Chin. Ann. of Math.
23B:4(2002),531-538.

BRILL-NOETHER MATRIX FOR RANK TWO VECTOR BUNDLES**

TAN Xiaojiang*

Abstract

Let X be an arbitrary smooth irreducible complex projective curve, $E \mapsto X$ a rank two vector bundle generated by its sections. The author first represents E as a triple $\{D_1, D_2, f\}$, where D_1, D_2 are two effective divisors with $d = \deg(D_1) + \deg(D_2)$, and $f \in H^0(X, [D_1] \mid_{D_2})$ is a collection of polynomials. E is the extension of $[D_2]$ by $[D_1]$ which is determined by f. By using f and the Brill-Noether matrix of $D_1 + D_2$, the author constructs a $2g \times d$ matrix W_E whose zero space gives $\mathrm{Im}\{H^0(X, [D_1]) \mapsto H^0(X, [D_1] \mid_{D_1})\} \oplus \mathrm{Im}\{H^0(X, E) \mapsto H^0(X, [D_2]) \mapsto H^0(X, [D_2] \mid_{D_2})\}$. From this and $H^0(X, E) = H^0(X, [D_1]) \oplus \mathrm{Im}\{H^0(X, E) \mapsto H^0(X, [D_2])\}$, it is got in particular that $\dim H^0(X, E) = \deg(E) - \mathrm{rank}(W_E) + 2$.

Keywords Brill-Noether matrix, Vector bundle, Effective divisor
2000 MR Subject Classification 14H20
Chinese Library Classification O187 **Document Code** A
Article ID 0252-9599(2002)04-0531-08

§1. Introduction

Let X be a smooth projective curve of genus g over C, $D = n_1p_1 + \cdots + n_kp_k$ a given effective divisor with $d = \deg(D) = n_1 + \cdots + n_k$. For $i = 1, \cdots, k$, let z_i be a local coordinate at p_i with $z_i(p_i) = 0$. Let $\mu = \{\mu_i = \sum\limits_{k=-n_i}^{-1} b_{ik} z_i^k\}$ be a given collection of Laurent tails (or principal parts). The Mittag-Leffler problem is to ask which collections of Laurent tails come from a global meromorphic function on X. Let w be a holomorphic form on X, assume at p_i, $w = f_i(z_i)dz_i$, then the residue of $\mu \cdot w$ at p_i is defined to be

$$\mathrm{Res}_{p_i}(\mu \cdot w) = \frac{1}{2\pi i} \int_\gamma \mu_i \cdot w,$$

where γ is any curve homotopic to $\{\mid z_i \mid = \epsilon\}$ in a small neighborhood of p_i.

The residue of $\mu \cdot w$ on X is defined to be

$$\mathrm{Res}(\mu \cdot w) = \sum_{i=1}^{k} \mathrm{Res}_{p_i}(\mu \cdot w).$$

Manuscript received June 7, 2000. Revised Novermber 10, 2000.

*School of Mathematical Sciences, Peking University, Beijing 100871, China.
E-mail: tan@math.pku.edu.cn

**Project supported by the National Natural Science Foundation of China.

It is well known that μ comes from a global meromorhpic function if and only if $\operatorname{Re}s(\mu\cdot w)=0$ for all holomorphic forms w on X.

Now let $\{w_1,\cdots,w_g\}$ be a linear basis of the space of all holomorphic forms on X, for each i assume at p_i, $w_t(z_i)=f_{ti}(z_i)dz_i$ for $t=1,\cdots,g$, let W_D be the matrix of the restrictions of $\{w_1,\cdots,w_g\}$ on D, that is,

$$
\begin{aligned}
W_D &= \begin{bmatrix} w_1\mid_D \\ w_2\mid_D \\ \vdots \\ w_g\mid_D \end{bmatrix} \\
&= \begin{bmatrix} f_{11}(p_1) & \cdots & \frac{1}{(n_1-1)!}f_{11}^{(n_1-1)}(p_1) & f_{12}(p_2) & \cdots & \frac{1}{(n_2-1)!}f_{12}^{(n_2-1)}(p_2) & \cdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{g1}(p_1) & \cdots & \frac{1}{(n_k-1)!}f_{g1}^{(n_1-1)}(p_1) & f_{g2}(p_2) & \cdots & \frac{1}{(n_2-1)!}f_{g2}^{(n_2-1)}(p_2) & \cdots \end{bmatrix}.
\end{aligned}
$$

For a collection of Laurent tails $\mu=\{\mu_i=\sum\limits_{k=-n_i}^{-1}b_{ik}z_i^k\}$, we denote it as a d-dimensional vector

$$\mu=(b_{1-1},b_{1-2},\cdots,b_{1-n_1},b_{2-1},\cdots,b_{2-n_2},\cdots)\in C^d.$$

Then μ comes from a global meromorphic function if and only if $W_D\cdot\mu^t=0$. From this one can get Riemann-Roch theorem easily.

