Chin. Ann. of Math.
23B:4(2002),539-544.

GENERALIZED SIMPLE NONCOMMUTATIVE TORI**

C. G. PARK*

Abstract

The generalized noncommutative torus T_ρ^k of rank n was defined in [4] by the crossed product $A_{\frac{m}{k}} \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \cdots \times_{\alpha_n} \mathbb{Z}$, where the actions α_i of $\mathbb{Z}$ on the fibre $M_k(\mathbb{C})$ of a rational rotation algebra $A_{\frac{m}{k}}$ are trivial, and $C^*(k\mathbb{Z} \times k\mathbb{Z}) \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \cdots \times_{\alpha_n} \mathbb{Z}$ is a completely irrational noncommutative torus A_ρ of rank n. It is shown in this paper that T_ρ^k is strongly Morita equivalent to A_ρ, and that $T_\rho^k \otimes M_{p^\infty}$ is isomorphic to $A_\rho \otimes M_k(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of k is a subset of the set of prime factors of p.

Keywords Noncommutative torus, Equivalence bimodule, UHF-algebra, Cuntz algebra, Crossed product
2000 MR Subject Classification 46L05, 46L87
Chinese Library Classification O177.5 **Document Code** A
Article ID 0252-9599(2002)04-0539-06

§0. Introduction

Given a locally compact abelian group G and a multiplier ω on G, one can associate to them the twisted group C^*-algebra $C^*(G, \omega)$. $C^*(\mathbb{Z}^n, \omega)$ is said to be a noncommutative torus of rank n and denoted by A_ω. The multiplier ω determines a subgroup S_ω of G, called its symmetry group, and the multiplier ω is called totally skew if the symmetry group S_ω is trivial. And A_ω is called completely irrational if ω is totally skew (see [1]). It was shown in [1] that if G is a locally compact abelian group and ω is a totally skew multiplier on G, then $C^*(G, \omega)$ is a simple C^*-algebra.

Boca[3] showed that almost all completely irrational noncommutative tori are isomorphic to inductive limits of circle algebras, where the term "circle algebra" denotes a C^*-algebra which is a finite direct sum of C^*-algebras of the form $C(\mathbb{T}^1) \otimes M_q(\mathbb{C})$. We will assume that each completely irrational noncommutative torus appearing in this paper is an inductive limit of circle algebras.

In [6], the authors showed that two separable C^*-algebras A and B are stably isomorphic if and only if they are strongly Morita equivalent, i.e., there exists an A-B-equivalence bimodule defined in [15]. In [5], M. Brabanter constructed an $A_{\frac{m}{k}}$-$C(\mathbb{T}^2)$-equivalence bimodule. Modifying his construction, we are going to construct a T_ρ^k-A_ρ-equivalence bimodule.

It was shown in [2, Theorem 1.5] that each completely irrational noncommutative torus has real rank 0, where the "*real rank* 0" means that the set of invertible self-adjoint elements

Manuscript received April 17, 2000. Revised August 9, 2001.
*Department of Mathematics, Chungnam National Uinversity, Taejon 305-764, South Korea.
E-mail: cgpark@math.chungnam.ac.kr
**Project supported by Grant No.1999-2-102-001-3 from the Interdisciplinary Research Program Year of the KOSEF.

is dense in the set of self-adjoint elements. Combining Theorem 1.1 in the next section and [7, Corollary 3.3] yields that the generalized noncommutative torus T_ρ^k has real rank 0, since the completely irrational noncommutative torus A_ρ has real rank 0. And the Lin and Rørdam results [13, Proposition 2 and Proposition 3] say that the generalized noncommutative torus T_ρ^k is an inductive limit of circle algebras, since $T_\rho^k \otimes \mathcal{K}(\mathcal{H}) \cong A_\rho \otimes \mathcal{K}(\mathcal{H})$ is an inductive limit of circle algebras. Combining the Elliott classification theorem[11, Theorem 7.1] and the Ji and Xia result[12, Theorem 1.3] yields that the completely irrational noncommutative tori A_ω of rank n and the generalized noncommutative tori T_ρ^k of rank n are classified by the ranges of the traces, and so one can completely classify them up to isomorphism or up to strong Morita equivalence. Hence some completely irrational noncommutative tori A_ω of rank n are isomorphic to some generalized noncommutative tori T_ρ^k of rank n.

