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COMPLETE LIE ALGEBRAS WITH l-STEP NILPOTENT RADICALS***

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Abstract

The authors first give a necessary and sufficient condition for some solvable Lie algebras with l-step nilpotent radicals to be complete, and then construct a new class of infinite dimensional complete Lie algebras by using the modules of simple Lie algebras. The quotient algebras of this new constructed Lie algebras are non-solvable complete Lie algebras with l-step nilpotent radicals.

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§1. Introduction

A Lie algebra $\mathfrak{g}$ is called a complete Lie algebra, if $\mathfrak{g}$ satisfies the two conditions

$$C(\mathfrak{g}) = 0, \quad \mathrm{Der}\mathfrak{g} = \mathrm{ad}\mathfrak{g}.$$

It is well-known that semisimple Lie algebras over field of characteristic 0, the Borel subalgebras and the parabolic subalgebras of complex semisimple Lie algebras are complete Lie algebras. In [1,2] some complete Lie algebras with commutative nilpotent radicals and other complete Lie algebras whose nilpotent radicals are the direct sum of abelian Lie algebras and Heisenberg algebras were given. In [3] solvable complete Lie algebras were studied. However, up to now there are a great deal of complete Lie algebras unknown. So looking for complete Lie algebras is still an important task. In this paper we first give a necessary and sufficient condition for some solvable Lie algebras with l-step nilpotent radicals to be complete. Then we give a method to construct non-solvable complete Lie algebras. Throughout the paper, unless specially pointed out, we always discuss finite dimensional Lie algebras over the complex field $\mathbf{C}$.

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§2. Solvable Complete Lie Algebras with l-Step Nilpotent Radicals

A Lie algebra L is said to be nilpotent if L^n is zero for some positive integer n. If $L^l \neq 0 = L^{l+1}$, we say L is l-step nilpotent.

Lemma 2.1.[4] *Let N be a nilpotent Lie algebra. Then the following assertions are equivalent*:

(1) $\{x_1, \cdots, x_n\}$ *is a minimal system of generators*;

(2) $\{x_1 + [N,N], \cdots, x_n + [N,N]\}$ *is a basis of* $N/[N,N]$.

Definition 2.1. *Let N be a nilpotent Lie algebra and $\mathfrak{h}$ a subalgebra of* $\mathrm{Der}\, N$. *If all elements of $\mathfrak{h}$ are semisimple linear transformations of N, then $\mathfrak{h}$ is called a torus on N.*

Suppose $\mathfrak{h}$ is a torus on N. Clearly N is decomposed into a direct sum of root spaces for $\mathfrak{h}$:

$$N = \bigoplus_{\beta \in \mathfrak{h}^*} N_\beta,$$

where $\mathfrak{h}^*$ is the dual space of the vector space $\mathfrak{h}$ and

$$N_\beta = \{x \in N : h.x = \beta(h)x, \quad \forall h \in \mathfrak{h}\}.$$

Definition 2.2. *Let $\mathfrak{h}$ be a maximal torus on N. One calls $\mathfrak{h}$-msg a minimal system of generators which consists of root vectors for $\mathfrak{h}$.*

Lemma 2.2.[4] *Let $\mathfrak{h}$ be a maximal torus on $N, \{x_1, \cdots, x_n\}$ an $\mathfrak{h}$-msg and $\{\beta_1, \cdots, \beta_n\}$ the corresponding roots, then $\{\beta_1, \cdots, \beta_n\}$ is a basis for the vector space $\mathfrak{h}^*$.*

Lemma 2.3.[4] *Let N be a nilpotent Lie algebra and $\mathfrak{h}_1, \mathfrak{h}_2$ two maximal tori on N. Then* $\dim\mathfrak{h}_1 = \dim\mathfrak{h}_2 \leq \dim N/[N,N]$.

