \quad v \in H(V), \tag{2.11}$$

where $W(u,v) = \frac{1}{4}\left[W(u+v,u+v) - W(u-v,u-v)\right]$ is the inner product of $u, v \in H(V)$. If Δu is continuous on $V\backslash V_0$, then Δu is actually the standard Laplacian given by Kigami[21] (see the argument in [14]).

Given the Laplacian Δ as in (2.11), we solve the eigenvalue problem

$$-\Delta u = \lambda u, \quad (du)|_{V_0} = 0. \tag{2.12}$$

Using the standard technique[27,30] and (2.10), we obtain that (2.12) has a sequence of eigenfunctions $\{\varphi_n\}_{n\geq 0}$ in $H^1(V)$ which forms an orthonormal basis of $L^2(V)$ and corresponds to non-negative eigenvalues $\{\lambda_n\}_{n\geq 0}$, that is, $\varphi_0 = 1, \|\varphi_n\|_2 = 1$ for $n \geq 1$ and

$$W(\varphi_i, \varphi_j) = \int_V \varphi_i(x)\varphi_j(x)\, d\mu(x) = 0, \quad i \neq j, \tag{2.13}$$

$$W(\varphi_n, v) = \lambda_n \int_V \varphi_n(x)v(x)\, d\mu(x) \text{ for all } v \in H(V), n \geq 0, \tag{2.14}$$

with $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots, \lambda_n \to \infty$ as $n \to \infty$. From (2.14) we see that

$$W(\varphi_n, \varphi_n) = \lambda_n, n \geq 0. \tag{2.15}$$

For $n \geq 1$, there exists a point $x_0 \in V$ such that $|\varphi_n(x_0)| \leq 1$ since $\|\varphi_n\|_2 = 1$. It follows from (2.8) and (2.15) that

$$\sup_{x\in V} |\varphi_n(x)| \leq |\varphi_n(x_0)| + c\sup_{x\in V} |x-x_0|^\beta W(\varphi_n,\varphi_n)^{\frac{1}{2}} \leq M\lambda_n^{\frac{1}{2}} \quad \text{for all } n \geq 1, \tag{2.16}$$

for some $M > 0$.

Motivated by [9] on classical domains, we define the heat kernel $K : (0,\infty)\times V \times V \to \mathbb{R}$ by

$$\begin{aligned} K(t,x,y) &= \sum_{n=0}^{\infty} e^{-\lambda_n t}\, \varphi_n(x)\varphi_n(y) \\ &= 1 + \sum_{n=1}^{\infty} e^{-\lambda_n t}\, \varphi_n(x)\varphi_n(y), \quad t > 0 \text{ and } x,y \in V \end{aligned} \tag{2.17}$$

(see also [2,8,19]). Note that a Weyl's formula $c_1\lambda^{d_s/2} \leq \rho(\lambda) \leq c_2\lambda^{d_s/2}$ holds for all large λ (see [22]), where $c_1, c_2 > 0$ and $d_s \in [1,2)$ is the spectral dimension of V and $\rho(\lambda)$ is the number of eigenvalues not greater than λ. Thus

$$b_1 n^{2/d_s} \leq \lambda_n \leq b_2 n^{2/d_s} \quad \text{for all } n \geq 1 \tag{2.18}$$

for some $b_1, b_2 > 0$. From (2.16) and (2.18), we see that the series on the right-hand side of (2.17) is uniformly convergent for all $x, y \in V$ and all $t \geq \eta > 0$. Thus K is well-defined on $(0,\infty)\times V\times V$. Moreover, K is uniformly Hölder continuous in $x, y \in V$ for all $t \geq \eta > 0$, that is,

$$|K(t,x_2,y_2) - K(t,x_1,y_1)| \leq c_3(|x_2-x_1|^\beta + |y_2-y_1|^\beta) \tag{2.19}$$

for all $x_1, x_2, y_1, y_2 \in V$ and all $t \geq \eta > 0$, by virtue of (2.8) and (2.16).

