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# NEW WAVELET BASES FOR NON-HOMOGENEOUS SYMBOLIC SPACE $OpS^m_{1,1}$ AND RELATED KERNEL-DISTRIBUTION SPACES**

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**Abstract**

This paper constructs several classes of new wavelet bases, which are unconditional bases for related operator spaces. Using these bases, the author analyzes non-homogeneous symbolic space $OpS^m_{1,1}$ and two related kernel-distribution spaces, and characterizes them in two wavelet coefficients spaces. Besides, some properties for singular integral operators are studied.

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## §1. Problems and Introduction

Usual wavelet bases can serve as bases in most of function spaces, and the norm of a function is equivalent to a norm which is defined by the absolute value of its wavelet coefficients. Can one develop some unconditional wavelet bases for operator spaces? Furthermore, in 1950's Calderón established a formal relation between a symbol and a kernel-distribution; after this period, the symbol school and the kernel-distribution school remain almost in two isolated classes. How can one establish an internal relation between a symbol and a kernel-distribution? In this paper, we choose the non-homogeneuos space $OpS^m_{1,1}$ as an example to answer these two questions.

A symbolic operator $T=\sigma(x,D)$ is defined as follows:

$$\sigma(x,D)f(x)=(2\pi)^{-n}\int e^{ix\xi}\sigma(x,\xi)\hat{f}(\xi)d\xi. \tag{1.1}$$

Hörmander has given a classification for symbolic operators, here we study $\sigma(x,\xi)\in S^m_{1,1}$ or $\sigma(x,D)\in OpS^m_{1,1}$:

**Definition 1.1.** *$\sigma(x,\xi)\in S^m_{1,1}$ or $\sigma(x,D)\in OpS^m_{1,1}$, if*

$$|\partial^\alpha_x\partial^\beta_\xi\sigma(x,\xi)|\leq C_{m,\alpha,\beta}(1+|\xi|)^{m+|\alpha|-|\beta|},\ \forall\alpha,\beta\in N^n. \tag{1.2}$$

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According to kernel's theorem of Schwartz, a linear operator, which is continuous from $S(R^n)$ to $S'(R^n)$ , corresponds to a kernel-distribution $K(x,y)$ or $k(x,z)$; $K(x,y)$ and $k(x,z)$ are defined as follows:

$$Tf(x) = \int K(x,y)f(y)dy = \int k(x,z)f(x-z)dz. \tag{1.3}$$

Note $C^N(R^n) = C_c^N(R^n) = \{f(x), \sum_{|\alpha|\le N} ||\partial_x^\alpha f(x)||_\infty < +\infty\}, C_c^\infty(R^n) = \bigcap_{N\ge 1} C_c^N(R^n)$.

In this paper, we study $Op\hat{K}^m, OpK^m, Op\hat{k}^m$ and $OpK^m$ which are defined below.

**Definition 1.2.** (i) *$T \in Op\hat{K}^m$ or $K(x,y) \in \hat{K}^m$, if $K(x,y)$ satisfies the following conditions:*

*If $|x-y| \le 1$, then*

$$|\partial_x^\alpha \partial_y^\beta K(x,y)| \le C_{\alpha,\beta,N}|x-y|^{-n-m-|\alpha|-|\beta|-N},$$
$$\forall N > 0, \forall \alpha, \beta \in N^n \text{ and } n+m+|\alpha|+|\beta|+N > 0. \tag{1.4}$$

*If $|x-y| > 1$, then*

$$|\partial_x^\alpha \partial_y^\beta K(x,y)| \le C_{\alpha,\beta,N}|x-y|^{-N}, \forall \alpha, \beta \in N^n. \tag{1.5}$$

*and if $m \ge 0$, $K(x,y)$ satisfies also the following condition:*

*There exists a positive real number $q$, such that for all the cube $Q, |Q| \le 1, \forall f(x) \in C_0^q(Q), g(x) \in C_0^{q+m}(Q)$, one has*

$$|\langle Tf, g\rangle| \le C|Q|^{1-\frac{m}{n}}(||f||_\infty + |Q|^{\frac{q}{n}}||f||_{\dot{C}^q})(||g||_\infty + |Q|^{\frac{q+m}{n}}||g||_{\dot{C}^{q+m}}). \tag{1.6}$$

(ii) *$T \in OpK^m$ or $K(x,y) \in K^m$, if $K(x,y) \in \hat{K}^m$, and $T$ satisfies the following condition:*

$$\forall \alpha \in N^n, \int K(x,y)(x-y)^\alpha dy \in C_c^\infty(R^n). \tag{1.7}$$

(iii) *$T \in Op\hat{k}^m$ or $k(x,z) \in \hat{k}^m$, if $k(x,z)$ satisfies the following conditions:*

*If $|z| \le 1$, then*

$$|\partial_x^\alpha \partial_z^\beta k(x,z)| \le C_{\alpha,\beta,N}|z|^{-n-m-|\alpha|-|\beta|-N}, \forall N > 0, \forall \alpha, \beta \in N^n \text{ and } n+m+|\alpha|+|\beta|+N > 0. \tag{1.8}$$

*If $|z| > 1$, then*

$$|\partial_x^\alpha \partial_y^\beta k(x,z)| \le C_{\alpha,\beta,N}|z|^{-N}, \forall \alpha, \beta \in N^n. \tag{1.9}$$

*and if $m \ge 0$, $k(x,z)$ satisfies also the following condition:*

*$\exists N > 0$, such that $\forall x_0 \in R^n, \forall R, 0 < R < 1$, $\forall f(x) \in C_0^N(B(x_0,R))$, $\forall g(z) \in C_0^N(B(0,R))$, one has*

$$|\langle k(x,z), f(x)g(z)\rangle| \le CR^{n-m}(||f||_\infty + R^N||f||_{\dot{C}^N})(||g||_\infty + R^N||g||_{\dot{C}^N}). \tag{1.10}$$

(iv) *$T \in Opk^m$ or $k(x,z) \in k^m$, if $k(x,z) \in \hat{k}^m$, and $T$ satisfies the following condition:*

$$\forall f(z) \in C_0^\infty(B(0,1)), \text{ one has } |\langle \partial_x^\beta k(x,z), f(z)\rangle| \le A_\beta, \forall \beta \in N^n. \tag{1.11}$$

In 1950's, Calderón found a formal relation between (1.1) and (1.3):

$$K(x,y) = (2\pi)^{-n}\int \sigma(x,\xi)e^{i(x-y)\xi}d\xi \text{ or } k(x,z) = (2\pi)^{-n}\int \sigma(x,\xi)e^{iz\xi}d\xi. \tag{1.12}$$

But it is difficult to establish an internal relation between a symbol space and a kernel-distribution space. Here, we construct some groups of orthonormals bases in $L^2(R^n \times R^n)$, and then use them to analyze the symbolic operator spaces $OpS_{1,1}^m$ and four kernel-distribution spaces $Op\hat{K}^m, OpK^m, Op\hat{k}^m$ and $Opk^m$. With these results, we establish an isometry between $OpS_{1,1}^m$ and two kernel-distribution spaces $OpK^m$ and $Opk^m$. As an application, a result in Chapter 9 of [7] and some other results in Chapter 7 of [8] are corollaries of our theorems.

## §2. Several Groups of Orthonormals Bases

### 2.1. Usual Wavelet Bases

First, we introduce some preliminaries on Meyer's wavelets which will be used in the following sections.