It is moreover shown that $T_\rho^k \otimes M_{p^\infty}$ is isomorphic to $A_\rho \otimes M_k(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of k is a subset of the set of prime factors of p, that $\mathcal{O}_{2u} \otimes T_\rho^k$ is isomorphic to $\mathcal{O}_{2u} \otimes A_\rho \otimes M_k(\mathbb{C})$ if and only if k and $2u-1$ are relatively prime, and that $\mathcal{O}_\infty \otimes T_\rho^k$ is not isomorphic to $\mathcal{O}_\infty \otimes A_\rho \otimes M_k(\mathbb{C})$ if $k>1$, where $\mathcal{O}_u$ and $\mathcal{O}_\infty$ denote the Cuntz algebra and the generalized Cuntz algebra, respectively.

§1. Generalized Noncommutative Tori

Let T_ρ^k be a generalized noncommutative torus given in the abstract.

Theorem 1.1. *T_ρ^k is stably isomorphic to $A_\rho \otimes M_k(\mathbb{C})$.*

Proof. By [5, Theorem 3], $A_{\frac{m}{k}}$ is strongly Morita equivalent to $C^*(k\mathbb{Z}\times k\mathbb{Z}) \otimes M_k(\mathbb{C})$, where $\widehat{k\mathbb{Z}\times k\mathbb{Z}}$ is the primitive ideal space of $A_{\frac{m}{k}}$. So $A_{\frac{m}{k}} \otimes \mathcal{K}(\mathcal{H})$ is isomorphic to $C^*(k\mathbb{Z}\times k\mathbb{Z}) \otimes M_k(\mathbb{C}) \otimes \mathcal{K}(\mathcal{H})$. The generalized noncommutative torus T_ρ^k of rank n is realized as the crossed product $A_{\frac{m}{k}} \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \cdots \times_{\alpha_n} \mathbb{Z}$, where α_i act trivially on the fibre $M_k(\mathbb{C})$ of $A_{\frac{m}{k}}$. So

$$\begin{aligned} T_\rho^k \otimes \mathcal{K}(\mathcal{H}) &\cong (A_{\frac{m}{k}} \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \cdots \times_{\alpha_n} \mathbb{Z}) \otimes \mathcal{K}(\mathcal{H}) \\ &\cong (A_{\frac{m}{k}} \otimes \mathcal{K}(\mathcal{H})) \times_{\widetilde{\alpha_3}} \mathbb{Z} \times_{\widetilde{\alpha_4}} \cdots \times_{\widetilde{\alpha_n}} \mathbb{Z}, \end{aligned}$$

where $\widetilde{\alpha_i}$ are the canonical extensions of α_i such that $\widetilde{\alpha_i}$ act trivially on $M_k(\mathbb{C}) \otimes \mathcal{K}(\mathcal{H})$. Thus

$$\begin{aligned} T_\rho^k \otimes \mathcal{K}(\mathcal{H}) &\cong (C^*(k\mathbb{Z}\times k\mathbb{Z}) \otimes M_k(\mathbb{C}) \otimes \mathcal{K}(\mathcal{H})) \times_{\widetilde{\alpha_3}} \mathbb{Z} \times_{\widetilde{\alpha_4}} \cdots \times_{\widetilde{\alpha_n}} \mathbb{Z} \\ &\cong (C^*(k\mathbb{Z}\times k\mathbb{Z}) \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \cdots \times_{\alpha_n} \mathbb{Z}) \otimes M_k(\mathbb{C}) \otimes \mathcal{K}(\mathcal{H}). \end{aligned}$$

Hence T_ρ^k is stably isomorphic to $(C^*(k\mathbb{Z}\times k\mathbb{Z}) \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \cdots \times_{\alpha_n} \mathbb{Z}) \otimes M_k(\mathbb{C}) \cong A_\rho \otimes M_k(\mathbb{C})$, as desired.

So T_ρ^k is stably isomorphic to A_ρ, which implies that T_ρ^k is strongly Morita equivalent to A_ρ. We are going to construct a T_ρ^k-A_ρ-equivalence bimodule.