The weak Laplacian Δ in (2.11) is equivalent to the infinitesimal generator of the semigroup $\{P_t, t > 0\}$, that is,

$$\|(P_t u - u)/t - \Delta u\|_2 \to 0 \quad \text{as } t \to 0 \tag{2.20}$$

if and only if $\Delta u \in L^2(V)$ exists, where

$$P_t u(x) = \int_V K(t,x,y)u(y)\, d\mu(y), \quad u \in L^2(V). \tag{2.21}$$

Proposition 2.1. *Let K be defined in* (2.17). *Then*

$$K(t,x,y) \geq 0 \quad \text{for } t > 0 \text{ and } x, y \in V, \text{ and} \tag{2.22}$$

$$\int_V K(t,x,y)\, d\mu(y) = 1 \quad \text{for } t > 0 \text{ and } x \in V. \tag{2.23}$$

Proof. Let $t > 0$ be fixed. We show that for $u \in L^2(V)$ with $u \geq 0$,

$$P_t u(x) \geq 0, \quad x \in V. \tag{2.24}$$

To see this, let $u(x) = \sum\limits_{n=0}^{\infty} a_n \varphi_n(x) \in L^2(V)$, where $a_n = \int_V \varphi_n(x)u(x)\, d\mu(x)$. It follows from (2.17) and (2.21) that

$$P_t u(x) = \sum_{n=0}^{\infty} a_n\, e^{-\lambda_n t}\varphi_n(x). \tag{2.25}$$

Thus

$$\int_V P_t u(x)u(x)\, d\mu(x) = \sum_{n=0}^{\infty} a_n^2\, e^{-\lambda_n t} \geq 0$$

for all $u \in L^2(V)$, giving (2.24). Therefore, (2.22) follows immediately from (2.24), (2.21) and the continuity of K on $(0,\infty) \times V \times V$. Noting that

$$\int_V \varphi_0(x)\, d\mu(x) = 1 \quad \text{and} \quad \int_V \varphi_n(x)\, d\mu(x) = 0, \quad n \geq 1,$$

we see that for $t > 0$ and $x \in V$,

$$\int_V K(t,x,y)\, d\mu(y) = \sum_{n=0}^{\infty} e^{-\lambda_n t}\varphi_n(x) \int_V \varphi_n(x)\, d\mu(x) = 1,$$

giving (2.23).

Proposition 2.2. *Let Δu exist and be continuous on $V \backslash V_0$. If u reaches its maximum* (*minimum*) *at $x_0 \in V \backslash V_0$, then*

$$\Delta u(x_0) \leq 0 (\geq 0). \tag{2.26}$$

Proof. See [19, Lemma 5.2.4].

§3. Existence and Uniqueness

Let V be a p.c.f. self-similar fractal which has a regular harmonic structure and satisfies the separation condition (2.7). Let V_0 be the boundary of V. Consider

$$u_t = \Delta u + bu(u-a)(1-u), \quad t > 0, x \in V \backslash V_0, \tag{3.1}$$

with initial and zero Neumann conditions

$$u|_{t=0} = u_0(x), \quad x \in V,$$
$$(du)|_{V_0} = 0, \quad t > 0, \tag{3.2}$$

where $b > 0, a \in (0,1)$, and (du) denotes the Neumann derivative. We assume that u_0 satisfies $(du_0)|_{V_0} = 0$.

A function $u : (0,T) \times V \to \mathbb{R}$ is said to be a (local) weak solution of (3.1), (3.2) for $T > 0$ if u satisfies

$$u(t,x) = P_t u_0(x) + b\int_0^t d\tau \int_V K(t-\tau,x,y) f(u(\tau,y))\, d\mu(y), \tag{3.3}$$

for $t \in (0,T)$ and $x \in V$, where $f(u) = u(u-a)(1-u)$ and

$$P_t u_0(x) = \int_V K(t,x,y) u_0(y)\, d\mu(y), \quad t > 0 \text{ and } x \in V. \tag{3.4}$$

A function $u : (0,T) \times V \to \mathbb{R}$ with $(du)|_{V_0} = 0$ is said to be a (local) strong solution of (3.1), (3.2) for $T > 0$ if Δu exists on $V \backslash V_0$ and u satisfies (3.1) pointwise on $(0,T)\times(V\backslash V_0)$, and the initial condition

$$\lim_{t\to 0}\int_V |u(t,x) - u_0(x)|^2\, d\mu(x) = 0. \tag{3.5}$$

Lemma 3.1 (Local Existence). *Let $u_0 \in C(V)$. Then* (3.1), (3.2) *has a local weak solution on $(0,t_1)\times V$ for some $t_1 > 0$ that is continuous up to $\{0\}\times V$.*

Proof. Local existence of weak solution of (3.1), (3.2) follows from the standard approximation procedure. We sketch the proof for completeness.