Definition 2.3. *Let N be a nilpotent Lie algebra and $\mathfrak{h}$ a maximal torus on N. We call* $\dim\mathfrak{h}$ *and $n = \dim N/[N,N]$ the rank and type of N respectively. If the rank of N is the same as the type of N, then N is said to be of maximal rank.*

Now fix l and n. Let V be a vector space with a basis $\{y_1, \cdots, y_n\}$, form the tensor algebra $T(V)$ (view as Lie algebra via the bracket operation), and let L be the Lie subalgebra generated by $\{y_1, \cdots, y_n\}$. Set

$$N(l,n) = L/L^{l+1}$$

and let x_i denote the image of y_i under the canonical surjection $L \to N(l,n)$. Then $\{x_1, \cdots, x_n\}$ is a minimal system of generators of $N(l,n)$. We have the following well-known theorem.

Theorem 2.1. *$N(l,n)$ is an l-step niltopent Lie algebra of type n and any other l-step nilpotent Lie algebra of type n is a quotient of $N(l,n)$.*

We introduce semisimple derivations $d_1, \cdots, d_n$ of $T(V)$ such that

$$d_i(y_j) = \delta_{ij} y_j, \quad i,j = 1,2,\cdots,n.$$

It is easy to prove by induction that

$$d_i(L^k) \subseteq L^k, \quad k \in \mathbf{Z}_+.$$

So we have the semisimple derivations $h_1, \cdots, h_n$ of $N(l,n)$ such that

$$h_i(x_j) = \delta_{ij} x_j, \quad i,j = 1,2,\cdots,n.$$

Let

$$\mathfrak{h} = \mathbf{C}h_1 + \cdots + \mathbf{C}h_n.$$

Clearly we have $[\mathfrak{h}, \mathfrak{h}] = 0$, $\dim\mathfrak{h} = n$, so $\mathfrak{h}$ is a maximal torus on $N(l, n)$ and $N(l, n)$ is a maximal rank nilpotent Lie algebra.

Let us recall here the definition of the holomorph of a Lie algebra L. Let L be a Lie algebra, $\mathrm{Der}L$ the derivation algebra of L. Set

$$\mathfrak{h}(L) = \mathrm{Der}L + L.$$

Define the bracket on $\mathfrak{h}(L)$ by

$$[D + x, E + y] = [D, E] + Dy - Ex + [x, y],$$

where $D, E \in \mathrm{Der}L, x, y \in L$. It is easy to prove that $\mathfrak{h}(L)$ is a Lie algebra which is called the holomorph of L. Then it is clear that we have the following lemma.

Lemma 2.4. *Let L be a Lie algebra, S a subalgebra of* $\mathrm{Der}\, L$. *Set*

$$\mathfrak{g} = S + L.$$

Then $\mathfrak{g}$ is a subalgebra of $\mathfrak{h}(L)$.

Set

$$R = \mathfrak{h} + N(l, n).$$

Since $\mathfrak{h}$ is a subalgebra of $\mathrm{Der}N(l, n)$, by Lemma 2.4, R becomes a Lie algebra.

Lemma 2.5.[3] *Let R' be a solvable Lie algebra. Then R' is a complete Lie algebra with maximal rank nilpotent radical N if and only if*

$$R' = \mathfrak{h} + N,$$

where $\mathfrak{h}$ is a maximal torus on N.

Theorem 2.2. *A Lie algebra $\mathfrak{g}$ is a solvable complete Lie algebra with l-step nilpotent radical of maximal rank n if and only if there exists an ideal I of R such that $I \subseteq [N(l, n), N(l, n)], (N(l, n)/I)^l \neq 0$, and $\mathfrak{g} \cong R/I$.*

Proof. By Lemma 2.5, the sufficiency is clear.