It was shown in [5, Proposition 1] that $A_{\frac{m}{k}}$ is the C^*-algebra of matrices $(f_{ij})_{i,j=1}^k$ of functions f_{ij} with

$$f_{ij} \in C^*(k\mathbb{Z}\times k\mathbb{Z}) \text{ if } i,j \in \{1,2,\cdots,k-1\} \text{ or } (i,j)=(k,k),$$

$$f_{ik} \in \Omega \quad \& \quad f_{ki} \in \Omega^* \text{ if } i \in \{1,2,\cdots,k-1\},$$

where Ω and Ω^* are the $C^*(k\mathbb{Z}\times k\mathbb{Z})$-modules defined as

$$\Omega = \{f \in C(\widehat{k\mathbb{Z}} \times [0,1]) \mid f(z,1) = z^s f(z,0), \quad \forall z \in \widehat{k\mathbb{Z}}\},$$

$$\Omega^* = \{f \in C(\widehat{k\mathbb{Z}} \times [0,1]) \mid f^* \in \Omega\}$$

for an integer s such that $sm = 1 \pmod{k}$.

But the generalized noncommutative torus T_ρ^k has a matrix representation induced from the matrix representation of the rational rotation subalgebra $A_{\frac{m}{k}}$.

Proposition 1.1. *The generalized noncommutative torus T_ρ^k is isomorphic to the C^*-algebra of matrices $(g_{ij})_{i,j=1}^k$ of g_{ij} with*

$$g_{ij} \in A_\rho \text{ if } i,j \in \{1,2,\cdots,k-1\} \text{ or } (i,j)=(k,k),$$

$$g_{ik} \in \widetilde{\Omega} \quad \text{and} \quad g_{ki} \in \widetilde{\Omega}^* \text{ if } i \in \{1,2,\cdots,k-1\},$$

where $\widetilde{\Omega}$ and $\widetilde{\Omega}^*$ are the A_ρ-modules defined as $\widetilde{\Omega} = A_\rho \cdot \Omega$ and $\widetilde{\Omega}^* = A_\rho \cdot \Omega^*$. Here Ω and Ω^* are the $C^*(k\mathbb{Z} \times k\mathbb{Z})$-modules defined above.

Proof. One obtains from the definition of T_ρ^k that the isomorphism between $A_{\frac{m}{k}}$ and the C^*-algebra of matrices $(f_{ij})_{i,j=1}^k$ of f_{ij} satisfying the condition given above gives an isomorphism between T_ρ^k and the C^*-algebra of matrices $(g_{ij})_{i,j=1}^k$ of g_{ij} satisfying the condition given in the statement. Note that $\widetilde{\Omega}$ and $\widetilde{\Omega}^*$ are the A_ρ-modules defined by canonically replacing $C^*(k\mathbb{Z} \times k\mathbb{Z})$ in $\Omega = C^*(k\mathbb{Z} \times k\mathbb{Z}) \cdot \Omega$ with $A_\rho \cong C^*(k\mathbb{Z} \times k\mathbb{Z}) \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \cdots \times_{\alpha_n} \mathbb{Z}$, since the entries in the matrix representation of $A_{\frac{m}{k}}$ have a $C^*(k\mathbb{Z} \times k\mathbb{Z})$-module structure, and T_ρ^k may be obtained by canonically replacing $C^*(k\mathbb{Z} \times k\mathbb{Z})$ with $A_\rho \cong C^*(k\mathbb{Z} \times k\mathbb{Z}) \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \cdots \times_{\alpha_n} \mathbb{Z}$.

Theorem 1.2. *T_ρ^k is strongly Morita equivalent to A_ρ.*

Proof. Let X be the complex vector space $\Big(\bigoplus_1^{k-1} \widetilde{\Omega}\Big) \oplus A_\rho$. We will consider the elements of X as $(k,1)$ matrices where the first $(k-1)$ entries are in $\widetilde{\Omega}$ and the last entry is in A_ρ. If $x \in X$, denote by x^* the $(1,k)$ matrix resulting from x by transposition and involution so that $x^* \in \Big(\bigoplus_1^{k-1} \widetilde{\Omega}^*\Big) \oplus A_\rho$. The space X is a left T_ρ^k-module if module multiplication is defined by matrix multiplication $F \cdot x$, where $F = (g_{ij})_{i,j=1}^k \in T_\rho^k$ and $x \in X$. If $g \in A_\rho$ and $x \in X$, then $x \cdot [g]$ defines a right A_ρ-module structure on X. Now we define a T_ρ^k-valued and an A_ρ-valued inner products $\langle \cdot, \cdot \rangle_{T_\rho^k}$ and $\langle \cdot, \cdot \rangle_{A_\rho}$ on X by

$$\langle x, y \rangle_{T_\rho^k} = x \cdot y^* \qquad \text{and} \qquad \langle x, y \rangle_{A_\rho} = x^* \cdot y$$

if $x, y \in X$ and we have matrix multiplication on the right. By the same reasoning as the proof given in [5, Theorem 3], equipped with this structure, X becomes a T_ρ^k-A_ρ-equivalence bimodule, as desired.