Let $u_0(t,x) = P_t u_0(x)$. Define

$$u_{m+1}(t,x) = P_t u_0(x) + b\int_0^t d\tau \int_V K(t-\tau,x,y) f(u_m(\tau,y))\, d\mu(y), \quad m \geq 0. \tag{3.6}$$

It is not hard to check that

$$|u_m(t,x)| \leq 2M_0 \quad \text{on } (0,t_1)\times V \tag{3.7}$$

for all $m \geq 0$, where $M_0 = \sup_{x\in V} |u_0(x)|$ and $t_1 = 1/b(1+M_0)(a+M_0)$, and that

$$\begin{aligned}\|u_{m+1}(t) - u_m(t)\|_\infty &\equiv \sup_{x\in V} |u_{m+1}(t,x) - u_m(t,x)| \\ &\leq \frac{(Lt)^m}{m!} \sup_{t\in(0,t_1)} \|u_1(t) - u_0(t)\|_\infty, \quad t \in (0,t_1),\end{aligned}$$

where $L = b\max_{|s|\leq 2M_0} |f'(s)|$. Therefore $\{u_m\}$ converges to some function u on $(0,t_1)\times V$, and u is the weak solution of (3.1), (3.2) on letting $m\to\infty$ in (3.6) and using the dominated convergence theorem.

It remains to show that such a u is continuous on $[0,t_1)\times V$. Let

$$\begin{aligned} z(t,x) &\equiv \int_0^t d\tau \int_V K(t-\tau,x,y) f(u(\tau,y))\, d\mu(y) \\ &= \int_0^{t-h} d\tau \int_V K(t-\tau,x,y) f(u(\tau,y))\, d\mu(y) \\ &\quad + \int_{t-h}^t d\tau \int_V K(t-\tau,x,y) f(u(\tau,y))\, d\mu(y) \\ &\equiv z_1^h(t,x) + z_2^h(t,x), \quad 0 < h < t. \end{aligned} \tag{3.8}$$

We see that $z_1^h(t,x)$ is continuous on $(0,t_1)\times V$ for fixed $h \in (0,t)$ since K is continuous on $(0,\infty)\times V\times V$. Note that there is some $c_3 > 0$ such that

$$|z(t,x) - z_1^h(t,x)| = |z_2^h(t,x)| \leq c_3\, h$$

for all $h \in (0,t)$ and all $(t,x) \in (0,t_1) \times V$. Thus z is continuous on $(0,t_1) \times V$. Clearly z is continuous at $\{0\} \times V$. Thus z is continuous on $[0,t_1) \times V$. On the other hand, it is easy to see that $P_t u_0(x)$ is continuous on $(0,t_1) \times V$ since K is continuous on $(0,\infty) \times V \times V$ (this does not require the continuity of the initial data u_0). In order to prove that $P_t u_0(x)$ is continuous at $\{0\} \times V$, we shall use the continuity of u_0. We first assume that $u_0 \in H^1(V)$, and write $u_0(x) = \sum_{n=0}^{\infty} a_n \varphi_n(x)$. We see that

$$\sum_{n=0}^{\infty} a_n^2 \lambda_n < \infty.$$

It follows from (2.8) that

$$\begin{aligned}\lim_{t\to 0} \sup_{x\in V} |P_t u_0(x) - u_0(x)|^2 &\le c_4 \lim_{t\to 0} W(P_t u_0 - u_0, P_t u_0 - u_0) \\ &\le c_4 \lim_{t\to 0} \sum_{n=0}^{\infty} (e^{-\lambda_n t} - 1)^2 a_n^2 \lambda_n = 0.\end{aligned} \tag{3.9}$$