Let $\theta(\xi)\in D(R)$ be an even function, $\theta(\xi)\in[0,1]$; and if $\xi\in[-\frac{2\pi}{3},\frac{2\pi}{3}]$, then $\theta(\xi)=1$; if $\xi\notin[-\frac{4\pi}{3},\frac{4\pi}{3}]$, then $\theta(\xi)=0$; furthermore, if $0\le\xi\le 2\pi$, then $\theta^2(\xi)+\theta^2(2\pi-\xi)=1$. Let $\theta_1(\xi)=(\theta^2(\frac{\xi}{2})-\theta^2(\xi))^{\frac{1}{2}}$. Then Meyer's wavelets $\varphi(x)$ and $\psi(x)$ are defined by $\hat{\varphi}(\xi)=\theta(\xi)$ and $\hat{\psi}(\xi)=\theta_1(\xi)e^{-\frac{i\xi}{2}}$. Denote also $\Phi^{(0)}(x)=\varphi(x),\Phi^{(1)}(x)=\psi(x)$. Furthermore, $\forall\varepsilon\in\{0,1\}^n, x\in R^n,\xi\in R^n$, denote $\Phi^{(\varepsilon)}(x)=\prod\limits_{i=1}^{n}\Phi^{(\varepsilon_i)}(x_i),\hat{\Phi}^{(\varepsilon)}(\xi)=\prod\limits_{i=1}^{n}\hat{\Phi}^{(\varepsilon_i)}(\xi_i)$. $\forall\varepsilon\in\{0,1\}^n,\mu(\varepsilon)$ is a function which is defined as follows: $\mu(\varepsilon)=0$, if $\varepsilon=0;\mu(\varepsilon)=1$, if $\varepsilon\neq 0$; and if $\varepsilon\neq 0$, the smallest index such that $\varepsilon_i\neq 0$ is denoted by $i_\varepsilon$. Denote also $x_\varepsilon=(x_1,\cdots,x_{-1+i_\varepsilon},y_1,x_{1+i_\varepsilon},\cdots,x_n)$. $\forall\varepsilon\in\{0,1\}^n\backslash\{0\}$ and $f(x)$, denote $I^0_\varepsilon f(x)=f(x)$ and $\forall N\in Z, N>0, I^N_\varepsilon f(x)=\int_{-\infty}^{x_{i_\varepsilon}}\int_{-\infty}^{y_N}\cdots\int_{-\infty}^{y_2}f(x_\varepsilon)dy_1\cdots dy_N$.

Meyer's wavelets have the following properties:

**Proposition 2.2.** (i) $\forall\alpha\in N^n,\sum\limits_l l^\alpha\Phi^{(0)}(x-l)=x^\alpha$.

(ii) $\forall N\in Z, N>0$, *there exists a series of functions* $\{\Phi^{(\varepsilon,k,N)}(x)\}_{k\in Z^n}$ *which satisfy* $\operatorname{Supp} I_\varepsilon^{N\mu(\varepsilon)}\Phi^{(\varepsilon,k,N)}(x)\subset[-M_N,M_N]^n$ *and* $\|\Phi^{(\varepsilon,k,N)}(x)\|_{C^N}\le C_N$ *, and a series of numbers* $\{a_k^{(\varepsilon,N)}\}_{k\in Z^n}$ *which satisfy:* $|a_k^{(\varepsilon,N)}|\le\frac{C_N}{(1+|k|)^N}$ *such that*

$$\Phi^{(\varepsilon)}(x)=\sum_{k\in Z^n}a_k^{(\varepsilon,N)}\Phi^{(\varepsilon,k,N)}(x-k).$$

**Proof.** (i) According to the knowledge of wavelets, one has $\forall\alpha\in N^n,\sum\limits_l l^\alpha\Phi^{(0)}(x-l)=P_\alpha(x)$. Then $\forall k\in Z^n$, one has

$$k^\alpha=\int\sum_l l^\alpha\Phi^{(0)}(x-l)\Phi^{(0)}(x-k)dx=\int P_\alpha(x)\Phi^{(0)}(x-k)dx=P_\alpha(k).$$

So one has $P_\alpha(x)=x^\alpha$.

(ii) $\forall N>0$, I. Daubechies has constructed a function $\Phi_N(x)\in C^N(R^n)$ such that $\operatorname{Supp}\Phi_N(x)\subset[-M_N,M_N]^n$ and $\sum\limits_{k\in Z^n}\Phi_N(x-k)=1$ (cf. [5]). If $\varepsilon=0$, applying directly $\sum\limits_{k\in Z^n}\Phi_N(x-k)=1$ to $\Phi^{(0)}(x)$, and choosing $\Phi^{(\varepsilon,k,N)}(x)=(1+|k|)^N\ \Phi_N(x)\Phi^{(0)}(x+k)$ and $a_k^{(\varepsilon,N)}=\frac{1}{(1+|k|)^N}$, one gets the conclusion for $\varepsilon=0$. If $\varepsilon\neq 0$, one has $I^N_\varepsilon\Phi^{(\varepsilon)}(x)\in S(R^n)$. Applying $\sum\limits_{k\in Z^n}\Phi_N(x-k)=1$ to $I^N_\varepsilon\Phi^{(\varepsilon)}(x)$, and choosing $\tilde{\Phi}^{(\varepsilon,k,N)}(x)=(1+|k|)^{2N}$ $\Phi_{2N}(x)I^N_\varepsilon\Phi^{(0)}(x+k)$, $\Phi^{(\varepsilon,k,N)}(x)=\partial^N_{x_{i_\varepsilon}}\tilde{\Phi}^{(\varepsilon,k,N)}(x)$ and $a_k^{(\varepsilon,N)}=\frac{1}{(1+|k|)^{2N}}$, one gets the conclusion for $\varepsilon\neq 0$.

Using $\Phi^{(\varepsilon)}(x)$, Meyer constructed a usual wavelet bases for function spaces. Let $\Gamma_n=\{\lambda=(\varepsilon,j,k),\varepsilon\in\{0,1\}^n,j\ge 0,k\in Z^n$, and if $j>0$, then $\varepsilon\neq 0\}$. $\forall\lambda=(\varepsilon,j,k)\in\Gamma_n$, denote $\Phi_\lambda(x)=2^{\frac{jn}{2}}\Phi^{(\varepsilon)}(2^jx-k)$; $\forall f(x)\in S'(R^n)$, denote $a_\lambda=a^{(\varepsilon)}_{j,k}=\langle f(x),\Phi_\lambda(x)\rangle$; furthermore $\forall\Psi(x)\in S(R^n)$ such that $\forall\alpha\in N^n,\int x^\alpha\Psi(x)dx=0$, denote $\Psi_{j,k}(x)=2^{\frac{jn}{2}}\Psi(2^jx-k)$, and $a^\Psi_{j,k}=\langle f(x),\Psi_{j,k}(x)\rangle$. Then one has

**Proposition 2.2.** (i) $\{\Phi_\lambda(x)\}_{\lambda\in\Gamma_n}$ *is an orthogonal basis in* $L^2(R^n)$.

(ii) $\forall f(x) \in C_c^N(R^n), \forall j \geq 0, k \in Z^n$, *then* $|a_{j,k}^{\Psi}| \leq C2^{-j(\frac{n}{2}+N)}$; *and vice versa, if* $\forall j \geq 0, k \in Z^n, |a_{j,k}| \leq C2^{-j(\frac{n}{2}+N)}$, *then* $\sum_{j\geq 0, k\in Z^n} a_{j,k}\Psi_{j,k}(x) \in C_c^{N'}(R^n), \forall 0 \leq N' < N$.

(iii) $f(x) \in C_c^{\infty}(R^n) \Leftrightarrow \forall \lambda = (\varepsilon, j, k) \in \Gamma_n, \forall q > 0, |a_{j,k}^{(\varepsilon)}| \leq C_q 2^{-jq}$.

One can find the proof of this proposition in [7]. In fact, the usual wavelet bases can analyze most of function spaces (cf. [10]).

**2.2. New Wavelet Bases**

Suppose that $T$ is an operator which continue from $S(R^n)$ to $S'(R^n)$. One hopes that the absolute value of the element in the matrix $\{\langle T\Phi_\lambda(x), \Phi_{\lambda'}(x)\rangle\}_{(\lambda,\lambda')\in\Gamma_n\times\Gamma_n}$ can reflect all the information of an operator $T$. Let $T^*$ be the conjugate operator of $T$. Meyer used such a method to analyze $OpB = \{T : T \in OpS_{1,1}^0, T^* \in OpS_{1,1}^0\}$; but if one does not know the infomation of $T^*$, one cannot use this method. Afterward, Beylkin, Coifman and Rokhlin used B-C-R algorithm to analyze an operator. The main idea of this method is to regard an operator as a distribution in $2n$ dimension, and one analyzes this distribution with usual wavelet bases $\{\Phi_\lambda(x, y)\}_{\lambda\in\Gamma_{2n}}$ in $2n$ dimension; but these ideas cannot provide unconditional wavelet bases for $OpS_{1,1}^m$. Here, we present several kinds of new wavelet bases (B-C-R wavelet bases), these new orthonormal bases are adapted to analysis of symbol or kernel-distribution of an operator.