The matrix W_D is called the Brill-Noether matrix of D. It plays a key role in the study of Brill-Noether theory for special divisors (or special line bundles).

Now if we let $[D]$ be the line bundle defined by D, let $[D]\mid_D$ be the skyscraper sheaf of the restriction of $[D]$ on D, $H^0(X,[D])$ be the space of holomorphic sections of $[D]$, then the Mitteg-Leffler problem could be given equivalently that a vector $\mu\in H^0(X,[D]\mid_D)=C^d$ is in the image of the restriction map $H^0(X,[D])\mapsto H^0(X,[D]\mid_D)$ if and only if $W_D\cdot\mu^t=0$, or the same, we have

$$\operatorname{Ker}(W_D)=\{\mu\mid\mu\in C^d,W_D\cdot\mu^t=0\}\cong\operatorname{Im}\{H^0(X,[D])\mapsto H^0(X,[D]\mid_D)\}\qquad(*)$$

and in particular, we get

$$\dim H^0(X,[D])=\deg(D)-\operatorname{rank}(W_D)+1.\qquad(**)$$

An effective divisor D is called special if $H^1(X,[D])\neq 0$, or if $\dim H^0(X,[D])>\deg(D)-g+1$. Let X^d be the d-fold symmetric product of X, X^d is a d-dimensional complex manifold, and it is the parameter space of all effective divisors of degree d. To study special divisors for given d and r, one defines

$$C_d^r=\{D\in X^d\mid\dim H^0(X,[D])\geq r+1\}.$$

It is called Brill-Noether variety. By (**), locally it can be given by

$$C_d^r=\{D\in X^d\mid\operatorname{rank}(W_D)\leq d-r\}.$$

This means C_d^r is a determinantal variety. From it, one gets the expected dimension of C_d^r is $\rho(g,d,r)+r=g-(r+1)(g-d+r)+r$. It was conjectured by Brill-Noether and proved by Griffiths-Harris that for generic X, C_d^r do have the expected dimension.

We refer to [1] for details of the Brill-Noether matrix and its applications in the study of Brill-Noether theory for special line bundles.

A vector bundle E on X is called special if both $H^0(X,E)$ and $H^1(X,E)$ are non-zero. To study the Brill-Noether theory for special vector bundles, it is nature to ask in what sense we can construct a Brill-Noether matrix W_E for E. In this paper, for rank two vector bundles E generated by its sections (E is then called effective vector bundle), we define a matrix W_E for which W_E shares the same properties $(*)$ and $(**)$ for E as W_D for line bundle $[D]$.

Before giving our main theorem, we will first introduce some basic notations. Let E be a rank two vector bundle on X with $H^0(X,E) \neq 0$, let $s \in H^0(X,E)$ be a non-zero section, L_1 be the line subbundle of E generated by s, let $L_2 = E/L_1$. we then have a splitting of E,

$$0 \mapsto L_1 \mapsto E \mapsto L_2 \mapsto 0.$$

If $\mathrm{Im}\{H^0(X,E) \mapsto H^0(X,L_2)\} = 0$, then $H^0(X,E) = H^0(X,L_1)$, and the study of $\dim H^0(X,E)$ is reduced to the study of line bundles. So to study the Brill-Noether for rank two vector bundles, we need only to consider those vector bundles E that $\mathrm{Im}\{H^0(X,E) \mapsto H^0(X,L_2)\} \neq 0$.

