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HILBERTIAN INVARIANCE PRINCIPLE FOR EMPIRICAL PROCESS ASSOCIATED WITH A MARKOV PROCESS

JIANG Yiwen* WU Liming**

Abstract

The authors establish the Hilbertian invariance principle for the empirical process of a stationary Markov process, by extending the forward-backward martingale decomposition of Lyons-Meyer-Zheng to the Hilbert space valued additive functionals associated with general non-reversible Markov processes.

Keywords Forward-backward martingale decomposition, Functional central limit theorem or Donsker's invariance principle, Empirical process
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§1. Introduction

1.1. Motivation and Several Known Results

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}}, (X_t)_{t\in\mathbb{T}}, (\theta_t)_{t\in\mathbb{T}}, (\mathbb{P}_x)_{x\in E})$ be a Markov process valued in a Polish space E, with transition probability semigroup $(P_t)_{t\in\mathbb{T}}$ and with an invariant and ergodic probability measure μ on $(E, \mathcal{B})$, which is unknown. Here $\mathbb{T} = \mathbb{N}$ (discrete time) or $\mathbb{R}^+$ (continuous time). For any initial measure ν, set $\mathbb{P}_\nu(\cdot) := \int_E \mathbb{P}_x(\cdot)\nu(dx)$ and write $\mathbb{E}^\nu(\cdot) := \int_\Omega (\cdot) d\mathbb{P}_\nu$.

Let $f : E \to \mathbb{R}$ be a fixed $\mathcal{B}$-measurable function (our observable). A natural question from the point of view of non-parametric statistics is to estimate the distribution function $F(u) := \mu[f(x) \le u] = \mathbb{P}_\mu(f(X_0) \le u)$ by the observed (X_t). By an extension of the Kolmogorov-Smirnov theorem, we have $\mathbb{P}_\mu$-a.s.

$$\sup_{u\in\mathbb{R}} |F_T^*(u) - F(u)| \longrightarrow 0,$$

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*Department of Mathematics, Wuhan University, Wuhan 430072, China.
E-mail: gyiwen@ public.wh.hb.cn
Department of Mathematics, Military Economic Academy, 430035 Wuhan, China.
Laboratoire de Mathématiques Appliquées. CNRS-UMR 6620, Université Blaise Pascal, 63177 Aubière, France. **E-mail: Li-Ming.Wu@ math.univ-bpclermont.fr and Department of Mathematics, Wuhan University, Wuhan 430072, China.

as T goes to infinity, where

$$F_T^*(u) := \begin{cases} \frac{1}{n}\sum_{k=0}^{n-1} \mathbf{1}_{[f(X_k)\le u]}, & \text{if } 0 < T = n \in \mathbb{T} = \mathbb{N}; \\ \frac{1}{T}\int_0^T \mathbf{1}_{[f(X_s)\le u]} ds, & \text{if } 0 < T \in \mathbb{T} = \mathbb{R}^+ \end{cases}$$

is the empirical distribution. For many statistical purposes a basic question is to establish the corresponding central limit theorem (CLT in short) of functional type, i.e., to prove that as T goes to infinity,

$$\xi_T(u) := \sqrt{T}(F_T^*(u) - F(u)) \tag{1.1}$$

converges in law to some gaussian process $(\xi(u))$ on some appropriate space $\mathbf{B}$ of functions on $\mathbb{R}$.

If $(X_n)_{n\in\mathbb{N}}$ is i.i.d., it is well known that the above functional CLT (FCLT in short) holds on $\mathbf{B} = \{h(u) : \mathbb{R} \to \mathbb{R} \mid h$ is càdlàg and bounded $\}$ equipped with the sup norm topology, and $(\xi(u))_{u\in\mathbb{R}}$ is the Brownian Bridge, i.e., a continuous Gaussian process with

$$\mathbb{E}\xi(u)\xi(v) = F(u \wedge v) - F(u)F(v)$$

(see [8]).

How to extend the preceeding FCLT to Markov processes is an old question. When E is countable, the reader is referred to [15], [2] (in the latter paper the moderate and large deviations are furnished). X. Chen[1] got very fine and complete results for general irreducible Markov processes. Their main tool is atom's decomposition due to Nummelin (see [12] and [1]), which allows to reduce the question to the i.i.d. case (with many serious technical difficulties).