D. Poguntke[14] proved that the noncommutative torus A_ω of rank n is stably isomorphic to a noncommutative torus A_ρ of rank n which has a trivial bundle structure. One can construct an A_ω-A_ρ-equivalence bimodule by the same trick as the proof given in Theorem 1.2.

The noncommutative torus A_ω of rank n is the universal object for unitary ω-representations of $\mathbb{Z}^n$, so A_ω is realized as $C^*(u_1, \cdots, u_n \mid u_i u_j = e^{2\pi i \theta_{ji}} u_j u_i)$, where u_i are unitaries and θ_{ji} are real numbers for $1 \le i,j \le n$.

We are going to show that $[1_{T_\rho^k}] \in K_0(T_\rho^k)$ is primitive.

Theorem 1.3. *Let T_ρ^k be a generalized noncommutative torus of rank n. Then $K_0(T_\rho^k) \cong K_1(T_\rho^k) \cong \mathbb{Z}^{2^{n-1}}$, and $[1_{T_\rho^k}] \in K_0(T_\rho^k)$ is primitive.*

Proof. By Theorem 1.1, T_ρ^k is stably isomorphic to $A_\rho \otimes M_k(\mathbb{C})$. By the Elliott theorem[10, Theorem 2.2],

$$K_0(T_\rho^k) \cong K_0(A_\rho) \cong \mathbb{Z}^{2^{n-1}}, \quad K_1(T_\rho^k) \cong K_1(A_\rho) \cong \mathbb{Z}^{2^{n-1}}.$$

So it is enough to show that $[1_{T_\rho^k}] \in K_0(T_\rho^k)$ is primitive. The proof is by induction on m. Assume that $m=2$. The result was obtained in [10, Theorem 2.2].

So assume that the result is true for all generalized noncommutative tori of rank $m = i-1$. Write $\mathbb{S}_i = C^*(\mathbb{S}_{i-1}, u_i)$, where $\mathbb{S}_i = C^*(A_{\frac{m}{k}}, u_3, \ldots, u_i)$. Then the inductive hypothesis applies to $\mathbb{S}_{i-1}$. Also, we can think of $\mathbb{S}_i$ as the crossed product by an action α_i of $\mathbb{Z}$ on $\mathbb{S}_{i-1}$, where the generator of $\mathbb{Z}$ corresponds to u_i, which acts on $C^*(u_1^k, u_2^k, \cdots, u_{i-1})$ by conjugation (sending u_j to $u_i u_j u_i^{-1} = e^{2\pi i\theta_{ji}} u_j, j \neq 1, 2$, and sending u_j^k to $u_i u_j^k u_i^{-1} = e^{2\pi ik\theta_{ji}} u_j^k, j = 1, 2$), and which acts trivially on $M_k(\mathbb{C})$. Here $C^*(u_1^k, u_2^k) \cong C^*(k\mathbb{Z} \times k\mathbb{Z})$. Note that this action is homotopic to the trivial action, since we can homotope θ_{ji} to 0. Hence $\mathbb{Z}$ acts trivially on the K-theory of $\mathbb{S}_{i-1}$. The Pimsner-Voiculescu exact sequence for a crossed product gives an exact sequence

$$K_0(\mathbb{S}_{i-1}) \xrightarrow{1-(\alpha_i)_*} K_0(\mathbb{S}_{i-1}) \xrightarrow{\Phi} K_0(\mathbb{S}_i) \to K_1(\mathbb{S}_{i-1}) \xrightarrow{1-(\alpha_i)_*} K_1(\mathbb{S}_{i-1})$$

and similarly for K_1, where the map Φ is induced by inclusion. Since $(\alpha_i)_* = 1$ and since the K-groups of $\mathbb{S}_{i-1}$ are free abelian, this reduces a split short exact sequence

$$\{0\} \to K_0(\mathbb{S}_{i-1}) \xrightarrow{\Phi} K_0(\mathbb{S}_i) \to K_1(\mathbb{S}_{i-1}) \to \{0\}$$