Definition 1.1.[2] *A rank two vector bundle E is said to be generated by its sections if there exists a splitting of E,*

$$0 \mapsto L_1 \mapsto E \mapsto L_2 \mapsto 0,$$

such that L_1 and L_2 are line bundles and both $H^0(X,L_1) \neq 0$ and

$$\mathrm{Im}\{H^0(X,E) \mapsto H^0(X,L_2)\} \neq 0.$$

Now let E be a rank two vector bundle generated by its sections, let

$$0 \mapsto L_1 \mapsto E \mapsto L_2 \mapsto 0$$

be a given splitting of E. E is then an extension of L_2 by L_1, it is determined by an element $e \in H^1(X, L_1 \otimes L_2^*)$. Now let $s_1 \in H^0(X,L_1), s_2 \in \mathrm{Im}\{H^0(X,E) \mapsto H^0(X,L_2)\}$ with both $s_1 \neq 0$ and $s_2 \neq 0$. Assume $D_1 = \mathrm{div}(s_1) = m_1p_1 + \cdots + m_tp_t + \cdots + m_sp_s, D_2 = \mathrm{div}(s_2) = n_tp_t + \cdots + n_sp_s + \cdots + n_kp_k, d_1 = \deg(D_1), d_2 = \deg(D_2)$ and let $D = D_1 + D_2, d = d_1 + d_2 = \deg(E)$. Choose a local coordinate cover $\mathcal{U} = \{U_i, z_i\}_{i=1}^n$ such that $U_i = \{\mid z_i \mid < 1\}_{i=1}^n$, and for $i = 1, \cdots, k$, we have $p_i \in U_i$, with $z_i(p_i) = 0$. Since s_2 can be lifted to a section of E, we have $s_2 \cdot e = 0$, that is, $e \in \mathrm{Im}\{H^0(X, L_1\mid_{D_2}) \mapsto H^1(X, L_1 \otimes L_2^*)\}$, where $L_1\mid_{D_2}$ is the skyscraper sheaf of the restriction of L_1 on D_2, and the map is induced from sequence

$$0 \mapsto L_1 \otimes L_2^* \mapsto^{\cdot s_2} L_1 \mapsto L_1\mid_{D_2} \mapsto 0.$$

Let e be the image of some $f \in H^0(X, L_1\mid_{D_2})$. f is then determined uniquely up to $\mathrm{Im}\{H^0(X,L_1) \mapsto H^0(X, L_1\mid_{D_2})\}$. It can be represented as a collection of polynomials $f = \{f_i(z_i)\}_{i=t}^k$, where $f_i(z_i)$ is a polynomial of z_i and $\deg(f_i(z_i)) < n_i$. So from E we get a triple $\{D_1, D_2, f\}$. Conversely, for a given triple $\{D_1, D_2, f\}$, where D_1 and D_2 are two effective divisors and $f \in H^0(X, [D_1]\mid_{D_2})$, let $e \in H^1(X, [D_1 - D_2])$ be the image of f, and E be the extension of $[D_2]$ by $[D_1]$ which is determined by e. E is then a rank two vector bundle generated by its sections (by s_{D_1} and s_{D_2}, where s_{D_1} and s_{D_2} are the canonical sections of $[D_1]$ and $[D_2]$). So to give a rank two vector bundle E generated by its sections will be the same as to give a triple $\{D_1, D_2, f\}$. From now on we will write E as $E = \{D_1, D_2, f\}$.

Now let $D_1 = m_1p_1 + \cdots + m_tp_t + \cdots + m_sp_s, D_2 = n_tp_t + \cdots + n_sp_s + \cdots + n_kp_k, d_1 = \deg(D_1), d_2 = \deg(D_2)$ and let $D = D_1 + D_2, d = d_1 + d_2 = \deg(E)$. Choose a local coordinate cover $\mathcal{U} = \{U_i, z_i\}_{i=1}^n$ such that $U_i = \{| z_i |< 1\}$, and for $i = 1, \cdots, k$ we have $p_i \in U_i$, with $z_i(p_i) = 0$. Assume L_1 and L_2 are two given line bundles, $L_1 |_{D_1}$ and $L_2 |_{D_2}$ are the skyscraper sheaves of the restrictions of L_1 on D_1 and L_2 on D_2.