When dealing with infinite dimensional Markov processes such as systems of infinite particles or infinite dimensional stochastic differential equations, the powerful tool of atom's decomposition is no longer valid: other tools are required. Below we review two lines of development which largely inspire and motivate this work.

In the reversible (or symmetric) case (i.e., $P_t^* = P_t$ where P_t^* is the adjoint operator of P_t in $L^2(\mu)$), Kipnis and Varadhan[6] showed that under the natural minimal condition

$$\lim_{t\to+\infty} \frac{\mathbb{E}^\mu \left(S_t(f)\right)^2}{t} := \sigma^2(f) < \infty, \tag{1.2}$$

the additive functional

$$S_t(f) := \sum_{k=0}^{n-1} f(X_k) \text{ or } \int_0^t f(X_s)ds \ \text{ according to } t = n \in \mathbb{T} = \mathbb{N} \text{ or } t \in \mathbb{T} = \mathbb{R}^+ \tag{1.3}$$

satisfies the FCLT, i.e., $T^{-1/2}(S_{Tt}(f))_{t\in[0,1]}$ converges, as T goes to infinity, in law to the Brownian Motion $(B_t)_{t\in[0,1]}$ with $\mathbb{E}B_1^2 = \sigma^2(f)$. Moreover they showed that (1.2) is equivalent to the following finite energy condition for $f \in L^2(\mu)$,

$$|\langle f, \phi\rangle| \le C\sqrt{\langle A\phi, \phi\rangle}, \quad \forall \phi \in L_0^2(E, \mu; \mathbb{R}) \cap \mathbb{D}(A), \tag{1.4}$$

where $A = I - P_1$ if $\mathbb{T} = \mathbb{N}$ or $-\mathcal{L}$ if $\mathbb{T} = \mathbb{R}^+$, $\mathcal{L}$ being the generator of $(P_t)_{t\in\mathbb{R}^+}$ in $L^2(\mu)$.

The main tool used in this important work is the martingale decomposition approach of Gordin[4] and the control of $\sup_{t\le T} |g(X_t) - g(X_0)|$ by means of the Dirichlet form $\mathcal{E}(g, g)$.

The extension of this beautiful result to the non-symmetric case has attracted some attention. Indeed, for the simple exclusion process with an asymmetric mean zero probability kernel, Varadhan[16] established the CLT of $S_t(f)$ for all $f \in L_0^2(E, \mu)$ satisfying (1.4), and proved even the FCLT for some special f related to the movement of a tagged particle, by exploiting the quasi-symmetry (or strong sector property of its generator) of this process. Here $(X_t)_{t\in\mathbb{T}}$ is said to be quasi-symmetric (or to satisfy the sector condition), if there is some constant $K \geq 1$ such that

$$\langle A\phi, \varphi\rangle \leq K\sqrt{\langle A\phi, \phi\rangle \cdot \langle A\varphi, \varphi\rangle}, \quad \forall \phi, \varphi \in \mathbb{D}(A). \tag{1.5}$$

Later for general quasi-symmetric Markov processes, Osada and Saitoh[13] got the finite dimensional CLT for general additive functional $(S_t(f))$ under condition (1.4) (see [13, (1.6)]). And they obtained the corresponding FCLT for quite general additive functionals related to reflected diffusions. And Wu[17] established the equivalence between (1.2) and (1.4), the FCLT, and the functional law of iterated logarithm of $S_t(f)$ for f satisfying (1.2) or (1.4). The main tool in that work is an extension of the forward-backward martingale decomposition of Lyons-Meyer-Zheng from the symmetric case to the non-symmetric case.

In a completely different line of development many studies are realized for an associated stationary and ergodic sequence $(X_n, n \in \mathbb{Z})$ of uniform random variables on $[0,1]$, and $f(x) = x$. Shao and Yu[14] proved the weak $\mathbb{D}[0,1]$ convergence of ξ_n assuming the $\mathrm{Cov}(X_0, X_n) = O(n^{-a})$ with $a > \frac{3+\sqrt{33}}{2} \simeq 4.373$. Further, B. Morel and C. Suquet[9] realized that the optimal condition for the weak $L^2[0,1]$ convergence of ξ_n to a Gaussian random element is $\sum_{k\geq 1}(\frac{2}{3} - \mathbb{E}\max(X_0, X_k)) < \infty$. This motivates our main result below.