Now suppose $\mathfrak{g}$ is a complete Lie algebra as in the theorem. Then by Lemma 2.5 we have

$$\mathfrak{g} = \mathfrak{h}' + N.$$

Let $y_1, y_2, \cdots, y_n$ be an $\mathfrak{h}'$-msg and $\alpha_1, \alpha_2, \cdots, \alpha_n$ the corresponding roots. Then

$$[h, y_i] = \alpha_i(h)y_i, \quad i = 1, 2, \cdots, n.$$

By Lemma 2.2, $\alpha_1, \alpha_2, \cdots, \alpha_n$ are linearly independent. So there exist $h'_1, \cdots h'_n$ such that

$$\alpha_i(h'_j) = \delta_{ij}, \quad i, j = 1, 2, \cdots n.$$

Let $\varphi : R \to \mathfrak{g}$ be a linear map such that

$$h_i \to h'_i, x_j \to y_j, \quad i, j = 1, 2, \cdots, n.$$

Then φ is a homomorphism. Let $I = \ker\varphi$, then $\mathfrak{g} \cong R/I$. It is clear that I satisfies the conditions of the theorem.

§3. Non-Solvable Complete Lie Algebras

In this section we first construct a class of infinite dimensional Lie algebras and a class of finite dimensional Lie algebras with l-step nilpotent radicals. Then we prove that these Lie algebras are complete.

Let S be a semisimple Lie algebra, V a finite dimensional S-module. Then we have S-modules

$$\mathbf{C}, V, V\otimes V, V\otimes V\otimes V, \cdots,$$

where $\mathbf{C}$ is a trivial S-module. So the tensor algebra

$$T(V) = \mathbf{C} + V + V\otimes V + \cdots$$

becomes an S-module. For any x belonging to S, u, v belonging to V, by the definition of tensor module we have obviously

$$x.(u\otimes v) = (x.u)\otimes v + u\otimes (x.v).$$

So

$$S \subseteq \operatorname{Der} T(V).$$

Since S is a semisimple Lie algebra, V is completely reducible. Let

$$V = V_{\lambda_1} + V_{\lambda_2} + \cdots + V_{\lambda_s}$$

be the irreducible decomposition of V, λ_i be the highest weight of $V_{\lambda_i}, i = 1, 2, \cdots, s$.

We introduce semisimple derivations $d_1, d_2, \cdots, d_s$ of $T(V)$ such that

$$d_i|_{V_{\lambda_j}} = \delta_{ij}\mathbf{I}_{V_{\lambda_j}}, \quad i, j = 1, 2, \cdots, s.$$

Clearly it is well defined. Let

$$D = \mathbf{C}d_1 + \mathbf{C}d_2 + \cdots + \mathbf{C}d_s.$$

Then $D \subseteq \operatorname{Der} T(V), [D, D] = 0$.

Lemma 3.1. *Let (ρ, V) be an irreducible representation of S. If $x \in S, \rho(x) = kI_V, k \in \mathbf{C}$, then $k = 0$.*

Proof. If $\rho(x) \neq 0$, then $\rho(x)$ is a central element of $\rho(S)$. But $\rho(S)$ is semisimple, it is a contradiction.

Lemma 3.2. *As the subspaces of* $\operatorname{Der} T(V)$, *S and D have the following properties*:

(1) $[s_1 + u_1, s_2 + u_2] = [s_1, s_2], \quad s_1, s_2 \in S, u_1, u_2 \in D$;

(2) $S \cap D = 0$.

Proof. (1) It is straightforward.

(2) Suppose $c \in S \cap D$. Then

$$c = k_1 d_1 + k_2 d_2 + \cdots + k_s d_s.$$

If $c \neq 0$, then there exists $k_i \neq 0$. Therefore

$$c|_{V_{\lambda_i}} = k_i I_{V_{\lambda_i}} \neq 0.$$

But $c \in S$, by Lemma 3.1, it is impossible.

By Lemma 3.2, $S + D$ becomes a Lie algebra. So $T(V)$ becomes an $S + D$-module and $S + D \subseteq \operatorname{Der} T(V)$.