Definition 1.2. *If* $g = \{g_i(z_i)\}_{i=1}^s \in H^0(X, L_1 |_{D_1})$, *and* $f = \{f_i(z_i)\}_{i=t}^k \in H^0(X, L_2 |_{D_2})$, *we define* $g + f = h \in H^0(X, (L_1 \otimes L_2) |_{D_1+D_2})$ *to be an element* $g + f = h = \{h_i(z_i)\}_{i=1}^k$ *such that*

$$h_i(z_i) = g_i(z_i)$$

if $1 \le i \le t - 1$;

$$h_i(z_i) = g_i(z_i) + z_i^{m_i} \cdot f_i(z_i)$$

if $t \le i \le s$; *and*

$$h_i(z_i) = f_i(z_i)$$

if $s + 1 \le i \le k$. (*Note here* $g + f$ *may not equal* $f + g$).

From this definition, we get

$$H^0(X, (L_1 \otimes L_2) |_{D_1+D_2}) = H^0(X, L_1 |_{D_1}) \oplus H^0(X, L_2 |_{D_2}).$$

That means for any $v \in H^0(X, (L_1 \otimes L_2) |_{D_1+D_2})$, we can find uniquely $v_1 \in H^0(X, L_1 |_{D_1})$ and $v_2 \in H^0(X, L_2 |_{D_2})$ such that $v = v_1 + v_2$.

Definition 1.3. *From above direct sum decomposition, we define two projection maps*

$$P_1 : H^0(X, (L_1 \otimes L_2) |_{D_1+D_2}) \mapsto H^0(X, L_1 |_{D_1}),$$
$$P_2 : H^0(X, (L_1 \otimes L_2) |_{D_1+D_2}) \mapsto H^0(X, L_2 |_{D_2})$$

to be

$$P_1(v) = v_1, P_2(v) = v_2.$$

Main theorem of this paper is

Theorem 1.1. *Let* $E = \{D_1, D_2, f\}$ *be a rank two vector bundle generated by its sections. By using the Brill-Noether matrix* W_D *of* $D = D_1 + D_2$ *and the polynomials of* f, *we can construct a matrix* W_E *such that*

$$\begin{aligned}\mathrm{Ker}(W_E) &\cong P_1(\mathrm{Ker}\{P_2 : \mathrm{Ker}(W_E) \mapsto H^0(X, [D_2] |_{D_2})\}) \\ &\oplus \mathrm{Im}\{P_2 : \mathrm{Ker}(W_E) \mapsto H^0(X, [D_2] |_{D_2})\} \cong \mathrm{Im}\{H^0(X, [D_1]) \\ &\mapsto H^0(X, [D_1] |_{D_1})\} \oplus \mathrm{Im}\{H^0(X, E) \mapsto H^0(X, [D_2]) \mapsto H^0(X, [D_2] |_{D_2})\},\end{aligned}$$

and since $H^0(X, E) = H^0(X, [D_1]) \oplus \mathrm{Im}\{H^0(X, E) \mapsto H^0(X, [D_2])\}$, *we get in particular that*

$$\dim H^0(X, E) = \deg(E) - \mathrm{rank}(W_E) + 2.$$

Here $\mathrm{Ker}(W_E) = \{\mu \in C^d = H^0(X, ([D_1 + D_2]) |_{D_1+D_2}) \mid W_E \cdot \mu^t = 0\}$ *and* $d = \deg(D)$.

§2. Proof of Theorem 1.1

Let $D = n_1p_1 + \cdots + n_kp_k$ be a given effective divisor, $d = \deg(D) = n_1 + \cdots + n_k$. For each i, assume z_i is a local coordinate at p_i with $z_i(p_i) = 0$. If w is a holomorphic form on X, supposing at p_i, $w = f_i(z_i)dz_i$, we then use

$$w |_D = \Big(f_1(p_1), \cdots, \frac{1}{(n_1 - 1)!} f_1^{(n_1-1)}(p_1), f_2(p_2), \cdots, \frac{1}{(n_2 - 1)!} f_2^{(n_2-1)}(p_2), \cdots\Big)$$

to denote the restriction of w on D. But for a line bundle L, if $s \in H^0(X, L)$ and at p_i, $s = f_i(z_i)$ for $i = 1, \cdots, k$, then the restriction of s on D will be given by

$$s\mid_D = \Big(\frac{1}{(n_1-1)!} f_1^{(n_1-1)}(p_1), \cdots, f_1(p_1), \frac{1}{(n_2-1)!} f_2^{(n_2-1)}(p_2), \cdots, f_2(p_2), \cdots\Big).$$