Thus $P_t u_0(x)$ is continuous at $\{0\} \times V$ if $u_0 \in H^1(V)$. Let $u_0 \in C(V)$. There exists a sequence of $\{u_0^n\}$ in $H^1(V)$ such that

$$\lim_{n\to\infty} \sup_{x\in V} |u_0^n(x) - u_0(x)| = 0 \tag{3.10}$$

since $H(V)$ is dense in $C(V)$. Note that by (2.23),

$$|P_t u_0(x) - P_t u_0^n(x)| \le \sup_{y\in V} |u_0^n(y) - u_0(y)| \tag{3.11}$$

for all $t > 0$ and $x \in V$. Therefore

$$\begin{aligned}&\lim_{t\to 0} \sup_{x\in V} |P_t u_0(x) - u_0(x)| \\ &\le \lim_{t\to 0} \sup_{x\in V} [|P_t u_0(x) - P_t u_0^n(x)| + |P_t u_0^n(x) - u_0^n(x)| + |u_0^n(x) - u_0(x)|] \\ &\le 2 \sup_{x\in V} |u_0^n(x) - u_0(x)| \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

proving the continuity of $P_t u_0(x)$ at $\{0\} \times V$ if $u_0 \in C(V)$. Thus $u(t,x) = P_t u_0(x) + b\, z(t,x)$ is continuous on $[0,t_1) \times V$.

The initial data u_0 is said to satisfy the regularity condition if

$$\frac{\partial}{\partial t} P_t u_0(x) \quad \text{is bounded on } (0,\infty) \times V. \tag{3.12}$$

An example when (3.12) holds is

$$u_0(x) = \int_V K(\gamma, x, y) w_0(y)\, d\mu(y),$$

where $\gamma > 0$ and $w_0(y) \in L^1(V)$.

Lemma 3.2 (Regularity). *Suppose that the initial data u_0 satisfies the regularity condition* (3.12) *and that u is a weak solution of* (3.1), (3.2) *in $(0,t_1) \times V$ for some $t_1 > 0$ and continuous up to $\{0\} \times V$. Then $\frac{\partial u}{\partial t}(t,x)$ exists on $(0,t_1) \times V$. Moreover,*

$$\Delta u(t,x) = \frac{\partial u}{\partial t}(t,x) - bf(u(t,x)) \quad \text{on } (0,t_1) \times V. \tag{3.13}$$

Proof. The proof given here is the same in spirit as in [13]. We first show that u is uniformly Lipschitz continuous in t, that is,

$$|u(t+h,x) - u(t,x)| \le c_5 h \quad \text{for all } (t,x) \in (0,t_1 - h) \times V, \tag{3.14}$$

for $h > 0$ small. To see this, we have from (3.3) that for $(t,x) \in (0, t_1 - h) \times V$,

$$\begin{aligned} & u(t+h,x) - u(t,x) \\ &= P_{t+h}u_0(x) - P_t u_0(x) + b\int_t^{t+h} d\tau \int_V K(\tau,x,y) f(u(t+h-\tau,y))\, d\mu(y) \\ &\quad + b\int_0^t d\tau \int_V K(\tau,x,y)\left[f(u(t+h-\tau,y)) - f(u(t-\tau,y))\right]\, d\mu(y). \end{aligned} \tag{3.15}$$

Using the boundedness of u and (3.12), it follows from (3.15) that, setting

$$g(t) = \sup_{x\in V} |u(t+h,x) - u(t,x)| \quad \text{for} \quad t \in (0, t_1 - h),$$

$$g(t) \le M_1 h + c_6 \int_0^t g(t-\tau)\, d\tau \quad \text{for some} \quad M_1, c_6 > 0.$$

Applying Gronwall's inequality, we see that there exists some $c_5 > 0$ such that $g(t) \le c_5\, h$ for all $(t,x) \in (0, t_1 - h) \times V$, giving (3.14). Therefore, $\frac{\partial u}{\partial t}(t,x)$ exists for all $x \in V$ and almost every $t \in (0, t_1)$. From this, it is easy to see that the function z given in (3.8) satisfies that $\frac{\partial z}{\partial t}$ exists pointwise on $(0,t_1) \times V$, and

$$\begin{aligned} \frac{\partial z}{\partial t}(t,x) &= \int_V K(t,x,y) f(u_0(y))\, d\mu(y) \\ &\quad + \int_0^t d\tau \int_V K(t-\tau,x,y) \frac{\partial f}{\partial u}(u(\tau,y)) \frac{\partial u}{\partial \tau}(\tau,y)\, d\tau \quad \text{on } (0,t_1)\times V. \end{aligned}$$

Thus $\frac{\partial u}{\partial t}(t,x)$ exists on $(0,t_1)\times V$.