and similarly for K_1. So $K_0(\mathbb{S}_i)$ and $K_1(\mathbb{S}_i)$ are free abelian of rank $2 \cdot 2^{i-2} = 2^{i-1}$. Furthermore, since the inclusion $\mathbb{S}_{i-1} \to \mathbb{S}_i$ sends $1_{\mathbb{S}_{i-1}}$ to $1_{\mathbb{S}_i}$, $[1_{\mathbb{S}_i}]$ is the image of $[1_{\mathbb{S}_{i-1}}]$, which is primitive in $K_0(\mathbb{S}_{i-1})$ by inductive hypothesis. Hence the image is primitive, since the Pimsner-Voiculescu exact sequence is a split short exact sequence of torsion-free groups.

Therefore, $K_0(T_\rho^k) \cong K_1(T_\rho^k) \cong \mathbb{Z}^{2^{n-1}}$, and $[1_{T_\rho^k}] \in K_0(T_\rho^k)$ is primitive.

It was shown in [4, Lemma 4.1] that $\mathrm{tr}(K_0(T_\rho^k)) = \frac{1}{k} \cdot \mathrm{tr}(K_0(A_\rho))$. By [11, Theorem 7.1] and [12, Theorem 1.3], T_ρ^k is stably isomorphic to A_ρ, which was shown in Theorem 1.1. And it was also shown in [4, Theorem 4.2] that if A_ω is a completely irrational noncommutative torus of rank n with $\mathrm{tr}(K_0(A_\omega)) = \frac{1}{k} \cdot \mathrm{tr}(K_0(A_\rho))$ for A_ρ a completely irrational noncommutative torus of rank n then A_ω is isomorphic to a generalized noncommutative torus T_ρ^k of rank n.

Corollary 1.1. *Let p be a positive integer. Then $T_\rho^k \otimes M_p(\mathbb{C})$ is not isomorphic to $A \otimes M_{sp}(\mathbb{C})$ for A a C^*-algebra if s is greater than* 1.

Proof. Assume that $T_\rho^k \otimes M_p(\mathbb{C})$ is isomorphic to $A \otimes M_{sp}(\mathbb{C})$. Then the unit $1_{T_\rho^k} \otimes I_p$ maps to the unit $1_A \otimes I_{sp}$. So $[1_{T_\rho^k} \otimes I_p] = [1_A \otimes I_{sp}]$. Thus there is a projection $e \in T_\rho^k$ such that $p[1_{T_\rho^k}] = (sp)[e]$. But $K_0(T_\rho^k) \cong \mathbb{Z}^{2^{n-1}}$ is torsion-free, so $[1_{T_\rho^k}] = s[e]$. This contradicts Theorem 1.3.

Therefore, $T_\rho^k \otimes M_p(\mathbb{C})$ is not isomorphic to $A \otimes M_{sp}(\mathbb{C})$ if $s > 1$.

We have obtained that $[1_{T_\rho^k}] \in K_0(T_\rho^k)$ is primitive. This result is very useful to investigate the structure of the tensor products of generalized noncommutative tori with UHF-algebras and Cuntz algebras.

§2. Tensor Products of Generalized Noncommutative Tori with UHF-Algebras and Cuntz Algebras

Using the fact that $[1_{T_\rho^k}] \in K_0(T_\rho^k)$ is primitive, we investigate the structure of $T_\rho^k \otimes M_{p^\infty}$ for M_{p^∞} a UHF-algebra of type p^∞.

Theorem 2.1. *$T_\rho^k \otimes M_{p^\infty}$ is isomorphic to $A_\rho \otimes M_k(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of k is a subset of the set of prime factors of p.*

Proof. Assume that the set of prime factors of k is a subset of the set of prime factors of p. To show that $T_\rho^k \otimes M_{p^\infty}$ is isomorphic to $A_\rho \otimes M_k(\mathbb{C}) \otimes M_{p^\infty}$, it is enough to show that $T_\rho^k \otimes M_{k^\infty} \cong A_\rho \otimes M_k(\mathbb{C}) \otimes M_{k^\infty}$. But there exist the C^*-algebra homomorphisms which are the canonical inclusions $T_\rho^k \otimes M_{k^g}(\mathbb{C}) \hookrightarrow A_\rho \otimes M_k(\mathbb{C}) \otimes M_{k^g}(\mathbb{C})$ and the A_ρ-module maps $A_\rho \otimes M_{k^g}(\mathbb{C}) \hookrightarrow T_\rho^k \otimes M_{k^g}(\mathbb{C})$:

$$T_\rho^k \hookrightarrow A_\rho \otimes M_k(\mathbb{C}) \hookrightarrow T_\rho^k \otimes M_k(\mathbb{C}) \hookrightarrow A_\rho \otimes M_{k^2}(\mathbb{C}) \hookrightarrow \cdots .$$