Also if L_1 and L_2 are two line bundles, $f = \{f_i(z_i)\}_{i=1}^k \in H^0(X, L_1\mid_D), g = \{g_i(z_i)\}_{i=1}^k \in H^0(X, L_2\mid_D)$, we define $f \cdot g \in H^0(X, L_1 \otimes L_2\mid_D)$ by

$$f \cdot g = \{f_i(z_i) g_i(z_i) \,(\mathrm{mod}(z_i^{n_i}))\}_{i=1}^k.$$

For a polynomial $p(z)$ of degree n with $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, we define an $(n+1) \times (n+1)$ matrix N_p to be

$$N_p = \begin{bmatrix} a_0 & a_1 & \cdots & a_n \\ 0 & a_0 & \cdots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 \end{bmatrix};$$

for a collection of polynomials $h = \{h_i(z_i)\}_{i=1}^k \in H^0(X, L\mid_D)$, we define a $d \times d$ matrix N_h to be

$$N_h = \begin{bmatrix} N_{h_1} & 0 & \cdots & 0 \\ 0 & N_{h_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N_{h_k} \end{bmatrix}.$$

Now let $E = \{D_1, D_2, f\}$ be a given rank two vector bundle generated by its sections, and assume $D_1 = m_1 p_1 + \cdots + m_t p_t + \cdots + m_s p_s$, $D_2 = n_t p_t + \cdots + n_s p_s + \cdots + n_k p_k$ with $p_i \neq p_j$ if $i \neq j$. For $i = 1, \cdots, k$, let z_i be a local coordinate at p_i with $z_i(p_i) = 0$. Let $f = \{f_i(z_i)\}_{i=t}^k \in H^0(X, [D_1]\mid_{D_2})$. Now corresponding to each i, for $i = 1, \cdots, k$, we define, from m_i, n_i and $f_i(z_i)$, two matrixes M_i, N_i to be

$$M_i = [0]_{m_i \times m_i}, \quad N_i = [I]_{m_i \times m_i},$$

if $1 \le i \le t-1$;

$$M_i = \begin{bmatrix} 0_{m_i \times m_i} & 0_{m_i \times n_i} \\ 0_{n_i \times m_i} & I_{n_i \times n_i} \end{bmatrix}, \quad N_i = \begin{bmatrix} I_{m_i \times m_i} & 0_{m_i \times n_i} \\ 0_{n_i \times m_i} & N_{f_i} \end{bmatrix},$$

if $t \le i \le s$;

$$M_i = [I]_{n_i \times n_i}, \quad N_i = N_{f_i},$$

if $s+1 \le i \le k$.

From M_i and N_i we define two $d \times d$ matrixes M_E, N_E to be

$$M_E = \begin{bmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_k \end{bmatrix}, \quad N_E = \begin{bmatrix} N_1 & 0 & \cdots & 0 \\ 0 & N_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N_k \end{bmatrix}.$$

Now let W_D be the Brill-Noether matrix for $D = D_1 + D_2$. We define W_E by

$$W_E = \begin{bmatrix} W_D \cdot M_E \\ W_D \cdot N_E \end{bmatrix}.$$

W_E is a $2g \times d$ matrix, we claim W_E is the Brill-Noether matrix for E, that is, it satisfies our main Theorem.

A special and interesting case is $D_1 = 0$; in this case $M_E = I_{d\times d}, N_E = N_f$, and our theorem can be given as

Theorem 2.1. *Let $E = \{D_1, D_2, f\}$ be a given rank two vector bundle generated by its sections, assume $D_1 = 0$, and let W_D be the Brill-Noether matrix for divisor D_2. Then a vector $\mu \in H^0(X, [D_2] \mid_{D_2}) \cong C^d$ is in the image of map $H^0(X, E) \mapsto H^0(X, [D_2]) \mapsto H^0(X, [D_2] \mid_{D_2})$ if and only if*

$$\begin{bmatrix} W_D \\ W_D \cdot N_f \end{bmatrix} \mu^t = 0,$$

that is,

$$\mathrm{Im}\{H^0(X, E) \mapsto H^0(X, [D_2]) \mapsto H^0(X, [D_2] \mid_{D_2})\} \cong \mathrm{Ker}\Big(\begin{bmatrix} W_D \\ W_D \cdot N_f \end{bmatrix} \Big),$$

and we get in particular that

$$\mathrm{dim} H^0(X, E) = \deg(E) - \mathrm{rank} \begin{bmatrix} W_D \\ W_D \cdot N_f \end{bmatrix} + 2.$$

We start our proof from some very basic lemmas.