It remains to check (3.13). Clearly $\Delta P_t u_0(x) = \frac{\partial P_t u_0}{\partial t}(t,x)$ on $(0,\infty)\times V$. We claim that

$$\Delta z(t,x) = \frac{\partial z}{\partial t}(t,x) - f(u(t,x)) \quad \text{on } (0,t_1)\times V. \tag{3.16}$$

To see this, note that K satisfies the semigroup property

$$\int_V K(s_1,x,y)K(s_2,y,w)\, d\mu(y) = K(s_1+s_2,x,w) \qquad \text{for } s_1,s_2 > 0 \text{ and } x,w \in V.$$

Therefore, we have from (2.12) and (3.8) that for $h > 0$,

$$\begin{aligned} P_h z(t,x) &= \int_V K(h,x,y) z(t,y)\, d\mu(y) \\ &= \int_V K(h,x,y) \left[\int_0^t d\tau \int_V K(t-\tau,y,w) f(u(\tau,w))\, d\mu(w)\right] d\mu(y) \\ &= \int_0^t d\tau \int_V K(t+h-\tau,x,w) f(u(\tau,w))\, d\mu(w) \\ &= z(t+h,x) - \int_t^{t+h} d\tau \int_V K(t+h-\tau,x,w) f(u(\tau,w))\, d\mu(w). \end{aligned} \tag{3,.17}$$

Note that for $t > 0$,

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} d\tau \int_V K(t+h-\tau,x,w) f(u(\tau,w))\, d\mu(w) = f(u(t,x)).$$

Thus we have from (3.17) that for $t > 0$,

$$\Delta z(t,x) = \lim_{h\to 0} \frac{1}{h}\left[P_h z(t,x) - z(t,x)\right] = \frac{\partial z}{\partial t}(t,x) - f(u(t,x)),$$

giving (3.13).

Lemma 3.3 (A Priori Estimates). *Let u be a strong solution of* (3.1), (3.2) *in* $(0,T]\times V$ *for some* $T>0$ *and continuous at* $\{0\}\times V$. *Assume that the initial data* $u_0\in[0,1]$ *on* V. *Then*

$$0\le u(t,x)\le 1,\quad (t,x)\in[0,T]\times V. \tag{3.18}$$

Proof. Suppose that there exists a point $(t_0,x_0)\in(0,T]\times V$ such that $u(t_0,x_0)$ is the maximum of u over $[0,T]\times V$. We have that $u(t_0,x_0)\le 1$; otherwise, it follows from (2.26) that

$$0\le\frac{\partial u}{\partial t}(t_0,x_0)=(\Delta u+b\,u(u-a)(1-u))|_{(t_0,x_0)}<0,$$

giving a contradiction. Therefore, we have that $u(t,x)\le 1$ on $(t,x)\in[0,T]\times V$. In a similar way, we get that $u(t,x)\ge 0$ on $(t,x)\in[0,T]\times V$.

Theorem 3.1 (Global Existence and Uniqueness). *Assume that the initial data* $u_0\in C(V)$ *satisfies* (3.12) *and* $u_0\in[0,1]$ *on* V. *Then* (3.1), (3.2) *has a unique global strong solution with* $0\le u(t,x)\le 1$ *on* $(0,\infty)\times V$.

Proof. The global existence follows immediately from Lemmas 3.1, 3.2, 3.3. We show that the solution is unique. Suppose that there are two solutions u_1 and u_2. Let $v=u_2-u_1$ and $p(t)=\sup\limits_{x\in V}|u_2(t,x)-u_1(t,x)|,\ t>0$. By (3.3), we have

$$p(t)\le\text{const.}\int_0^t p(\tau)\,d\tau,\quad t>0,$$

giving $p(t)\equiv 0$ on $(0,\infty)$.

§4. Asymptotic Behavior as $t\to\infty$

In this section we discuss large time behavior of the solution of (3.1), (3.2). It is shown that the solution of (3.1), (3.2) decays in the L^2-norm to the solution of the associated ordinary differential equation if b is small.