Let $\Lambda_1 = \{(\varepsilon, \varepsilon', j, k, l), \varepsilon, \varepsilon' \in \{0,1\}^n, j \geq 0, k, l \in Z^n$, and if $j > 0, \varepsilon' \neq 0\}$, $\Lambda_2 = \{(\varepsilon, \varepsilon', j, j', k, k'), \varepsilon \in \{0,1\}^n\backslash\{0\}, \varepsilon' \in \{0,1\}^n, 0 \leq j' < j, k, k' \in Z^n$, and if $j' > 0, \varepsilon' \neq 0\}$, $\tilde{\Lambda}_2 = \{(\varepsilon, 0, j, k, l), \varepsilon \in \{0,1\}^n\backslash\{0\}, j > 0, k, l \in Z^n\}$. Denote then $\Lambda = \Lambda_1 \cup \Lambda_2, \tilde{\Lambda} = \Lambda_1 \cup \tilde{\Lambda}_2$. In fact, $\tilde{\Lambda} = \Gamma_{2n}$. Now we construct some groups of functions:

$$\tilde{K}_\lambda(x, y) = 2^{jn}\Phi^{(\varepsilon)}(2^j x - k)\Phi^{(\varepsilon')}(2^j y - l), \lambda \in \tilde{\Lambda}; \tag{2.1}$$

$$K_\lambda(x, y) = \begin{cases} 2^{jn}\Phi^{(\varepsilon)}(2^j x - k)\Phi^{(\varepsilon')}(2^j y - l), & \lambda \in \Lambda_1; \\ 2^{\frac{n(j+j')}{2}}\Phi^{(\varepsilon)}(2^j x - k)\Phi^{(\varepsilon')}(2^{j'} y - k'), & \lambda \in \Lambda_2; \end{cases} \tag{2.2}$$

$$\Phi_\lambda(x, \xi) = \begin{cases} \Phi^{(\varepsilon)}(2^j x - k)\hat{\Phi}^{(\varepsilon')}(2^{-j}\xi)e^{-i(x-2^{-j}l)\xi}, & \lambda \in \Lambda_1; \\ 2^{\frac{n(j-j')}{2}}\Phi^{(\varepsilon)}(2^j x - k)\hat{\Phi}^{(\varepsilon')}(2^{-j'}\xi)e^{-i(x-2^{-j'}k')\xi}, & \lambda \in \Lambda_2; \end{cases} \tag{2.3}$$

$$\tilde{\Phi}_\lambda(x, \xi) = \begin{cases} \Phi^{(\varepsilon)}(2^j x - k)\hat{\Phi}^{(\varepsilon')}(2^{-j}\xi)e^{i2^{-j}l\xi}, & \lambda \in \Lambda_1; \\ 2^{\frac{n(j-j')}{2}}\Phi^{(\varepsilon)}(2^j x - k)\hat{\Phi}^{(\varepsilon')}(2^{-j'}\xi)e^{i2^{-j'}k'\xi}, & \lambda \in \Lambda_2. \end{cases} \tag{2.4}$$

For these groups of functions, one has the following theorem.

**Theorem 2.1.** $\{\Phi_\lambda(x, \xi)\}_{\lambda\in\Lambda}, \{\tilde{\Phi}_\lambda(x, \xi)\}_{\lambda\in\Lambda}, \{\tilde{K}_\lambda(x, y)\}_{\lambda\in\Lambda}, \{K_\lambda(x, y)\}_{\lambda\in\Lambda}$ *belong to* $S(R^n \times R^n)$*, and they are four groups of orthonormal bases in* $L^2(R^n \times R^n)$.

**Proof.** According to the knowlege of wavelet's theory, $\{\tilde{K}_\lambda(x, y)\}_{\lambda\in\tilde{\Lambda}}$ are a group of orthonormal bases in $L^2(R^n \times R^n)$. Using this fact, one sees that $\{K_\lambda(x, y)\}_{\lambda\in\Lambda}$ are also a group of orthonormal bases in $L^2(R^n \times R^n)$. Then applying the following facts that $\Phi_\lambda(x, \xi) = \int K_\lambda(x, y)e^{i(y-x)\xi}dy$ and $\tilde{\Phi}_\lambda(x, \xi) = \int k_\lambda(x, z)e^{-iz\xi}dz$ , one gets that $\{\Phi_\lambda(x, \xi)\}_{\lambda\in\Lambda}$ and $\{\tilde{\Phi}_\lambda(x, \xi)\}_{\lambda\in\Lambda}$ are two groups of orthonormal bases in $L^2(R^n \times R^n)$.

## §3. $OpS_{1,1}^m$ and New Wavelet Bases

In this section, we apply the two groups of wavelet bases in (2.3) and (2.4) to analyze $OpS_{1,1}^m$. $\forall\lambda \in \Lambda$, denote $a_\lambda = \langle\sigma(x, \xi), \Phi_\lambda(x, \xi)\rangle$, and $b_\lambda = \langle\sigma(x, \xi), \tilde{\Phi}_\lambda(x, \xi)\rangle$. According to Theorem 2.1, $\{a_\lambda\}_{\lambda\in\Lambda}$ or $\{b_\lambda\}_{\lambda\in\Lambda}$ can reflect all the information of an operator in $OpS_{1,1}^m$. For the clarification of notations, note $a_\lambda = a_{j,k,l}^{(\varepsilon,\varepsilon')}$ and $b_\lambda = \tilde{a}_{j,k,l}^{(\varepsilon,\varepsilon')}$ , $\forall\lambda \in \Lambda_1$; and $a_\lambda = a_{j,j',k,k'}^{(\varepsilon,\varepsilon')}$ and $b_\lambda = \tilde{a}_{j,j',k,k'}^{(\varepsilon,\varepsilon')}, \forall\lambda \in \Lambda_2$. First, we present a definition of $OpN^m$ and $Op\tilde{N}^m$.

**Definition 3.1.** *$T \in OpN^m$ or $\{a_\lambda\}_{\lambda\in\Lambda} \in N^m$, if $\{a_\lambda\}_{\lambda\in\Lambda}$ satisfies the following conditions*:

$$\forall\lambda \in \Lambda_1, \forall N > 0, \text{ one has } |a_\lambda| \leq \frac{C_N 2^{jm}}{(1+|k-l|)^N}, \tag{3.1}$$

$$\forall\lambda \in \Lambda_2, \forall N > 0, \forall N' > 0, \text{ one has } |a_\lambda| \leq \frac{C_N 2^{jm-(j-j')N'}}{(1+|2^{j'-j}k-k'|)^N}. \tag{3.2}$$

*$T \in Op\tilde{N}^m$ or $\{b_\lambda\}_{\lambda\in\Lambda} \in \tilde{N}^m$, if $\{b_\lambda\}_{\lambda\in\Lambda}$ satisfies the following conditions*:

$$\forall\lambda \in \Lambda_1, \forall N > 0, \text{ one has } |b_\lambda| \leq \frac{C_N 2^{jm}}{(1+|l|)^N}; \tag{3.3}$$

$$\forall\lambda \in \Lambda_2, \forall N > 0, \forall N' > 0, \text{ one has } |b_\lambda| \leq \frac{C_N 2^{j'm-N'(j-j')}}{(1+|k'|)^N}. \tag{3.4}$$

In this section, we prove that $OpS^m_{1,1} = OpN^m = Op\tilde{N}^m$. In fact, one has

**Theorem 3.1.** (i) $T \in OpS^m_{1,1} \Leftrightarrow T \in OpN^m$. (ii) $T \in OpS^m_{1,1} \Leftrightarrow T \in Op\tilde{N}^m$.