First let $E = \{D_1, D_2, f\}$ be a given rank two vector bundle generated by its sections, and let $e \in H^1(X, [D_1 - D_2])$ be the image of f.

Lemma 2.1.[3] *A section $s \in H^0(X, [D_2])$ can be lifted to a section of E if and only if $s \cdot e = 0$.*

Proof. It is well known.

Lemma 2.2.[4] *A section $s \in H^0(X, [D_2])$ can be lifted to a section of E if and only if*

$$s \mid_{D_2} \cdot f \in \mathrm{Im}\{H^0(X, [D_1 + D_2]) \mapsto H^0(X, [D_1 + D_2] \mid_{D_2})\}.$$

Proof. From the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \mapsto & [D_1 - D_2] & \to^{\cdot s_2} & [D_1] & \mapsto & [D_1] \mid_{D_2} & \mapsto & 0 \\ & & \downarrow \cdot s & & \downarrow \cdot s & & \downarrow \cdot s \mid_{D_2} & & \\ 0 & \mapsto & [D_1] & \to^{\cdot s_2} & [D_1 + D_2] & \mapsto & [D_1 + D_2] \mid_{D_2} & \mapsto & 0 \end{array}$$

we get the following commutative diagram

$$\begin{array}{ccccccccc} \mapsto & H^0(X, [D_1]) & \mapsto & H^0(X, [D_1] \mid_{D_2}) & \mapsto & H^1(X, [D_1 - D_2]) & \mapsto \\ & \downarrow \cdot s & & \downarrow \cdot s \mid_{D_2} & & \downarrow \cdot s & \\ \mapsto & H^0(X, [D_1 + D_2]) & \mapsto & H^0(X, [D_1 + D_2] \mid_{D_2}) & \mapsto & H^1(X, [D_1]) & \mapsto \end{array}$$

$s \cdot e = 0$ means exactly $s \mid_{D_2} \cdot f \in \mathrm{Im}\{H^0(X, [D_1 + D_2]) \mapsto H^0(X, [D_1 + D_2] \mid_{D_2})\}$.

Proof of Theorem 1.1. Let $C^d = H^0(X, ([D_1 + D_2]) \mid_{D_1+D_2})$, and

$$H^0(X, ([D_1 + D_2]) \mid_{D_1+D_2}) = H^0(X, [D_1] \mid_{D_1}) \oplus H^0(X, [D_2] \mid_{D_2})$$

be the direct sum decomposition given in Definition 1.3. For

$$\mu = \{h_i(z_i)\}_{i=1}^k \in H^0(X, ([D_1 + D_2]) \mid_{D_1+D_2}),$$

by definition, we have $\deg(h_i(z_i)) = m_i - 1$ if $1 \le i \le t - 1$; $\deg(h_i(z_i)) = m_i + n_i - 1$, if

$t \le i \le s$; and $\deg(h_i(z_i)) = n_i - 1$ if $s+1 \le i \le k$. We denote μ as a d-dimensional vector

$$\begin{aligned}\mu = \Big(&\frac{1}{(m_1-1)!}h_1^{(m_1-1)}(p_1), \cdots, h_1(p_1), \cdots, \frac{1}{(m_{s-1}-1)!}h_{s-1}^{(m_{s-1}-1)}(p_{s-1}), \cdots,\\ &h_{s-1}(p_{s-1}), \frac{1}{(m_s+n_s-1)!}h_s^{(m_s+n_s-1)}(p_s), \cdots,\\ &h_s(p_s), \cdots, \frac{1}{(m_t+n_t-1)!}h_t^{(m_t+n_t-1)}(p_t), \cdots,\\ &h_t(p_t), \cdots, \cdots, \frac{1}{(n_{t+1}-1)!}h_{t+1}^{(n_{t+1}-1)}(p_{t+1}), \cdots,\\ &h_{t+1}(p_{s-1}), \cdots, \frac{1}{(n_k-1)!}h_k^{(k_k-1)}(p_k), \cdots, h_k(p_k)\Big).\end{aligned}$$