1.2. A Main Result

We are mainly interested in the FCLT of the empirical distribution $\xi_n(\cdot)$ (or $\xi_T(\cdot)$) in some Hilbert space as in [9]. In this paper, we assume $F(u) := \mu[f(x) \leq u]$ is continuous.

Let $\Gamma_n(u,v) := \mathbb{E}^\mu \xi_n(u)\xi_n(v)$ and $h_u(x) := \mathbf{1}_{[f(x)\leq u]}$, we have the following invariance principle:

Theorem 1.1. *Assume the strong sector condition* (1.5) (*i.e., our Markov process* $(X_t)_{t\in\mathbb{T}}$ *is quasi-symmetric*). *If* $\frac{2}{3} - \mathbb{E}^\mu\max(F(f)(X_0), F(f)(X_t))$ *is summable in the sense of Abel, i.e., if*

$$\sum_{k\geq 1}(1+\varepsilon)^{-k-1}\left(\frac{2}{3} - \mathbb{E}^\mu\max(F(f)(X_0), F(f)(X_k))\right) \quad \text{or}$$
$$\int_0^\infty e^{-\varepsilon t}\left(\frac{2}{3} - \mathbb{E}^\mu\max(F(f)(X_0), F(f)(X_t))\right) dt$$

(*according to* $\mathbb{T} = \mathbb{N}$ *or* $\mathbb{R}^+$) *converges in* $\mathbb{R}$ *as* $\varepsilon \downarrow 0$, *then we have*

(a) $\Gamma_T(u,v) := \mathbb{E}^\mu \xi_T(u)\xi_T(v)$ *weakly converges in* $L^2(\mathbb{R}, dF(u)dF(v))$ *to* $\Gamma(u,v)$ *as* T *goes to infinity, and* Γ *as a kernel operator*

$$\Gamma(f)(u) := \int \Gamma(u,v)f(v)dF(v)$$

is of the trace class on $L^2(\mathbb{R}, dF(u)dF(v))$;

(b) *for any initial measure* $\nu \ll \mu$, *the law of* ξ_n *on* $L^2(\mathbb{R}, dF)$ *under* $\mathbb{P}_\nu$ *converges weakly to the Gaussian measure on* $L^2(\mathbb{R}, dF)$ *with reproducing kernel given by* $\Gamma(u,v)$.

In order to prove Theorem 1.1, we shall extend the forward-backward martingale decomposition of Lyons-Meyer-Zheng[7,11] to the Hilbert space valued additive functionals. The second named author[17] generalized the forward-backward martingale decomposition of Lyons-Meyer-Zheng's type from the symmetric case to the general stationary situation for the partial sum $S_t(f)$ with f satisfying a finite energy condition for real f.

This paper is organized as follows. The next two sections are devoted to extending forward-backward martingale decomposition of Lyons-Meyer-Zheng's type in the discrete time and continuous time from the real valued case to the Hilbertian valued case. In section 4 we discuss the quasi-symmetric case. We complete the proof of Theorem 1.1 in the last section.

Throughout this paper, $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denote respectively the inner product and norm in $L^2(E,\mu;\mathbb{R})$, $\mathbb{E}(\cdot)$ the expectation w.r.t. $\mathbb{P} = \mathbb{P}_\mu$. $L_0^2(E,\mu;\mathbb{R}) := \{g \in L^2(E,\mu;\mathbb{R}); \langle g, 1\rangle = 0\}$. We say that $(A_t)_{t\in\mathbb{T}}$is an additive functional, if for all $s,t \in \mathbb{T}$,

$$A_{s+t}(\omega) = A_s(\omega) + A_t(\theta_s\omega),\ \mathbb{P}_\mu - a.s.$$

(i.e., in the loose sense). A typical additive functional is $(S_t(f))$ given in (1.3).

§2.Forward-Backward Martingale Decomposition and Invariance Principle: the Discrete Time Case

2.1. Some Preliminary Lemmas in the Real Valued Case

We begin by recalling some results in [17].