Fix a basis

$$X = \{x_{11}, \cdots, x_{1r_1}, \cdots, x_{s1}, \cdots, x_{sr_s}\}$$

of V consisting of weight vectors, where x_{i1} is a highest weight vector of $V_{\lambda_i}, i = 1, 2, \cdots, s$. Let L be the Lie subalgebra of $T(V)$ generated by X.

Lemma 3.3. *$L^l (l \in \mathbf{Z}_+)$ is an $S + D$-submodule of $T(V)$. In particular $S + D \subseteq \operatorname{Der} L^l$.*

Proof. We use induction for l.

When l=1, we have $L^l = L$. For any $x \in S + D$ and for any $u, v \in X$, because $x.u, x.v \in V, x.u, x.v$ can be written as the linear combinations of elements of X, we have

$$x.[u, v] = x.(u \otimes v - v \otimes u) = [x.u, v] + [u, x.v] \in L.$$

So it is true that $(S + D).L \subseteq L$.

Now suppose it is true for $l - 1$. For any $x \in S + D, u_1, u_2, \cdots, u_l \in L$, we have

$$x.[u_1, [u_2, \cdots, u_l] \cdots] = [x.u_1, [u_2, \cdots, u_l] \cdots] + [u_1, x.[u_2, \cdots, u_l] \cdots] \in L^l.$$

So we have proved the lemma.

Set

$$\tilde{\mathfrak{g}} = S + D + L$$

and define the bracket on $\tilde{\mathfrak{g}}$ by

$$[x + u, y + v] = [x, y] + x.v - y.u + [u, v].$$

By Lemma 2.4 and Lemma 3.3, $\tilde{\mathfrak{g}}$ becomes an infinite dimensional Lie algebra.

We will prove that $\tilde{\mathfrak{g}}$ is a complete Lie algebra. For that we need the following theorem.

Theorem 3.1. [2] *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. $\mathfrak{g}$ and $\mathfrak{h}$ satisfy the following conditions*:

(1) $\mathfrak{h}$ *is abelian*;

(2) *the decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is*

$$\mathfrak{g} = h + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where $\Delta \subset \mathfrak{h}^* - (0)$ and

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x, h \in \mathfrak{h}\};$$

(3) *there is a basis $\alpha_1, \alpha_2, \cdots, \alpha_n$ of $\mathfrak{h}^*$ in Δ such that*

$$\dim \mathfrak{g}_{\pm\alpha_i} \leq 1, [\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{-\alpha_i}] \neq 0, \ \textit{if} \ -\alpha_i \in \Delta, \quad i = 1, 2, \cdots, n;$$

(4) $\mathfrak{h}$ *and* $\{\mathfrak{g}_{\pm\alpha_i} : 1 \leq i \leq n\}$ *generate* $\mathfrak{g}$.

Then $\mathfrak{g}$ is a complete Lie algebra.

Theorem 3.2. $\tilde{\mathfrak{g}}$ *is an infinite dimensional complete Lie algebra.*

Proof. We prove that $\tilde{\mathfrak{g}}$ satisfies the conditions (1)–(4) of Theorem 3.1

Let $\mathfrak{h}_0$ be a Cartan subalgebra of S. The root space decomposition of S with respect to $\mathfrak{h}_0$ is

$$S = \mathfrak{h}_0 + \sum_{\alpha \in \Delta_0} S_\alpha.$$

It is easy to prove that

$$\mathfrak{h} = \mathfrak{h}_0 + D$$

is a Cartan subalgebra of $\tilde{\mathfrak{g}}$. Clearly it is abelian and for any $h \in \mathfrak{h}$, ad$h$ is semisimple on $\tilde{\mathfrak{g}}$. Now, for $\beta \in \mathfrak{h}_0^*$, by Lemma 3.2 we can extend β to a linear function on $\mathfrak{h}$ by setting

$$\beta(h) = \beta(h), \quad h \in \mathfrak{h}_0, \quad \beta(d_i) = 0, \quad i = 1, 2, \cdots, s.$$