Then the projections

$$\begin{aligned}P_1 &: H^0(X, ([D_1+D_2]) \mid_{D_1+D_2}) \mapsto H^0(X, [D_1] \mid_{D_1}),\\ P_2 &: H^0(X, ([D_1+D_2]) \mid_{D_1+D_2}) \mapsto H^0(X, [D_2] \mid_{D_2}),\end{aligned}$$

which are defined in Definition 1.3, could be given by

$$\begin{aligned}P_1(\mu) = \Big(&\frac{1}{(m_1-1)!}h_1^{(m_1-1)}(p_1), \cdots, h_1(p_1), \cdots, \frac{1}{(m_{s-1}-1)!}h_{s-1}^{(m_{s-1}-1)}(p_{s-1}), \cdots,\\ &h_{s-1}(p_{s-1}), \frac{1}{(m_s-1)!}h_s^{(m_s-1)}(p_s), \cdots, h_s(p_s), \cdots,\\ &\frac{1}{(m_t-1)!}h_t^{(m_t-1)}(p_t), \cdots, h_t(p_t)\Big) \in C^{d_1} = H^0(X, [D_1] \mid_{D_1}),\\ P_2(\mu) = \Big(&\frac{1}{(m_s+n_s-1)!}h_s^{(m_s+n_s-1)}(p_s), \cdots, \frac{1}{m_s!}h_s^{(m_s)}(p_s), \cdots,\\ &\frac{1}{(m_t+n_t-1)!}h_t^{(m_t+n_t-1)}(p_t), \cdots, \frac{1}{m_t!}h_t^{(m_t)}(p_t), \cdots,\\ &\cdots, \frac{1}{(n_{t+1}-1)!}h_{t+1}^{(n_{t+1}-1)}(p_{t+1}), \cdots, h_{t+1}(p_{t+1}),\\ &\cdots, \frac{1}{(n_k-1)!}h_k^{(n_k-1)}(p_k), \cdots, h_k(p_k)\Big) \in C^{d_2} = H^0(X, [D_2] \mid_{D_2}).\end{aligned}$$

Now let

$$\ker(W_E) = \{\mu \in H^0(X, [D_1+D_2] \mid_{D_1+D_2}) \mid W_E \cdot \mu^t = 0\}.$$

For $\mu \in \operatorname{Ker}(W_E)$, from $W_E \cdot \mu^t = 0$, we need to show

$$P_2(\mu) \in \operatorname{Im}\{H^0(X, E) \mapsto H^0(X, [D_2]) \mapsto H^0(X, [D_2] \mid_{D_2})\}.$$

But $W_E \cdot \mu^t = 0$ means exactly $W_D \cdot M_E \cdot \mu^t = 0$ and $W_D \cdot N_E \cdot \mu^t = 0$.

First from the definition of M_E, it is easy to see that $M_E \cdot \mu^t = P_2(\mu)$, so $W_E \cdot M_E \cdot \mu^t = 0$ is the same as $W_{D_2} \cdot P_2(\mu)^t = 0$, where W_{D_2} is the Brill-Noether matrix of D_2. But by the definition of Brill-Noether matrix, $W_{D_2} \cdot P_2(\mu) = 0$ means

$$P_2(\mu) \in \operatorname{Im}\{H^0(X, [D_2]) \mapsto H^0(X, [D_2] \mid_{D_2})\}.$$

Let $P_2(\mu)$ be the image of some $s \in H^0(X, [D_2])$, that is, $P_2(\mu) = s \mid_{D_2}$. But $W_D \cdot N_E \cdot \mu^t = 0$ means exactly

$$N_E \mu^t \in \operatorname{Im}\{H^0(X, [D_1+D_2]) \mapsto H^0(X, [D_1+D_2] \mid_{D_1+D_2})\}.$$

Let it be the image of $t \in H^0(X, [D_1 + D_2])$. Then by the definition of N_E, we have

$$P_2(N_E\mu^t) = \begin{bmatrix} N_{f_s} & 0 & \cdots & 0 \\ 0 & N_{f_{s+1}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N_{f_k} \end{bmatrix} \cdot P_2(\mu)^t;$$

this means exactly that we have $\{P_2(\mu) \cdot f\} = \{s \mid_{D_2} \cdot f\}$. What we get now is that

$$\{s \mid_{D_2} \cdot f\} = P_2(N_E \cdot \mu^t) = P_2(t \mid_{D_2}),$$

that is, $\{s \mid_{D_2} \cdot f\}$ is the restriction of t on D_2. By Lemma 2.2, this means s can be lifted to a section E. We get

$$P_2(\mu) = s \mid_{D_2} \in \mathrm{Im}\{H^0(X, E) \mapsto H^0(X, [D_2]) \mapsto H^0(X, [D_2] \mid_{D_2})\}.$$