Let $\mathbb{T} = \mathbb{N}$ and write $P = P_1$. Let P^* be the adjoint operator of P in $L^2(E,\mu;\mathbb{R})$ and $P^\sigma = \frac{P+P^*}{2}$, the symmetrization of P. Let $\mathbb{W}_0 := L_0^2(E,\mu;\mathbb{R}) = \{f \in L^2(E,\mu;\mathbb{R}); \langle f\rangle_\mu = 0\}$ equipped with norm $\|f\|_0 = \|f\|_{L^2(\mu)}$. It is easy to see that $\forall u \in L_0^2(E,\mu;\mathbb{R}) := \mathbb{W}_0$,

$$\langle (I-P^\sigma)u, u\rangle = 0 \Rightarrow u = 0. \tag{2.1}$$

Then (see [17]) $I - P^\sigma : \mathbb{W}_0 \to \mathbb{W}_0$ is injective, its inverse $R_0^\sigma : \mathbb{D}(R_0^\sigma)(\subset \mathbb{W}_0) \mapsto \mathbb{W}_0$ is a self-adjoint operator with domain $\mathbb{D}(R_0^\sigma) = \mathrm{Ran}(I - P^\sigma)$.

Definition 2.1. *Let $\mathbb{W}_1$ be the completion of the pre-Hilbert space $(\mathbb{W}_0 = L_0^2(E,\mu;\mathbb{R}),$ $\langle\cdot,\cdot\rangle_1)$ where the inner product is given by*

$$\langle u, v\rangle_1 := \langle (I-P^\sigma)u, v\rangle. \tag{2.2}$$

We define $(\mathbb{W}_{-1}, \|\cdot\|_{-1})$ as the dual Hilbert space of $(\mathbb{W}_1, \|\cdot\|_1)$ w.r.t. the canonical dual relation $\mathbb{W}_0' = \mathbb{W}_0$.

Lemma 2.1.[17] $\mathbb{W}_0 \subset \mathbb{W}_1$, $\mathbb{W}_{-1} \subset \mathbb{W}_0$ *are both continuous and dense imbedding, and for $f \in L_0^2(E,\mu;\mathbb{R}), f \in \mathbb{W}_{-1}$ iff* (1.3) *holds; and the minimal constant C in* (1.3) *equals to* $\|f\|_{-1}$.

By the ergodicity of P we can define the potential (or Poisson) operators

$$R_0 = (I-P)^{-1} : \mathbb{D}(R_0) = \mathrm{Ran}(I-P)(\subset \mathbb{W}_0) \to \mathbb{W}_0,$$
$$R_0^* = (I-P^*)^{-1} : \mathbb{D}(R_0^*) = \mathrm{Ran}(I-P^*)(\subset \mathbb{W}_0) \to \mathbb{W}_0.$$

Lemma 2.2.[17] $\mathbb{D}(R_0) \cup \mathbb{D}(R_0^*) \subset \mathbb{W}_{-1}$ *and*

$$\|f\|_{-1} \le \sqrt{2}\|R_0 f\|_0, \forall f \in \mathbb{D}(R_0),\ \textit{and}\quad \|f\|_{-1} \le \sqrt{2}\|R_0^* f\|_0, \forall f \in \mathbb{D}(R_0^*); \tag{2.3}$$

$$\|R_0 f\|_1 \le \|f\|_{-1},\quad \forall f \in \mathbb{D}(R_0),\ \textit{and}\quad \|R_0^* f\|_1 \le \|f\|_{-1},\quad \forall f \in \mathbb{D}(R_0^*). \tag{2.4}$$

2.2. Forward-Backward Martingale Decomposition for H-valued Additive Functionals

Let $\mathbb{H}$ be a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ and norm $\|\cdot\|_{\mathbb{H}}$; $(\mathbf{e}_i)_{i\in I}$ $(I = [1, \dim(\mathbb{H})] \cap \mathbb{N})$ denotes an orthonormal basis for $\mathbb{H}$. The $\mathbb{H}$-valued functions will be denoted by the bold letters $\mathbf{f}$, $\mathbf{g}$, $\cdots$. Let $\mathbf{L}_0^2(E,\mu;\mathbb{H}) := \{\mathbf{f} \in \mathbf{L}^2(E,\mu;\mathbb{H}); \mathbb{E}^\mu(\mathbf{f}) = \mathbf{0}\}$. The domain of operator R_0 on $\mathbf{L}_0^2(E,\mu;\mathbb{H})$ is denoted by $\mathbb{D}_{\mathbb{H}}(R_0)$. We denote the norm and inner product in $\mathbf{L}^2(E,\mu;\mathbb{H})$ by $|\!|\!|\cdot|\!|\!|$ and $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ respectively. We define