**Proof.** We can get easily that (1.2) implies (3.1), (3.2), (3.3) and (3.4). In fact, $\forall\lambda \in \Lambda_1$, we calculate $a_\lambda, (k-l)^\alpha a_\lambda, b_\lambda$ and $l^\alpha b_\lambda$; $\forall\lambda \in \Lambda_2$, we calculate $a_\lambda, (2^{j'-j}k-k')^\alpha a_\lambda, b_\lambda$, and $(k')^\alpha b_\lambda$. Applying the integration by parts for $x$ and $\xi$, we get what we want. Then we prove that (3.1) and (3.2) or (3.3) and (3.4) imply (1.2). Denote $\sigma_1(x,\xi) = \sum\limits_{\lambda\in\Lambda_1} a_\lambda \Phi_\lambda(x,\xi)$, $\sigma_2(x,\xi) = \sum\limits_{\lambda\in\Lambda_2} a_\lambda \Phi_\lambda(x,\xi)$, $\tilde{\sigma}_1(x,\xi) = \sum\limits_{\lambda\in\Lambda_1} a_\lambda \tilde{\Phi}_\lambda(x,\xi)$ and $\tilde{\sigma}_2(x,\xi) = \sum\limits_{\lambda\in\Lambda_2} a_\lambda \tilde{\Phi}_\lambda(x,\xi)$. Then we prove that $\sigma_1(x,\xi), \sigma_2(x,\xi), \tilde{\sigma}_1(x,\xi)$ and $\tilde{\sigma}_2(x,\xi)$ all belong to $OpS^m_{1,1}$.

For $\sigma_1(x,\xi)$,

$$\begin{aligned}
&\partial_x^\alpha \partial_\xi^\beta \sigma_1(x,\xi) = \partial_x^\alpha \partial_\xi^\beta \sum_{\lambda\in\Lambda_1} a_\lambda \Phi_\lambda(x,\xi) \\
&= \partial_x^\alpha \sum_{\lambda\in\Lambda_1} a_\lambda \Phi^{(\varepsilon)}(2^j x - k) \sum_{\beta_1+\beta_2=\beta} C_\beta^{\beta_1} 2^{-j|\beta|} (\partial_\xi^{\beta_1} \hat{\Phi}^{(\varepsilon')})(2^{-j}\xi)(2^j x - l)^{\beta_2} e^{-i(x-2^{-j}l)\xi} \\
&= \sum_{\lambda\in\Lambda_1} \sum_{\beta_1+\beta_2=\beta} \sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha \\ \alpha_2\leq\beta_2}} 2^{j(|\alpha|-|\beta|)} C_{\alpha,\beta}^{\alpha_1,\alpha_2,\beta_1} a_\lambda (2^j x - l)^{\beta_2-\alpha_2} \\
&\quad (\partial_x^{\alpha_1}\Phi^{(\varepsilon)})(2^j x - k)(2^{-j}\xi)^{\alpha_3}(\partial_\xi^{\beta_1}\hat{\Phi}^{(\varepsilon')})(2^{-j}\xi) e^{-i(x-2^{-j}l)\xi} \\
&= \sum_{\lambda\in\Lambda_1} \sum_{\beta_1+\beta_2=\beta} \sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha \\ \gamma_1+\gamma_2=\beta_2-\alpha_2}} 2^{j(|\alpha|-|\beta|)} C_{\alpha,\beta,\gamma}^{\alpha_1,\alpha_2,\beta_1,\gamma_1} (k-l)^{\gamma_1} a_\lambda (2^j x - k)^{\gamma_2} \\
&\quad (\partial_x^{\alpha_1}\Phi^{(\varepsilon)})(2^j x - k)(2^{-j}\xi)^{\alpha_3}(\partial_\xi^{\beta_1}\hat{\Phi}^{(\varepsilon')})(2^{-j}\xi) e^{-i(x-2^{-j}l)\xi}.
\end{aligned}$$

Applying (3.1), we get $\sigma_1(x,\xi) \in OpS^m_{1,1}$.

For $\sigma_2(x,\xi)$,

$$\begin{aligned}
&\partial_x^\alpha \partial_\xi^\beta \sigma_2(x,\xi) = \partial_x^\alpha \partial_\xi^\beta \sum_{\lambda\in\Lambda_2} a_\lambda \Phi_\lambda(x,\xi) \\
&= \partial_x^\alpha \sum_{\lambda\in\Lambda_2} a_\lambda 2^{\frac{n(j-j')}{2}} \Phi^{(\varepsilon)}(2^j x - k) \sum_{\beta_1+\beta_2=\beta} C_\beta^{\beta_1} 2^{-j'|\beta|} (\partial_\xi^{\beta_1}\hat{\Phi}^{(\varepsilon')})(2^{-j'}\xi) \\
&\quad (2^{j'}x - k')^{\beta_2} e^{-i(x-2^{-j'}k')\xi}
\end{aligned}$$

$$= \sum_{\lambda\in\Lambda_2}\sum_{\beta_1+\beta_2=\beta}\sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha\\ \alpha_2\le\beta_2}} 2^{(j-j')(\frac{n}{2}+|\alpha_1|)+j'(|\alpha|-|\beta|)}C_{\alpha,\beta}^{\alpha_1,\alpha_2,\beta_1}a_\lambda(2^{j'}x-k')^{\beta_2-\alpha_2}$$

$$(\partial_x^{\alpha_1}\Phi^{(\varepsilon)})(2^jx-k)(2^{-j'}\xi)^{\alpha_3}(\partial_\xi^{\beta_1}\hat\Phi^{(\varepsilon')})(2^{-j'}\xi)e^{-i(x-2^{-j'}k')\xi}$$

$$= \sum_{\lambda\in\Lambda_2}\sum_{\beta_1+\beta_2=\beta}\sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha\\ \gamma_1+\gamma_2=\beta_2-\alpha_2}} 2^{(j-j')(\frac{n}{2}+|\alpha_1|-\gamma_2)+j'(|\alpha|-|\beta|)}C_{\alpha,\beta,\gamma}^{\alpha_1,\alpha_2,\beta_1,\gamma_1}(2^{j'-j}k-k')^{\gamma_1}$$

$$a_\lambda(2^jx-k)^{\gamma_2}(\partial_x^{\alpha_1}\Phi^{(\varepsilon)})(2^jx-k)(2^{-j'}\xi)^{\alpha_3}(\partial_\xi^{\beta_1}\hat\Phi^{(\varepsilon')})(2^{-j'}\xi)e^{-i(x-2^{-j'}k')\xi}.$$

Applying (3.2), we get $\sigma_2(x,\xi)\in OpS^m_{1,1}$.

For $\tilde\sigma_1(x,\xi)$ and $\tilde\sigma_2(x,\xi)$, we can apply the same ideas as above and get $\tilde\sigma_1(x,\xi)$, $\tilde\sigma_2(x,\xi)\in OpS^m_{1,1}$.

## §4. Distrbution Spaces $\widehat{K}^m, \hat{k}^m$ and Unconditional Bases

In this section, we analyze two distribution spaces $\widehat{K}^m$ and $\widehat{k}^m$ with wavelet bases in (2.1). $\forall\lambda\in\tilde\Lambda$, denote $\tilde a_\lambda=\widetilde{a}^{(\varepsilon,\varepsilon')}_{j,k,l}=\langle K(x,y),\widetilde{K}_\lambda(x,y)\rangle$, and $\tilde b_\lambda=\tilde b^{(\varepsilon,\varepsilon')}_{j,k,l}=\langle k(x,z),\widetilde{K}_\lambda(x,z)\rangle$. We characterize $\widehat{K}^m$ and $\hat k^m$ in wavelet coefficients spaces.

**Definition 4.1.** (i) *$T\in Op\widehat{K}N^m$ or $\{\tilde a_\lambda\}_{\lambda\in\tilde\Lambda}\in\widehat{K}N^m$, if $\{\tilde a_\lambda\}_{\lambda\in\tilde\Lambda}$ satisfies the following condition*:

$$|\tilde a_\lambda|\le\frac{C_N2^{jm}}{(1+|k-l|)^N},\forall\lambda\in\tilde\Lambda. \tag{4.1}$$

(ii) *$T\in Op\hat kN^m$ or $\{\tilde b_\lambda\}_{\lambda\in\tilde\Lambda}\in\hat kN^m$, if $\{\tilde b_\lambda\}_{\lambda\in\tilde\Lambda}$ satisfies the following condition*:

$$|\tilde b_\lambda|\le\frac{C_N2^{jm}}{(1+|l|)^N},\forall\lambda\in\tilde\Lambda. \tag{4.2}$$

$\widehat{K}^m$ and $\hat k^m$ can be characterized by the above two spaces. In fact, one has

**Theorem 4.1.** (i)] $K(x,y)\in\widehat{K}^m\Leftrightarrow\{\tilde a_\lambda\}_{\lambda\in\tilde\Lambda}\in\widehat{K}N^m$.