The inductive limit of the odd terms

$$\cdots \to T_\rho^k \otimes M_{k^g}(\mathbb{C}) \to T_\rho^k \otimes M_{k^{g+1}}(\mathbb{C}) \to \cdots$$

is $T_\rho^k \otimes M_{k^\infty}$, and the inductive limit of the even terms

$$\cdots \to A_\rho \otimes M_{k^g}(\mathbb{C}) \to A_\rho \otimes M_{k^{g+1}}(\mathbb{C}) \to \cdots$$

is $A_\rho \otimes M_{k^\infty}$. Thus by the Elliott theorem[11, Theorem 2.1], $T_\rho^k \otimes M_{k^\infty}$ is isomorphic to $A_\rho \otimes M_{k^\infty}$.

Conversely, assume that $T_\rho^k \otimes M_{p^\infty} \cong A_\rho \otimes M_k(\mathbb{C}) \otimes M_{p^\infty}$. Then the unit $1_{T_\rho^k} \otimes 1_{M_{p^\infty}}$ maps to the unit $1_{A_\rho} \otimes 1_{M_{p^\infty}} \otimes I_k$. So

$$[1_{T_\rho^k} \otimes 1_{M_{p^\infty}}] = [1_{A_\rho} \otimes 1_{M_{p^\infty}} \otimes I_k],$$
$$[1_{T_\rho^k} \otimes 1_{M_{p^\infty}}] = [1_{T_\rho^k}] \otimes [1_{M_{p^\infty}}],$$
$$[1_{A_\rho} \otimes 1_{M_{p^\infty}} \otimes I_k] = k([1_{A_\rho}] \otimes [1_{M_{p^\infty}}]).$$

Under the assumption that the unit $1_{T_\rho^k} \otimes 1_{M_{p^\infty}}$ maps to the unit $1_{A_\rho} \otimes 1_{M_{p^\infty}} \otimes I_k$, if there is a prime factor q of k such that $q \nmid p$, then $[1_{M_{p^\infty}}] \neq q[e_\infty]$ for e_∞ a projection in M_{p^∞}. So there is a projection $e \in T_\rho^k$ such that $[1_{T_\rho^k}] = q[e]$. This contradicts Theorem 1.4. Thus the set of prime factors of k is a subset of the set of prime factors of p.

Therefore, $T_\rho^k \otimes M_{p^\infty}$ is isomorphic to $A_\rho \otimes M_k(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of k is a subset of the set of prime factors of p.

Let us study the structure of the tensor products of generalized noncommutative tori with (even) Cuntz algebras.

The Cuntz algebra $\mathcal{O}_u, 2 \leq u < \infty$, is the universal C^*-algebra generated by u isometries $s_1, \ldots, s_u$, i.e., $s_j^* s_j = 1$ for all j, with the relation $s_1 s_1^* + \cdots + s_u s_u^* = 1$. Cuntz[8,9] proved that $\mathcal{O}_u$ is simple and the K-theory of $\mathcal{O}_u$ is $K_0(\mathcal{O}_u) = \mathbb{Z}/(u-1)\mathbb{Z}$ and $K_1(\mathcal{O}_u) = 0$. He proved that $K_0(\mathcal{O}_u)$ is generated by the class of the unit.

Proposition 2.1. *Let u be a positive integer such that k and $u-1$ are not relatively prime. Then $\mathcal{O}_u \otimes T_\rho^k$ is not isomorphic to $\mathcal{O}_u \otimes A_\rho \otimes M_k(\mathbb{C})$.*