Definition 2.2. *Let* $\mathbf{f} \in L^2_0(E,\mu;\mathbb{H})$ *and* $f_i := \langle \mathbf{f}, \mathbf{e}_i\rangle_{\mathbb{H}}$. *We say that* $\mathbf{f} \in \mathbb{W}^{\mathbb{H}}_{-1}$, *if*

$$\|\mathbf{f}\|_{\mathbb{W}^{\mathbb{H}}_{-1}} := \sqrt{\sum_{i\in I} \|f_i\|^2_{-1}} < +\infty.$$

Lemma 2.3. *Let* $\mathbf{f} \in \mathbf{L}^2_0(E,\mu;\mathbb{H})$. *Then* $\mathbf{f} \in \mathbb{W}^{\mathbb{H}}_{-1}$ *iff there exists a constant* $C \ge 0$, *such that*

$$\langle\langle \mathbf{f}, \mathbf{g}\rangle\rangle \le C\sqrt{\langle A\mathbf{g}, \mathbf{g}\rangle_{\mathbb{H}}}, \quad \forall\ \mathbf{g} \in \mathbb{D}_{\mathbb{H}}(A) = \mathbf{L}^2(E,\mu;\mathbb{H}) \tag{2.5}$$

(*recalling that* $A = I - P$) *and*

$$\|\mathbf{f}\|_{\mathbb{W}^{\mathbb{H}}_{-1}} = \inf\{C \ge 0 \mid \quad (2.5)\ holds\ for\ C\}. \tag{2.6}$$

In particular, $\|\mathbf{f}\|_{\mathbb{W}^{\mathbb{H}}_{-1}}$ *does not depend on the choice of the ONB* $(\mathbf{e}_i)$.

Proof. We prove it only in the case where $\dim \mathbb{H} = +\infty$ $(I = \mathbb{N}^*)$.

Necessary part: Note that for all $\mathbf{g} \in \mathbb{D}_{\mathbb{H}}(A)$,

$$\begin{aligned}\langle\langle \mathbf{f}, \mathbf{g}\rangle\rangle &= \sum_{i\in I}\langle f_i, g_i\rangle \le \sum_{i\in I}\|f_i\|_{-1}\sqrt{\langle Ag_i, g_i\rangle}\\ &\le \sqrt{\sum_{i\in I}\|f_i\|^2_{-1}}\sqrt{\sum_{i\in I}\langle Ag_i, g_i\rangle} = \|\mathbf{f}\|_{\mathbb{W}^{\mathbb{H}}_{-1}}\sqrt{\langle A\mathbf{g}, \mathbf{g}\rangle}.\end{aligned}$$

Hence inequality (2.5) holds with some $C \le \|\mathbf{f}\|_{\mathbb{W}^{\mathbb{H}}_{-1}}$.

Sufficient part: Let $n \ge 1$ and $(\lambda_i)_{i=1,\cdots,n} \in \mathbb{R}^n_+$ such that $\sum_i \lambda_i^2 \le 1$. For each $f_i = \langle \mathbf{f}, \mathbf{e}_i\rangle$, and for any $g \in \mathbb{W}_0$, we have by the sufficiency assumption (2.5),

$$\langle f_i, g\rangle = \langle\langle \mathbf{f}, g\mathbf{e}_i\rangle\rangle \le C\sqrt{\langle Ag, g\rangle},$$

and it follows that $f_i \in \mathbb{W}_{-1}$. For any $\varepsilon > 0$, we can find $g_i \in \mathbb{W}_0$ such that $\|f_i\|_{-1} \le (1+\varepsilon)\langle f_i, g_i\rangle$ and $\langle Ag_i, g_i\rangle = 1$. Thus by (2.5) again,

$$\begin{aligned}\sum_{i=1}^n \lambda_i\|f_i\|_{-1} &\le (1+\varepsilon)\sum_{i=1}^n \lambda_i\langle f_i, g_i\rangle = (1+\varepsilon)\langle\langle \mathbf{f}, \sum_{i=1}^n \lambda_i g_i\mathbf{e}_i\rangle\rangle\\ &\le (1+\varepsilon)C\sum_{i=1}^n \lambda_i^2\langle Ag_i, g_i\rangle \le (1+\varepsilon)C.\end{aligned}$$

Taking the supremum over all such (λ_i), we get

$$\sqrt{\sum_{i=1}^n \|f_i\|^2_{-1}} \le (1+\varepsilon)C$$

for all $n \ge 1$ and $\varepsilon > 0$. So $\|\mathbf{f}\|_{\mathbb{W}^{\mathbb{H}}_{-1}} \le C$ by letting $\varepsilon \to 0,\ n \to \infty$.