We denote by δ_i the linear function on $\mathfrak{h}$ defined by

$$\delta_i|_{\mathfrak{h}_0} = 0, \quad \delta_i(d_j) = \delta_{ij}, \quad i, j = 1, 2, \cdots s.$$

Now we can describe the root space decomposition of $\tilde{\mathfrak{g}}$ with respect to $\mathfrak{h}$. Let the weight space decomposition of V_{λ_i} with respect to $\mathfrak{h}_0$ be

$$V_{\lambda_i} = \sum_{\lambda \in w(\lambda_i)} V_\lambda.$$

Then the root space decomposition of $\tilde{\mathfrak{g}}$ with respect to $\mathfrak{h}$ is

$$\tilde{\mathfrak{g}} = \mathfrak{h} + \sum_{\alpha \in \Delta_0} S_\alpha + \sum_{i=1}^{s} \sum_{\lambda \in w(\lambda_i)} V_{\lambda+\delta_i} + \sum_{i_1,\cdots,i_r} \tilde{\mathfrak{g}}_{\mu_{i_1}+\delta_{i_1}+\cdots+\mu_{i_r}+\delta_{i_r}},$$

where $\tilde{\mathfrak{g}}_{\mu_{i_1}+\delta_{i_1}+\cdots+\mu_{i_r}+\delta_{i_r}}$ is the root space of $\tilde{\mathfrak{g}}$ with root $\mu_{i_1}+\delta_{i_1}+\cdots+\mu_{i_r}+\delta_{i_r}$, which linearly spans by element $[v_{i_1j_1},[v_{i_2j_2},\cdots,v_{i_rj_r}]\cdots]$.

Suppose $\alpha_1,\alpha_2,\cdots,\alpha_n$ is a simple root system of Δ_0. Then

$$\alpha_1,\alpha_2,\cdots,\alpha_n,\lambda_1+\delta_1,\lambda_2+\delta_2,\cdots,\lambda_s+\delta_s$$

is a basis of $\mathfrak{h}^*$, and

$$S_{\pm\alpha_k}, V_{\lambda_1+\delta_1}, V_{\lambda_2+\delta_2},\cdots,V_{\lambda_s+\delta_s}$$

are 1-dimensional.

It is clear that $-(\lambda_{i_1}+\delta_{i_1}),\cdots,-(\lambda_{i_r}+\delta_{i_r})$ are not roots. On the other hand we have $-\alpha_k, k=1,2,\cdots,n$, are roots, and

$$[S_{\alpha_k}, S_{-\alpha_k}] \neq 0.$$

Clearly, $\{\mathfrak{h}, S_{\pm\alpha_k}, V_{\lambda_i}, k=1,2,\cdots,n; i=1,2,\cdots,s\}$ generate $\tilde{\mathfrak{g}}$.

From above discussion we have proved that $\tilde{\mathfrak{g}}$ satisfies the conditions (1)–(4) of Theorem 3.1, so $\tilde{\mathfrak{g}}$ is a complete Lie algebra.

Theorem 3.3. *$S+D+L/L^{l+1}$ is a complete Lie algebra with l-step nilpotent radical L/L^{l+1}, radical $D+L/L^{l+1}$ and Levi subalgebra S.*

Proof. Noticing that L^l is an ideal of L, by Lemmas 2.4 and 3.3 we know that $S+D+L/L^l$ becomes a Lie algebra. From the proof of Theorem 3.2 clearly we have the result.

Similarly we can prove the following theorems.

Theorem 3.4. *Let I be an ideal of $\tilde{\mathfrak{g}}$ and $I \subseteq L^2$. Then $S+D+L/I$ is a complete Lie algebra.*

Theorem 3.5. *The Borel subalgebra and the parabolic subalgebras of $S+D+L/L^{l+1}$ are complete Lie algebras.*

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