(ii) $k(x,z)\in\hat k^m\Leftrightarrow\{\tilde b_\lambda\}_{\lambda\in\tilde\Lambda}\in\hat kN^m$.

**Proof.** The ideas of proof for (i) is the same as that for (ii), so we prove only (ii). First, prove that $k(x,z)\in\hat k^m\Rightarrow\{\tilde b_\lambda\}_{\lambda\in\tilde\Lambda}\in\hat kN^m$. In fact, $\forall\lambda\in\Lambda_1$,

$$\tilde b^{(\varepsilon,\varepsilon')}_{j,k,l}=\iint k(x,z)2^{jn}\Phi^{(\varepsilon)}(2^jx-k)\Phi^{(\varepsilon')}(2^jz-l)dxdz.$$

Applying Proposition 2.1, we have

$$\begin{aligned}\tilde b^{(\varepsilon,\varepsilon')}_{j,k,l}&=2^{jn}\sum_{k_2\in Z^n}a^{(\varepsilon',N)}_{k_2}\iint k(x,z)\Phi^{(\varepsilon)}(2^jx-k)\Phi^{(\varepsilon',k_2,N)}(2^jz-l-k_2)dxdz\\&=2^{jn}\sum_{\substack{k_2\in Z^n\\|l+k_2|\le8}}a^{(\varepsilon',N)}_{k_2}\iint k(x,z)\Phi^{(\varepsilon)}(2^jx-k)\Phi^{(\varepsilon',k_2,N)}(2^jz-l-k_2)dxdz\\&\quad+2^{jn}\sum_{\substack{k_2\in Z^n\\|l+k_2|>8}}a^{(\varepsilon',N)}_{k_2}\iint k(x,z)\Phi^{(\varepsilon)}(2^jx-k)\Phi^{(\varepsilon',k_2,N)}(2^jz-l-k_2)dxdz\\&=I_1+I_2.\end{aligned}$$

For $I_1$, if $m<0$, we apply (1.8) and (1.9); if $m\ge0$, we apply (1.8), (1.9) and (1.10), and we get $|I_1|\le\frac{C_N2^{jm}}{(1+|l|)^N}$. For $I_2$, applying (1.8), (1.9) and the integration by parts for $z$, we get $|I_2| \le \frac{C_N 2^{jm}}{(1+|l|)^N}$.

$\forall\lambda \in \tilde{\Lambda}_2$, applying the same calculation, we get the same conclusion.

Then, we prove that $\{\tilde{b}_\lambda\}_{\lambda\in\tilde{\Lambda}} \in \hat{k}N^m \Rightarrow k(x,z) \in \hat{k}^m$.

We denote $k_1(x,z) = \sum_{\lambda\in\Lambda_1} \tilde{b}_\lambda \widetilde{K}_\lambda(x,z)$ and $k_2(x,z) = \sum_{\lambda\in\tilde{\Lambda}_2} \tilde{b}_\lambda \widetilde{K}_\lambda(x,z)$, and prove that $k_1(x,z)$ and $k_2(x,z)$ satisfy (1.8), (1.9) and (1.10). It is evident for (1.8) and (1.9). As for (1.10), we suppose $m \ge 0$. We choose $N > m$, then $\forall x_0 \in R^n, \forall f(x) \in C_0^N(B(x_0, 2^{-\mu})), \forall\alpha \in N^n, ||\partial_x^\alpha f||_\infty \le C_\alpha 2^{\mu|\alpha|}$, and $\forall g(z) \in C_0^N(B(0,2^{-\mu})), \beta \in N^n, ||\partial_z^\beta g||_\infty \le C_\alpha 2^{\mu|\alpha|}$ , we estimate $|\langle k_i(x,z), f(x)g(z)\rangle|, i = 1,2$.

For $i = 1$, we consider two cases for $j$, and calculate $I_1$ and $I_2$. In fact,

$$\begin{aligned}
I_1 &= \Big|\iint \sum_{\lambda\in\Lambda_1, j<\mu} \tilde{b}^{(\varepsilon,\varepsilon')}_{j,k,l} \Phi^{(\varepsilon)}_{j,k}(x)\Phi^{(\varepsilon')}_{j,l}(z) f(x)g(z)dxdz\Big| \\
&\le C\iint \sum_{j<\mu, l\in Z^n} \frac{2^{j(m+n)}}{(1+|l|)^{n+N}} |\Phi^{(\varepsilon)}(2^jx-k)||\Phi^{(\varepsilon')}(2^jz-l)||f(x)||g(z)|dxdz \\
&\le C\int \sum_{j<\mu} 2^{jm}|\Phi^{(\varepsilon)}(2^jx-k)||f(x)|dx \\
&\le C2^{\mu m}\int |f(x)|dx \le C2^{\mu(m-n)}.
\end{aligned}$$

For $I_2 = \Big|\iint \sum_{\lambda\in\Lambda_1, j\ge\mu} \tilde{b}^{(\varepsilon,0)}_{j,k,l} \Phi^{(\varepsilon)}_{j,k}(x)\Phi^{(\varepsilon')}_{j,l}(z) f(x)g(z)dxdz\Big|$, we make the integration by parts for $z$, and then making the same calculations, we get also $I_2 \le C2^{\mu(m-n)}$.

For $i = 2$, we consider also two cases for $j$, and calculate $I_1$ and $I_2$. In fact,

$$\begin{aligned}
I_1 &= \Big|\iint \sum_{\lambda\in\Lambda_1, j<\mu} \tilde{b}^{(\varepsilon,0)}_{j,k,l} \Phi^{(\varepsilon)}_{j,k}(x)\Phi^{(0)}_{j,l}(z) f(x)g(z)dxdz\Big| \\
&\le C\iint \sum_{j<\mu, l\in Z^n} \frac{2^{j(m+n)}}{(1+|l|)^{n+N}} |\Phi^{(\varepsilon)}(2^jx-k)||\Phi^{(0)}(2^jz-l)||f(x)||g(z)|dxdz \\
&\le C\int \sum_{j<\mu} 2^{jm}|\Phi^{(\varepsilon)}(2^jx-k)||f(x)|dx \le C2^{\mu(m-n)}.
\end{aligned}$$

For $I_2 = \Big|\iint \sum_{\lambda\in\Lambda_1, j\ge\mu} \tilde{b}^{(\varepsilon,0)}_{j,k,l} \Phi^{(\varepsilon)}_{j,k}(x)\Phi^{(0)}_{j,l}(z) f(x)g(z)dxdz\Big|$, we make the integration by parts for $x$, and then making the same calculations, we get also $I_2 \le C2^{\mu(m-n)}$.

## §5. Kernel-Distribution Spaces $K^m, k^m$ and Wavelets

In this section, we analyze $OpK^m$ and $Opk^m$ with wavelet bases in (2.1) and (2.2). First, we use wavelet bases in (2.1) to analyze these two spaces.

**Definition 5.1.** (i) *$T \in OpBN^m \Leftrightarrow T \in Op\widehat{K}N^m$ and $\{\tilde{a}_\lambda\}_{\lambda\in\tilde{\Lambda}}$ satisfies the following condition:*

$$\forall\lambda \in \tilde{\Lambda}_2, \forall q > 0, |\sum_l (k-l)^\alpha \tilde{a}^{(\varepsilon,0)}_{j,k,l}| \le C_\alpha 2^{-jq}. \tag{5.1}$$

(ii) *$T \in OpB\tilde{N}^m \Leftrightarrow T \in Op\widehat{K}N^m$ and $\{\tilde{b}_\lambda\}_{\lambda\in\tilde{\Lambda}}$ satisfies the following condition:*

$$\forall\lambda \in \tilde{\Lambda}_2, \forall q > 0, |\sum_l l^\alpha \tilde{b}^{(\varepsilon,0)}_{j,k,l}| \le C_\alpha 2^{-jq}. \tag{5.2}$$

**Theorem 5.1.** (i) $T \in OpK^m \Leftrightarrow T \in OpBN^m$. (ii) $T \in Opk^m \Leftrightarrow T \in OpB\tilde{N}^m$.