Theorem 2.1. *Let* $\mathbb{T} = \mathbb{N}$. *There exist three bounded linear mappings*

$$\begin{aligned}&\mathbf{G} : \mathbb{W}^{\mathbb{H}}_{-1} \to \mathbb{W}^{\mathbb{H}}_0 = \mathbf{L}^2_0(E,\mu;\mathbb{H}), \quad \langle \mathbf{Gf}, \mathbf{Gf}\rangle_{\mathbb{H}} \le 2(\|\mathbf{f}\|_{\mathbb{W}^{\mathbb{H}}_{-1}})^2;\\ &\mathbf{M}^{\rightarrow}_1 : \mathbb{W}^{\mathbb{H}}_{-1} \to \mathbf{L}^2(\Omega,\mathcal{F}_1,\mathbb{P}) \ominus \mathbf{L}^2(\Omega,\mathcal{F}_0,\mathbb{P}), \quad \mathbb{E}(\|\mathbf{M}^{\rightarrow}_1(\mathbf{f})\|_{\mathbb{H}})^2 \le 2(\|\mathbf{f}\|_{\mathbb{W}^{\mathbb{H}}_{-1}})^2;\\ &\mathbf{M}^{\leftarrow}_1 : \mathbb{W}^{\mathbb{H}}_{-1} \to \mathbf{L}^2(\Omega,\mathcal{G}_0,\mathbb{P}) \ominus \mathbf{L}^2(\Omega,\mathcal{G}_1,\mathbb{P}), \quad \mathbb{E}(\|\mathbf{M}^{\leftarrow}_1(\mathbf{f})\|_{\mathbb{H}})^2 \le 2(\|\mathbf{f}\|_{\mathbb{W}^{\mathbb{H}}_{-1}})^2,\end{aligned} \tag{2.7}$$

where $\mathcal{G}_k = \sigma(X_m; m \ge k)$ *is the future* σ*-field, such that the following forward-backward* *martingale decomposition holds* $\mathbb{P}$*-a.s. for every* $\mathbf{f} \in \mathbb{W}_{-1}^{\mathbb{H}}$,

$$2\sum_{k=0}^{n-1} \mathbf{f}(X_k) = \mathbf{M}_n^{\rightarrow}(\mathbf{f}) + \mathbf{M}_n^{\leftarrow}(\mathbf{f}) + \mathbf{G}\mathbf{f}(X_0) - \mathbf{G}\mathbf{f}(X_n), \quad \forall n \in \mathbb{N}, \tag{2.8}$$

where

$$\mathbf{M}_n^{\rightarrow}(\mathbf{f}) = \sum_{k=1}^{n} \theta_{k-1}\mathbf{M}_1^{\rightarrow}(\mathbf{f}), \quad \mathbf{M}_n^{\leftarrow}(\mathbf{f}) = \sum_{k=1}^{n} \theta_{k-1}\mathbf{M}_1^{\leftarrow}(\mathbf{f}).$$

In particular for each $\mathbf{f} \in \mathbb{W}_{-1}^{\mathbb{H}}$,

(a) *the maximal inequality below holds:*

$$\mathbb{E}^{\mu} \sup_{0\le k\le n} \|S_k(\mathbf{f})\|_{\mathbb{H}}^2 \le (24n+3)(\|\mathbf{f}\|_{\mathbb{W}_{-1}^{\mathbb{H}}})^2; \tag{2.9}$$

(b) *the family of the laws of*

$$t \to \frac{1}{\sqrt{n}} S_{[nt]}(\mathbf{f}) \in \mathbb{D}([0,1],\mathbb{H}), \quad n \ge 1,$$

on $\mathbb{D}([0,1],\mathbb{H})$ *under* $\mathbb{P}_\mu$ *is precompact for the weak convergence topology;*