Theorem 4.1. *Let $u\in[0,1]$ be a strong solution of* (3.1), (3.2) *and b be small. Then there exists constants $c_7,\alpha>0$ such that*

$$\int_V|u(t,x)-\bar u(t)|^2\,d\mu(x)\le c_7e^{-\alpha t} \tag{4.1}$$

for all $t>0$, where

$$\bar u(t)=\int_V u(t,x)\,d\mu(x). \tag{4.2}$$

Proof. Let

$$I(t)=\frac12\int_V|u(t,x)-\bar u(t)|^2\,d\mu(x),\quad t>0. \tag{4.3}$$

We claim that

$$2\lambda_1 I(t)\le W(u-\bar u,u-\bar u)\quad\text{for all } t>0, \tag{4.4}$$

where $\lambda_1>0$ is the least positive eigenvalue of (2.12). To see this, let $t>0$ be fixed. Since $u\in H^1(V)$, we write

$$u(t,x)=\sum_{n=0}^{\infty}a_n(t)\,\varphi_n(x),\quad x\in V,$$

where $a_n(t) = \int_V u(t,x)\varphi_n(x)\, d\mu(x)$. It follows that

$$\|u(t,x) - \bar{u}(t)\|_2^2 = \sum_{n=1}^{\infty} a_n(t)^2.$$

On the other hand,

$$W(u-\bar{u}, u-\bar{u}) = W(u,u) = \sum_{n=0}^{\infty} a_n(t)^2\, W(\varphi_n, \varphi_n) = \sum_{n=0}^{\infty} a_n(t)^2 \lambda_n \geq \lambda_1 \sum_{n=1}^{\infty} a_n(t)^2$$

since $\lambda_0 = 0$. Therefore, we have that for all $t > 0$,

$$\lambda_1 \|u(t,x) - \bar{u}(t)\|_2^2 \leq W(u-\bar{u}, u-\bar{u}),$$

giving (4.4).

From (4.3) and (4.4), we see that, using (4.2),

$$\begin{aligned} *I'(t) &= \int_V (u(t,x) - \bar{u}(t)) \left(\frac{\partial u}{\partial t}(t,x) - \frac{d\bar{u}}{dt}(t) \right) d\mu(x) \\ &= \int_V (u(t,x) - \bar{u}(t)) \frac{\partial u}{\partial t}(t,x)\, d\mu(x) \\ &= -W(u-\bar{u}, u-\bar{u}) + b \int_V (u(t,x) - \bar{u}(t))(f(u(t,x)) - f(\bar{u}(t)))\, d\mu(x) \\ &\leq -2\lambda_1 I(t) + 2bL_1 I(t) = -\alpha I(t), \quad t > 0, \end{aligned}$$

where $L_1 = \max_{0 \leq u \leq 1} |f'(u)|$ and $\alpha = 2(\lambda_1 - bL_1) > 0$ for b small. Therefore, we have

$$I(t) \leq c_7 e^{-\alpha t}, \quad t > 0,$$

giving (4.1) for b small.

Remark 4.1. Let $\bar{u}$ be given in (4.2). Then $\bar{u}$ satisfies

$$\frac{d\bar{u}}{dt}(t) = bf(\bar{u}(t)) + q(t), \quad \bar{u}(0) = \int_V u_0(x)\, d\mu(x), \tag{4.5}$$

where q satisfies $|q(t)| \leq c_8 e^{-\alpha t},\ t > 0$. To see this, note that

$$\begin{aligned} \frac{d\bar{u}}{dt}(t) &= \int_V \frac{\partial u}{\partial t}(t,x)\, d\mu(x) = \int_V [\Delta u(t,x) + b\, f(u(t,x))]\, d\mu(x) \\ &= b \int_V f(u(t,x))\, d\mu(x) = bf(\bar{u}) + b \int_V [f(u(t,x)) - f(\bar{u}(t))]\, d\mu(x), \end{aligned}$$

and (4.1) implies that

$$|q(t)| \leq b \int_V |f(u(t,x)) - f(\bar{u}(t))|\, d\mu(x) \leq c_8 e^{-\alpha t}.$$

Thus the solution of (3.1), (3.2) decays to the solution of the associated o.d.e. (4.5). Similar results on classical domains were obtained in [7].

Remark 4.2. From Theorem 4.1, we see that there exists $\{t_j\}_{j \geq 1}$ with $t_j \to \infty$ as $j \to \infty$ such that $u(t_j, x) \to A \quad$ as $j \to \infty$ for all $x \in V$, where A is a constant. And (4.5) says that $f(A) = 0$, which gives rise to $A = 0, a$ or 1, the only possible equilibrium state of (1.1) for b small.

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