**Proof.** (i) In order to prove (i), we apply Theorem 4.1, and prove that if $T \in Op\widehat{K}^m$, then (1.7) is equivalent to (5.1). In fact,

$$\begin{aligned}
&\int K(x,y)(x-y)^\alpha dy \\
&= \sum_{(\varepsilon,j,k)\in\Gamma_n} \sum_l 2^{nj} \tilde{a}_{j,k,l}^{(\varepsilon,0)} \Phi^{(\varepsilon)}(2^j x - k) \int (x-y)^\alpha \Phi^{(0)}(2^j y - l) dy \\
&= \sum_{(\varepsilon,j,k)\in\Gamma_n} \sum_l 2^{nj} \tilde{a}_{j,k,l}^{(\varepsilon,0)} \Phi^{(\varepsilon)}(2^j x - k) \int (x - 2^{-j} l - (y - 2^{-j} l))^\alpha \Phi^{(0)}(2^j y - l) dy \\
&= \sum_l \sum_{(\varepsilon,j,k)\in\Gamma_n} (x - 2^{-j} l)^\alpha \tilde{a}_{j,k,l}^{(\varepsilon,0)} \Phi^{(\varepsilon)}(2^j x - k) \\
&= \sum_\beta C_\alpha^\beta \sum_{(\varepsilon,j,k)\in\Gamma_n} 2^{-j|\alpha|} \sum_l (k-l)^\beta \tilde{a}_{j,k,l}^{(\varepsilon,0)} (2^j x - k)^{\alpha-\beta} \Phi^{(\varepsilon)}(2^j x - k).
\end{aligned}$$

That is to say, (5.1) implies (1.7).

Furthermore, applying proposition 2.1 in Section 2, we get

$$\begin{aligned}
\sum_l (2^j x - l)^\beta \Phi^{(0)}(2^j y - l) &= \sum_{l,\beta_1} (-1)^{\beta_1} (2^j x)^{\beta-\beta_1} l^{\beta_1} \Phi^{(0)}(2^j y - l) \\
&= \sum_{\beta_1} (-1)^{\beta_1} (2^j x)^{\beta-\beta_1} (2^j y)^{\beta_1} = (2^j x - 2^j y)^\beta.
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{(\varepsilon,j,k)\in\Gamma_n} \sum_l (k-l)^\alpha \tilde{a}_{j,k,l}^{(\varepsilon,0)} \\
&= \sum_{(\varepsilon,j,k)\in\Gamma_n} \sum_l (k-l)^\alpha \iint 2^{nj} \Phi^{(\varepsilon)}(2^j x - k) K(x,y) \Phi^{(0)}(2^j y - l) dxdy \\
&= \sum_{(\varepsilon,j,k)\in\Gamma_n} \sum_l \sum_\beta C_\alpha^\beta (k - 2^j x)^{\alpha-\beta} (2^j x - l)^\beta \iint 2^{nj} \Phi^{(\varepsilon)}(2^j x - k) K(x,y) \Phi^{(0)}(2^j y - l) dxdy \\
&= \sum_{(\varepsilon,j,k)\in\Gamma_n} \sum_l \sum_\beta C_\alpha^\beta (k - 2^j x)^{\alpha-\beta} \iint 2^{nj} \Phi^{(\varepsilon)}(2^j x - k) K(x,y) (2^j x - 2^j y)^\beta dxdy \\
&= \sum_{(\varepsilon,j,k)\in\Gamma_n} \sum_\beta (-1)^{\alpha-\beta} C_\alpha^\beta 2^{j(n+|\beta|)} \iint (2^j x - k)^{\alpha-\beta} \Phi^{(\varepsilon)}(2^j x - k)(x-y)^\beta dxdy.
\end{aligned}$$

Applying (ii) of Proposition 2.2, we get that (1.7) implies (5.1).

(ii) In order to prove (ii), we apply Theorem 4.1, and prove that if $T \in Op\hat{k}^m$, then (1.11) is equivalent to (5.2). $\forall f(z) \in C_0^\infty(B(0,1))$,

$$\begin{aligned}
&\langle \partial_x^\beta k(x,z), f(z) \rangle \\
&= \sum_{\substack{(\varepsilon,\varepsilon',j,k,l)\in\tilde{\Lambda} \\ \varepsilon'\neq 0}} 2^{j(n+|\beta|)} \check{b}_{j,k,l}^{(\varepsilon,\varepsilon')} (\partial_x^\beta \Phi^{(\varepsilon)})(2^j x - k) \langle \Phi^{(\varepsilon')}(2^j z - l), f(z) \rangle \\
&\quad + \sum_{(\varepsilon,j,k)\in\Gamma_n} \sum_l 2^{j(n+|\beta|)} \check{b}_{j,k,l}^{(\varepsilon,0)} (\partial_x^\beta \Phi^{(\varepsilon)})(2^j x - k) \langle \Phi^{(0)}(2^j z - l), f(z) \rangle \\
&= I_1 + I_2.
\end{aligned}$$

Then, if $T \in Op\hat{k}^m$, (5.2) implies (1.11).

$$\sum_l l^\beta \check{b}^{(\varepsilon,0)}_{j,k,l} = 2^{jn} \sum_l l^\beta \iint \Phi^{(\varepsilon)}(2^j x - k) k(x,z) \Phi^{(0)}(2^j z - l) dxdz$$
$$= 2^{j(n-N)} \sum_l l^\beta \iint \Phi^{(\varepsilon,N)}(2^j x - k) \partial^N_{x_{i_\varepsilon}} k(x,z) \Phi^{(0)}(2^j z - l) dxdz.$$

Then, if $T \in Op\hat{k}^m$, (1.11) implies (5.2).

Secondly, we prove that wavelet bases in (2.2) are unconditional bases for $K^m$ and $k^m$. $\forall \lambda \in \Lambda$, denote $ka_\lambda = \langle K(x,y), K_\lambda(x,y)\rangle$, and $kb_\lambda = \langle k(x,z), K_\lambda(x,z)\rangle$. For clarification of notations, we note also $ka_\lambda = a^{(\varepsilon,\varepsilon')}_{j,k,l}$ and $kb_\lambda = b^{(\varepsilon,\varepsilon')}_{j,k,l}$, if $\lambda \in \Lambda_1$; $ka_\lambda = a^{(\varepsilon,\varepsilon')}_{j,j',k,k'}$ and $kb_\lambda = b^{(\varepsilon,\varepsilon')}_{j,j',k,k'}$, if $\lambda \in \Lambda_2$. According to Theorem 2.1, $\{ka_\lambda\}_{\lambda\in\Lambda}$ and $\{kb_\lambda\}_{\lambda\in\Lambda}$ become two representations for the operators in $OpK^m$ and in $Opk^m$.

**Definition 5.2.** (i) *$T \in OpKN^m$, if $\{ka_\lambda\}_{\lambda\in\Lambda} \in N^m$.*

(ii) *$T \in Opk\tilde{N}^m$, if $\{kb_\lambda\}_{\lambda\in\Lambda} \in \tilde{N}^m$.*

One has the following theorem.

**Theorem 5.2.** (i) *$T \in OpK^m \Leftrightarrow T \in OpKN^m$.* (ii) *$T \in Opk^m \Leftrightarrow T \in Opk\tilde{N}^m$.*

Using Theorem 5.1, we study the relations between $OpKN^m$ and $OpBN^m$, the relations between $Opk\tilde{N}^m$ and $OpB\tilde{N}^m$. Theorem 5.2 is a direct conclusion of Theorem 5.1 and the following theorem.