Proof. Let p be a prime such that $p \mid k$ and $p \mid u-1$. Suppose that $\mathcal{O}_u \otimes T_\rho^k$ is isomorphic to $\mathcal{O}_u \otimes A_\rho \otimes M_k(\mathbb{C})$. Then the unit $1_{\mathcal{O}_u \otimes T_\rho^k}$ maps to the unit $1_{\mathcal{O}_u \otimes A_\rho} \otimes I_k$. So $[1_{\mathcal{O}_u \otimes T_\rho^k}] = [1_{\mathcal{O}_u \otimes A_\rho} \otimes I_k] = k[1_{\mathcal{O}_u \otimes A_\rho}]$. Hence there is a projection e in $\mathcal{O}_u \otimes T_\rho^k$ such that $[1_{\mathcal{O}_u \otimes T_\rho^k}] = k[e]$. But $[1_{\mathcal{O}_u \otimes T_\rho^k}] = [1_{\mathcal{O}_u}] \otimes [1_{T_\rho^k}]$ and $[1_{\mathcal{O}_u}]$ is a generator of $K_0(\mathcal{O}_u) \cong \mathbb{Z}/(u-1)\mathbb{Z}$ (see [9]). But $p \mid u-1$. $[1_{\mathcal{O}_u}] \neq p[e_*]$ for e_* a projection in $\mathcal{O}_u$. So $[1_{T_\rho^k}] = p[e']$ for e' a projection in T_ρ^k. This contradicts Theorem 1.3. Hence k and $u-1$ are relatively prime.

Therefore, $\mathcal{O}_u \otimes T_\rho^k$ is not isomorphic to $\mathcal{O}_u \otimes A_\rho \otimes M_k(\mathbb{C})$ if k and $u-1$ are not relatively prime.

The following result is useful to understand the structure of $\mathcal{O}_u \otimes T_\rho^k$.

Proposition 2.2.[16, Theorem 7.2] *Let A and B be unital simple inductive limits of even Cuntz algebras. If $\alpha : K_0(A) \to K_0(B)$ is an isomorphism of abelian groups satisfying $\alpha([1_A]) = [1_B]$, then there is an isomorphism $\phi : A \to B$ which induces α.*

Corollary 2.1. (1) *Let p be an odd integer such that p and $2u-1$ are relatively prime. Then $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{(2u-1)p+1} \otimes M_{p^\infty}$. That is, $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{p^\infty}$.*

(2) *$\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{(2u)^\infty}$.*

Theorem 2.2. *$\mathcal{O}_{2u} \otimes T_\rho^k$ is isomorphic to $\mathcal{O}_{2u} \otimes A_\rho \otimes M_k(\mathbb{C})$ if and only if k and $2u-1$ are relatively prime.*

Proof. Assume that k and $2u-1$ are relatively prime. Let $k = p2^v$ for some odd integer p. Then p and $2u-1$ are relatively prime. Then by Corollary 2.4 $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{p^\infty}$, and $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{(2u)^\infty} \cong \mathcal{O}_{2u} \otimes M_{(2u)^\infty} \otimes M_{(2^v)^\infty} \cong$

$\mathcal{O}_{2u} \otimes M_{(2^v)^\infty}$. So $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{p^\infty} \otimes M_{(2^v)^\infty} \cong \mathcal{O}_{2u} \otimes M_{k^\infty}$. Thus by Theorem 2.1 $\mathcal{O}_{2u} \otimes T_\rho^k$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{k^\infty} \otimes T_\rho^k$, which in turn is isomorphic to $\mathcal{O}_{2u} \otimes M_{k^\infty} \otimes A_\rho \otimes M_k(\mathbb{C})$. Hence $\mathcal{O}_{2u} \otimes T_\rho^k$ is isomorphic to $\mathcal{O}_{2u} \otimes A_\rho \otimes M_k(\mathbb{C})$.

The converse was proved in Proposition 2.1.

Therefore, $\mathcal{O}_{2u} \otimes T_\rho^k$ is isomorphic to $\mathcal{O}_{2u} \otimes A_\rho \otimes M_k(\mathbb{C})$ if and only if k and $2u-1$ are relatively prime.

However, we do not know whether or not $\mathcal{O}_{2u+1} \otimes T_\rho^k$ is isomorphic to $\mathcal{O}_{2u+1} \otimes A_\rho \otimes M_k(\mathbb{C})$ when k and $2u$ are relatively prime, since we do not know whether or not the result corresponding to Proposition 2.2 does hold for odd Cuntz algebras.

Cuntz[9] computed the K-theory of the generalized Cuntz algebra $\mathcal{O}_\infty$, generated by a sequence of isometries with mutually orthogonal ranges, $K_0(\mathcal{O}_\infty) = \mathbb{Z}$ and $K_1(\mathcal{O}_\infty) = 0$. He proved that $K_0(\mathcal{O}_\infty)$ is generated by the class of the unit.