Conversely, if $v \in \mathrm{Im}\{H^0(X, E) \mapsto H^0(X, [D_2]) \mapsto H^0(X, [D_2] \mid_{D_2})\}$, let it be the image of $s \in H^0(X, [D_2])$, i.e. $v = s \mid_{D_2}$. Then s can be lifted to a section of E, by Lemma 2.2, this means $\{s \mid_{D_2} \cdot f\}$ is the restriction on D_2 of some $t \in H^0(X, [D_1 + D_2])$, i,e, $\{s \mid_{D_2} \cdot f\} = t \mid_{D_2}$. Let $\mu = P_1(t \mid_{D_1+D_2}) + v$, where the $+$ is defined in Definition 1.3. Then since $v \in \{H^0(X, [D_2]) \mapsto H^0(X, [D_2] \mid_{D_2})\}$, and $W_D \cdot M_E \cdot \mu^t = 0$ is equivalent to $W_{D_2} v^t = 0$, we get $W_D \cdot M_E \cdot \mu^t = 0$. Since $N_E \cdot \mu^t = t \mid_{D_1+D_2}$, so $W_D \cdot N_E \cdot \mu^t = 0$, we then get $\mu \in \mathrm{Ker}(W_E)$ and $v = P_2(\mu)$. From this, we get

$$P_2(\mathrm{Ker}(W_E)) = \mathrm{Im}\{H^0(X, E) \mapsto H^0(X, [D_2]) \mapsto H^0(X, [D_2] \mid_{D_2})\}.$$

Now assume $\mu \in \mathrm{Ker}(P_2(\mathrm{Ker}(W_E))), v = P_1(\mu)$, and let 0 be the zero element in $H^0(X, [D_2] \mid_{D_2})$. Then $\mu = v + 0$ as defined in Definition 1.3. In this case, $W_E \cdot \mu^t = 0$ is exactly $W_{D_1} \cdot v^t = 0$, that means $v \in \mathrm{Im}\{H^0(X, [D_1]) \mapsto H^0(X, [D_1] \mid_{D_1})\}$. We get

$$P_1(\mathrm{Ker}(P_2(\mathrm{Ker}(W_E)))) = \mathrm{Im}\{H^0(X, [D_1]) \mapsto H^0(X, [D_1] \mid_{D_1})\}.$$

This gives

$$\begin{aligned} P_1(\mathrm{Ker}(P_2(\mathrm{Ker}(W_E)))) \oplus P_2(\mathrm{Ker}(W_E)) &= \mathrm{Im}\{H^0(X, [D_1]) \mapsto H^0(X, [D_1] \mid_{D_1})\} \\ &\oplus \mathrm{Im}\{H^0(X, E) \mapsto H^0(X, [D_2]) \mapsto H^0(X, [D_2] \mid_{D_2})\}. \end{aligned}$$

Since

$$\begin{aligned} H^0(X, E) &= \mathrm{Ker}\{H^0(X, E) \mapsto H^0(X, [D_2])\} \oplus \mathrm{Im}\{H^0(X, E) \mapsto H^0(X, [D_2])\} \\ &= H^0(X, [D_1]) \oplus \mathrm{Im}\{H^0(X, E) \mapsto H^0(X, [D_2])\}, \end{aligned}$$

we get in particular that $\dim H^0(X, E) = d - \mathrm{rank}(W_E) + 2.1$ This completes the proof.

Acknowledgements. The author is grateful to the referee, who carefully read this paper and gave many helpful suggestions.

References

[1] Arbarello, E., Cornaba, M., Griffiths, P. & Harris, J., Geometry of algebraic curves [M], Vol. I, Springer-Verlag, N. Y. 1984.

[2] Atiyah, M. F., Vector bundles on elliptic curves [J], *Proc. London Math. Soc.,* **7**(1957), 414–452.

[3] Lange, H. & Narasimhan, M. S., Maximal subbundles of rank two bundles on curves [J], *Math. Ann.,* **266**(1983), 55–72.

[4] Tan, X.J., Some results on the existence of rank two special stable vector bundles [J], *Manuscripta Math.,* **75**(1992), 365–373.