**Theorem 5.3.** (i) *$T \in OpKN^m \Leftrightarrow T \in OpBN^m$.*

(ii) *$T \in Opk\tilde{N}^m \Leftrightarrow T \in OpB\tilde{N}^m$.*

**Proof.** (i) First, we prove $T \in OpBN^m \Rightarrow T \in OpKN^m$.

$\forall \lambda \in \tilde{\Lambda}_2$, we have

$$a^{(\varepsilon,\varepsilon')}_{j,j',k,k'} = \langle \sum_l \tilde{a}^{(\varepsilon,0)}_{j,k,l} \Phi^{(0)}_{j,l}, \Phi^{(\varepsilon')}_{j',k'}\rangle = 2^{\frac{n(j+j')}{2}} \langle \sum_l \tilde{a}^{(\varepsilon,0)}_{j,k,l+k} \Phi^{(0)}(x-l), \Phi^{(\varepsilon')}(2^{j'-j}x + 2^{j'-j}k - k')\rangle.$$

We consider two cases for $2^{j'-j}k - k'$. If $|2^{j'-j}k - k'| > 1$,

$$|a^{(\varepsilon,\varepsilon')}_{j,j',k,k'}| \le C_N \int \frac{2^{jm}}{(1+|x|)^{2N}} \frac{1}{(1+|2^{j'-j}x + 2^{j'-j}k - k'|)^{2N}} dx$$
$$\le C_N \int_{|2^{j'-j}x| \le \frac{1}{2}|2^{j'-j}k - k'|} \frac{2^{jm}}{(1+|x|)^{2N}} \frac{1}{(1+|2^{j'-j}x + 2^{j'-j}k - k'|)^{2N}} dx$$
$$+ C_N \int_{|2^{j'-j}x| > \frac{1}{2}|2^{j'-j}k - k'|} \frac{2^{jm}}{(1+|x|)^{2N}} \frac{1}{(1+|2^{j'-j}x + 2^{j'-j}k - k'|)^{2N}} dx$$
$$\le \frac{C_{N,N'} 2^{jm-(j-j')N'}}{(1+|2^{j'-j}k - k'|)^N}.$$

If $|2^{j'-j}k - k'| \le 1$,

$$a^{(\varepsilon,\varepsilon')}_{j,j',k,k'} = 2^{\frac{n(j+j')}{2}} \langle \sum_l \tilde{a}^{(\varepsilon,0)}_{j,k,l+k} \Phi^{(0)}(x-l), \Phi^{(\varepsilon')}(2^{j'-j}x + 2^{j'-j}k - k')\rangle$$
$$= 2^{\frac{n(j+j')}{2}} \langle \sum_l \tilde{a}^{(\varepsilon,0)}_{j,k,l+k} \Phi^{(0)}(x-l), \Phi^{(\varepsilon')}(2^{j'-j}x + 2^{j'-j}k - k')$$
$$- \sum_{|\alpha|\le N} C^\alpha_N (\partial^\alpha_x \Phi^{(\varepsilon')})(2^{j'-j}k - k')(2^{j'-j}x)^\alpha\rangle$$
$$+ 2^{\frac{n(j+j')}{2}} \langle \sum_l \tilde{a}^{(\varepsilon,0)}_{j,k,l+k} \Phi^{(0)}(x-l), \sum_{|\alpha|\le N} C^\alpha_N (\partial^\alpha_x \Phi^{(\varepsilon')})(2^{j'-j}k - k')(2^{j'-j}x)^\alpha\rangle.$$

But

$$\begin{aligned}&2^{\frac{n(j+j')}{2}}\Big|\Big\langle\sum_l \tilde{a}^{(\varepsilon,0)}_{j,k,l+k}\Phi^{(0)}(x-l),\Phi^{(\varepsilon')}(2^{j'-j}x+2^{j'-j}k-k')\\&-\sum_{|\alpha|\le N}C^\alpha_N(\partial^\alpha_x\Phi^{(\varepsilon')})(2^{j'-j}k-k')(2^{j'-j}x)^\alpha\Big\rangle\Big|\\&\le 2^{\frac{n(j+j')}{2}}\int\sum_l|\tilde{a}^{(\varepsilon,0)}_{j,k,l+k}|\Phi^{(0)}(x-l)||2^{j'-j}x|^N dx\\&\le C_N 2^{mj+(\frac{n}{2}-N)(j-j')};\end{aligned}$$

$$\begin{aligned}&2^{\frac{n(j+j')}{2}}\Big|\Big\langle\sum_l \tilde{a}^{(\varepsilon,0)}_{j,k,l+k}\Phi^{(0)}(x-l),\sum_{|\alpha|\le N}C^\alpha_N(\partial^\alpha_x\Phi^{(\varepsilon')})(2^{j'-j}k-k')(2^{j'-j}x)^\alpha\Big\rangle\Big|\\&=\Big|\sum_{|\alpha|\le N}C^\alpha_N 2^{(\frac{n}{2}-|\alpha|)(j-j')}\sum_l l^\alpha\tilde{a}^{(\varepsilon,0)}_{j,k,l+k}(\partial^\alpha_x\Phi^{(\varepsilon')})(2^{j'-j}k-k')\Big|\le C_N 2^{mj+(\frac{n}{2}-N)(j-j')}.\end{aligned}$$

Then we prove $T\in OpKN^m\Rightarrow T\in OpBN^m$. In fact, $\forall\lambda\in\Lambda_2$,

$$\begin{aligned}\tilde{a}^{(\varepsilon,0)}_{j,k,l}&=\Big\langle\sum a^{(\varepsilon,\varepsilon')}_{j,j',k,k'}\Phi^{(\varepsilon')}_{j',k'},\Phi^{(0)}_{j,l}\Big\rangle\\&=\sum_{(\varepsilon',j',k')\in\Gamma_n}2^{\frac{n(j'-j)}{2}}a^{(\varepsilon,\varepsilon')}_{j,j',k,k'}\langle\Phi^{(\varepsilon')}(2^{j'-j}x+2^{j'-j}k-k'),\Phi^{(0)}(x+k-l)\rangle.\end{aligned}$$

Then

$$\begin{aligned}|\tilde{a}^{(\varepsilon,0)}_{j,k,l}|&\le\sum_{\varepsilon',j',k'}\frac{C_{N,N'}2^{jm-(j-j')N'}}{(1+|2^{j'-j}k-k'|)^N}|\langle\Phi^{(\varepsilon')}(2^{j'-j}x+2^{j'-j}k-k'),\Phi^{(0)}(x+k-l)\rangle|\\&\le\sum_{(\varepsilon',j',k')\in\Gamma_n}\frac{C_{N,N'}2^{jm-(j-j')N'}}{(1+|2^{j'-j}k-k'|)^N}\\&\quad\cdot\int\frac{1}{(1+|2^{j'-j}x+2^{j'-j}k-k'|)^{2N}}\frac{1}{(1+|x+k-l|)^{2N}}dx\\&\le\sum_{j'}C_{N,N'}2^{jm-(j-j')N''}\frac{1}{(1+|k-l|)^N}\le\frac{C_N2^{jm}}{(1+|k-l|)^N}.\end{aligned}$$

Furthermore,

$$\begin{aligned}&\Big|\sum_l(k-l)^\alpha\tilde{a}^{(\varepsilon,0)}_{j,k,l}\Big|\\&=\Big|\sum_{\varepsilon',j',k'}2^{\frac{n(j'-j)}{2}}a^{(\varepsilon,\varepsilon')}_{j,j',k,k'}\langle\Phi^{(\varepsilon')}(2^{j'-j}x+2^{j'-j}k-k'),\sum_l(k-l)^\alpha\Phi^{(0)}(x+k-l)\rangle\Big|\\&=\Big|\sum_{\varepsilon',j',k'}2^{\frac{n(j'-j)}{2}}a^{(\varepsilon,\varepsilon')}_{j,j',k,k'}\langle\Phi^{(\varepsilon')}(2^{j'-j}x+2^{j'-j}k-k'),x^\alpha\rangle\Big|\\&=\Big|\sum_{k'}2^{\frac{-nj}{2}}a^{(\varepsilon,0)}_{j,0,k,k'}\langle\Phi^{(0)}(2^{-j}x+2^{-j}k-k'),x^\alpha\rangle\Big|.\end{aligned}$$

Applying (3.2), we get (5.1).

(ii) The proof of (ii) is the same as (i).