Proposition 2.3. *$\mathcal{O}_\infty \otimes T_\rho^k$ is not isomorphic to $\mathcal{O}_\infty \otimes A_\rho \otimes M_k(\mathbb{C})$ if $k > 1$.*

Proof. Suppose $\mathcal{O}_\infty \otimes T_\rho^k$ is isomorphic to $\mathcal{O}_\infty \otimes A_\rho \otimes M_k(\mathbb{C})$. The unit $1_{\mathcal{O}_\infty \otimes T_\rho^k}$ maps to the unit $1_{\mathcal{O}_\infty \otimes A_\rho} \otimes I_k$. By the same trick as in the proof of Proposition 2.1, one can show that $[1_{\mathcal{O}_\infty \otimes T_\rho^k}] = k[e]$ for a projection $e \in \mathcal{O}_\infty \otimes T_\rho^k$. $[1_{\mathcal{O}_\infty \otimes T_\rho^k}] = [1_{\mathcal{O}_\infty}] \otimes [1_{T_\rho^k}]$ and $[1_{\mathcal{O}_\infty}]$ is a primitive element of $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$ (see [9]). So $[1_{T_\rho^k}] = k[e']$ for a projection $e' \in T_\rho^k$. This contradicts Theorem 1.3 if $k > 1$.

Therefore, $\mathcal{O}_\infty \otimes T_\rho^k$ is not isomorphic to $\mathcal{O}_\infty \otimes A_\rho \otimes M_k(\mathbb{C})$.

Acknowledgements. The author would also like to thank the reference for a number of valuable suggestions to a previous version of this paper.

References

[1] Baggett, L. & Kleppner, A., Multiplier representations of abelian groups [J], *J. Funct. Anal.,* **14**(1973), 299–324.
[2] Blackadar, B., Kumjian, A. & Rørdam, M., Approximately central matrix units and the structure of non-commutative tori [J], *K-Theory,* **6**(1992), 267–284.
[3] Boca, F., The structure of higher-dimensional noncommutative tori and metric Diophantine approximation [J], *J. Reine Angew. Math.,* **492**(1997), 179–219.
[4] Boo, D. & Park, C., The fundamental group of the automorphism group of a noncommutative torus [J], *Chin. Ann. Math.,* **21B:**4(2000), 441–452.
[5] Brabanter, M., The classification of rational rotation C^*-algebras [J], *Arch. Math.,* **43**(1984), 79–83.
[6] Brown, L., Green, P. & Rieffel, M., Stable isomorphism and strong Morita equivalence of C^*-algebras [J], *Pacific J. Math.,* **71**(1977), 349–363.
[7] Brown. L. & Pedersen, G., C^*-algebras of real rank zero [J], *J. Funct. Anal.,* **99**(1991), 131–149.
[8] Cuntz, J., Simple C^*-algebras generated by isometries [J], *Comm. Math. Phys.,* **57**(1977), 173–185.
[9] Cuntz, J., K-theory for certain C^*-algebras [J], *Ann. Math.,* **113**(1981), 181–197.
[10] Elliott, G. A., On the K-theory of the C^*-algebra generated by a projective representation of a torsion-free discrete abelian group [A], Operator algebras and group representations [M], (G. Arsene et al., ed.), vol. 1, Pitman, London, 1984, 157–184.
[11] Elliott, G. A., On the classification of C^*-algebras of real rank zero [J], *J. Reine Angew. Math.,* **443** (1993), 179–219.
[12] Ji, R. & Xia, J., On the classification of commutator ideals [J], *J. Funct. Anal.,* **78**(1988), 208–232.
[13] Lin, H. & Rørdam, M., Extensions of inductive limits of circle algebras [J], *J. London Math. Soc.,* **51** (1995), 603–613.
[14] Poguntke, D., The structure of twisted convolution C^*-algebras on abelian groups [J], *J. Operator Theory,* **38**(1997), 3–18.
[15] Rieffel, M., Morita equivalence for operator algebras [A], Operator algebras and applications [M], (R.V. Kadison ed.), Proc. Symp. Pure Math., Vol. 38, Amer. Math. Soc., Providence, R.I., 1982, 285–298.
[16] Rørdam, M., Classification of inductive limits of Cuntz algebras, *J. Reine Angew. Math.,* **XL:**4(1988), 257–338.