(c) *if moreover* $\mathbf{f}$ *belongs to the closure of* $\mathrm{Ran}_{\mathbb{H}}(I-P) := \{(I-P)\mathbf{f} : \mathbf{f} \in \mathbb{W}_0^{\mathbb{H}} = \mathbf{L}_0^2(E,\mu;\mathbb{H})\} = \mathbb{D}_{\mathbb{H}}(R_0)$ *in* $\mathbb{W}_{-1}^{\mathbb{H}}$*, then there is an additive square integrable* $\mathbb{H}$ *-valued martingale* $(\mathcal{F}_n)$*-* $(\mathbf{M}_n(\mathbf{f}))_{n\ge 0}$ *and an* $\mathbb{H}$ *-valued additive functional* $(\Delta_n(\mathbf{f}))_{n\ge 0}$ *such that*

$$S_n(\mathbf{f}) = \Delta_n(\mathbf{f}) + \mathbf{M}_n(\mathbf{f}), \tag{2.10}$$

$$\frac{1}{n}\mathbb{E}^{\mu} \max_{k\le n} \|\Delta_k(\mathbf{f})\|_{\mathbb{H}}^2 \to 0. \tag{2.11}$$

In particular, for any initial measure $\nu \ll \mu$*, as n goes to infinity, the law of* $\left(\frac{1}{\sqrt{n}} S_{nt}(\mathbf{f})\right)_{t\in[0,1]}$ *under* $\mathbb{P}_\nu$ *converges weakly in* $\mathbb{D}([0,1],\mathbb{H})$ *to the law of an* $\mathbb{H}$*-valued BM* $(\mathbf{B}_t)$ *where the covariance of* $\mathbf{B}_1$ *is given by*

$$\mathbb{E}(\langle \mathbf{B}_1, \mathbf{h}_1\rangle_{\mathbb{H}} \langle \mathbf{B}_1, \mathbf{h}_2\rangle_{\mathbb{H}}) = \langle \Gamma \mathbf{h}_1, \mathbf{h}_2\rangle_{\mathbb{H}}, \quad \forall\ \mathbf{h}_1, \mathbf{h}_2 \in \mathbb{H},$$

where

$$\begin{aligned}\langle \Gamma \mathbf{h}_1, \mathbf{h}_2\rangle_{\mathbb{H}} &:= \mathbb{E}(\langle \mathbf{M}_1(\mathbf{f}), \mathbf{h}_1\rangle_{\mathbb{H}} \langle \mathbf{M}_1(\mathbf{f}), \mathbf{h}_2\rangle_{\mathbb{H}}) \\ &= \lim_{n\to\infty} \frac{1}{n}\mathbb{E}(\langle S_n(\mathbf{f}), \mathbf{h}_1\rangle_{\mathbb{H}} \langle S_n(\mathbf{f}), \mathbf{h}_2\rangle_{\mathbb{H}}), \quad \forall\ \mathbf{h}_1, \mathbf{h}_2 \in \mathbb{H}.\end{aligned} \tag{2.12}$$

Here $\mathbb{D}([0,1],\mathbb{H})$ *is the space of all* $\mathbb{H}$*-valued càdlàg functions on* $[0,1]$ *equipped with the Skorokhod topology.*

Proof. When $\mathbb{H} = \mathbb{R}$, parts (a), (b), (c) are due to [17, Theorem 2.5].

For every $\mathbf{f} = \sum\limits_{i=1}^{\infty} f_i \mathbf{e}_i \in \mathbb{H}$, put

$$\mathbf{G}(\mathbf{f}) := \sum_{i=1}^{\infty} G(f_i)\mathbf{e}_i,\ \mathbf{M}_1^{\rightarrow}(\mathbf{f}) := \sum_{i=1}^{\infty} M_1^{\rightarrow}(f_i)\mathbf{e}_i,\ \mathbf{M}_1^{\leftarrow}(\mathbf{f}) := \sum_{i=1}^{\infty} M_1^{\leftarrow}(f_i)\mathbf{e}_i,$$

where $G, M_1^{\rightarrow}, M_1^{\leftarrow}$ are defined in [17, Theorem 2.5]. They are convergent, because