## §6. Conclusion and Some Remarks

**6.1. Characterisation of $OpS^m_{1,1}$ in Kernel-Distribution Spaces and in Wavelet Coefficient Spaces**

After Calderón established a formal relation between a symbol and a kernel-distribution, the symbolic school and the Calderón-Zygmund school remain almost in two classes isolated. Using the Littlewood-Paley analysis, E. M. Stein established, in [8], a relation for $OpS^m_{1,1}$ and the kernel-distribution spaces in the following sense:

**Proposition 6.1.** *$T \in OpS^0_{1,1}$, then $K(x, y)$ satisfies* (1.4) *and* (1.5) *or $k(x, z)$ satisfies* (1.8) *and* (1.9).

There are some properties which are trivial for a symbolic operator, such as (1.6) and (1.7) or (1.10) and (1.11), but they can not be analyzed by the Littlewood-Paley analysis. After the apparition of wavelets, Meyer used a wavelet method. $\forall \lambda = (\varepsilon, j, k) \in \Gamma_n, \lambda' = (\varepsilon', j', k') \in \Gamma_n$, denote $a_{\lambda,\lambda'} = a^{(\varepsilon,\varepsilon')}_{j,j',k,k'} = \langle T\Phi_\lambda, \Phi_{\lambda'}\rangle$. Meyer analyzed the matrix $\{a_{\lambda,\lambda'}\}_{(\lambda,\lambda')\in\Gamma_n\times\Gamma_n}$, he obtained the following results.

**Proposition 6.2.** *$T \in OpS^0_{1,1}$ and*

$$T^* \in OpS^0_{1,1} \Leftrightarrow$$

$$\forall N > 0, |a^{(\varepsilon,\varepsilon')}_{j,j',k,k'}| \le C_N 2^{-|j-j'|(\frac{n}{2}+N)} \Big(\frac{2^{-j} + 2^{-j'}}{2^{-j} + 2^{-j'} + |k2^{-j} - k'2^{-j'}|}\Big)^{n+N}, \ \forall \lambda, \lambda' \in \Gamma_n.$$

But if one does not know the information of the conjugate operator $T^*$ of an operator $T$, one cannot use this method. In [1] and [9], a new algorithm was developped, one can analyze the Calderón-Zygmund operator. In this paper, we use some special wavelet bases to analyze an operator in $OpS^m_{1,1}, OpK^m$, and $Opk^m$, the main conclusions of this paper are

**Theorem 6.1.** *$OpK^m = OpKN^m = OpN^m = OpS^m_{1,1} = Op\tilde{N}^m = Opk\tilde{N}^m = Opk^m$.*

**Proof.** According to the formal relation in (1.12), $\Phi_\lambda(x, \xi) = \int K_\lambda(x, y)e^{i(y-x)\xi}dy$, and $\tilde{\Phi}_\lambda(x, \xi) = \int k_\lambda(x, z)e^{iz\xi}dz$ , we have $OpKN^m = OpN^m$ and $Opk\tilde{N}^m = Op\tilde{N}^m$. Then applying Theorem 3.1 and Theorem 5.1, we get the proof of this theorem.

**Remark 6.1.** If one does not know the infomation of $T^*$, one cannot use Meyer's method in [7] to analyze an operator in $OpS^m_{1,1}$. As for B-C-R algorithm in [1], the norm of an operator in $OpS^m_{1,1}$ can not be decided by the absolute value of wavelet coefficients.

**Remark 6.2.** In [8], E. M. Stein has got Proposition 6.1. But if one knows only the information outside the diagonal set $\{x = y\}$ for $K(x, y)$, or the information outside the set $\{z = 0\}$ for $k(x, z)$, generally, one cannot say that $K(x, y)$ or $k(x, z)$ is a distribution in $S(R^n \times R^n)$; so $K(x, y)$ or $k(x, z)$ cannot be the kernel-distribution of a symbolic operator.

**6.2. Some Properties for Singular Integral Operators**

In this subsection, we consider some properties for singular integral operators. First, we see that (1.7) is very important for an operator. In fact, David and Journé considered the action of an operator to the polynomial function 1, they established their famous $T1$ theorem; Bony considered the action of an operator to the polynomial function 1, he found a para-product of an operator and established his para-product theory. If an operator maps all the polynomial functions into $S(R^n)$, then $T$ satisfies the following condition:

(6.1) $T$ continues from $S(R^n)$ to $S(R^n)$.

The auxiliary results of this paper are the following

**Theorem 6.2.** (i) $T \in Op\widehat{K}^m \Leftrightarrow T \in Op\hat{k}^m$.

(ii) *If $T \in Op\widehat{K}^m$ or $T \in Op\hat{k}^m$, then the condition* (1.7) *implies* (6.1).

**Proof.** (i) First, we prove that $T \in Op\widehat{K}^m \Leftrightarrow T \in Op\hat{k}^m$. It is evident that (1.4) and (1.5)$\Leftrightarrow$ (1.8) and (1.9). Then we apply (i) and (ii) of Theorem 4.1 to finish the proof of (i). First we prove that $T \in Op\widehat{K}^m$ implies (1.10). In fact, $\forall x_0 \in R^n, \forall 0 < R < 1, \forall f(x) \in$

$C_0^N(B(x_0,R)), \forall g(z) \in C_0^N(B(0,R))$, we have

$$\begin{aligned}
&\Big|\iint k(x,z)f(x)g(z)dxdz\Big| = \Big|\iint K(x,x-z)f(x)g(z)dxdz\Big| \\
&= \Big|\iint \sum_{\varepsilon,\varepsilon'}\sum_{j,k,l} \tilde{a}^{(\varepsilon,\varepsilon')}_{j,k,l}\Phi^{(\varepsilon)}_{j,k}(x)\Phi^{(\varepsilon')}_{j,l}(x-z)f(x)g(z)dxdz\Big| \\
&\le \Big|\iint \sum_{\varepsilon,\varepsilon'}\sum_{\substack{j,k,l\\ j<-\log_2 R}} \tilde{a}^{(\varepsilon,\varepsilon')}_{j,k,l}\Phi^{(\varepsilon)}_{j,k}(x)\Phi^{(\varepsilon')}_{j,l}(x-z)f(x)g(z)dxdz\Big| \\
&\quad + \Big|\iint \sum_{\substack{\varepsilon\in\{0,1\}^n\\ \varepsilon'\in\{0,1\}^n\setminus\{0\}}}\sum_{\substack{j,k,l\\ j\ge-\log_2 R}} \tilde{a}^{(\varepsilon,\varepsilon')}_{j,k,l}\Phi^{(\varepsilon)}_{j,k}(x)\Phi^{(\varepsilon')}_{j,l}(x-z)f(x)g(z)dxdz\Big| \\
&\quad + \Big|\iint \sum_{\varepsilon\in\{0,1\}^n\setminus\{0\}}\sum_{\substack{j,k,l\\ j\ge-\log_2 R}} \tilde{a}^{(\varepsilon,0)}_{j,k,l}\Phi^{(\varepsilon)}_{j,k}(x)\Phi^{(0)}_{j,l}(x-z)f(x)g(z)dxdz\Big| \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

For $I_1$, if $n+m>0$, then $|I_1| \le CR^{n-m}||f||_\infty||g||_\infty$; if $n+m\le 0$, then $|I_1| \le CR^{2n}(1-\log_2 R)||f||_\infty||g||_\infty \le CR^{n-m}||f||_\infty||g||_\infty$. For $I_2$, applying the integration by parts for $z$, we get $|I_2| \le C_N R^{n-m+N}||f||_\infty||g||_{\dot{C}^N}$. For $I_3$, applying the integration by parts for $x$, and then for $z$, we get $|I_3| \le C_N R^{n-m+N}||f||_{\dot{C}^N}||g||_\infty$ . So $T$ satisfys the condition (1.10).

Then, making the same calculations as above, we get that $T \in Op\hat{k}^m$ implys (1.6).

(ii) According to Theorem 6.1, if $T \in Op\widehat{K}^m$ or $T \in Op\hat{k}^m$, and $T$ satisfies the condition (1.7), then $T \in OpS^m_{1,1}$, so $T$ satisfies (6.1).

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