$$\mathbb{E}\|\mathbf{G}(\mathbf{f})\|_{\mathbb{H}}^2 = \mathbb{E}\sum_{i=1}^{\infty}(G(f_i))^2 \le 2\sum_{i=1}^{\infty}\|f_i\|_{-1}^2 \le 2(\|\mathbf{f}\|_{\mathbb{W}_{-1}^{\mathbb{H}}})^2 < \infty,$$

$$\mathbb{E}\|\mathbf{M}_1^{\rightarrow}(\mathbf{f})\|_{\mathbb{H}}^2 = \mathbb{E}\sum_{i=1}^{\infty}(M_1^{\rightarrow}(f_i))^2 \le 2\sum_{i=1}^{\infty}\|f_i\|_{-1}^2 = 2(\|\mathbf{f}\|_{\mathbb{W}_{-1}^{\mathbb{H}}})^2 < +\infty,$$

$$\mathbb{E}\|\mathbf{M}_1^{\leftarrow}(\mathbf{f})\|_{\mathbb{H}}^2 = \mathbb{E}\sum_{i=1}^{\infty}(M_1^{\leftarrow}(f_i))^2 \le 2\sum_{i=1}^{\infty}\|f_i\|_{-1}^2 = 2(\|\mathbf{f}\|_{\mathbb{W}_{-1}^{\mathbb{H}}})^2 < +\infty.$$

Thus summing the forward-backward martingale decomposition for $S_n(f_i)\mathbf{e}_i$ proved in [17], we get (2.8) for the $\mathbb{H}$-valued $\mathbf{f} \in \mathbb{W}_{-1}^{\mathbb{H}}$.

(a) (Following [17]) For the $\mathbb{H}$-valued martingale $\mathbf{M}_n^{\rightarrow}(\mathbf{f})$, by Doob's maximal inequality

$$\mathbb{E}\max_{k\le n}\|M_k^{\rightarrow}(\mathbf{f})\|_{\mathbb{H}}^2 \le 4\mathbb{E}\|\mathbf{M}_n^{\rightarrow}(\mathbf{f})\|_{\mathbb{H}}^2 \le 4n\mathbb{E}\|\mathbf{M}_1^{\rightarrow}(\mathbf{f})\|_{\mathbb{H}}^2 \le 8n\|\mathbf{f}\|_{\mathbb{W}_{-1}^{\mathbb{H}}}^2,$$

$$\begin{aligned}\mathbb{E}\max_{k\le n}\|M_k^{\leftarrow}(\mathbf{f})\|_{\mathbb{H}}^2 &\le 2\mathbb{E}\|\mathbf{M}_n^{\leftarrow}(\mathbf{f})\|_{\mathbb{H}}^2 + 2\mathbb{E}\max_{k\le n}\|\mathbf{M}_n^{\leftarrow}(\mathbf{f}) - M_{n-k}^{\leftarrow}(\mathbf{f})\|_{\mathbb{H}}^2\\ &\le 2\mathbb{E}\|\mathbf{M}_n^{\leftarrow}(\mathbf{f})\|_{\mathbb{H}}^2 + 8\mathbb{E}\|\mathbf{M}_n^{\leftarrow}(\mathbf{f})\|_{\mathbb{H}}^2 \le 20n\|\mathbf{f}\|_{\mathbb{W}_{-1}^{\mathbb{H}}}^2,\end{aligned}$$

$$\begin{aligned}\mathbb{E}\max_{k\le n}\|\mathbf{Gf}(X_k) - \mathbf{Gf}(X_0)\|_{\mathbb{H}}^2 &\le 2\mathbb{E}\|\mathbf{Gf}(X_0)\|_{\mathbb{H}}^2 + 2\mathbb{E}\max_{k\le N}\|\mathbf{Gf}(X_k)\|_{\mathbb{H}}^2\\ &\le 2\mathbb{E}\|\mathbf{Gf}(X_0\|_{\mathbb{H}}^2 + \mathbb{E}\sum_{k=1}^{n}\|\mathbf{Gf}(X_k)\|_{\mathbb{H}}^2\\ &= 2(n+1)\|\mathbf{Gf}\|_{\mathbb{H}}^2 \le 4(n+1)\||\mathbf{f}\|_{\mathbb{W}_{-1}^{\mathbb{H}}}^2.\end{aligned}$$

By the three estimations above, we get the inequality (2.9).

(b) Since $\mathbf{Gf} \in \mathbf{L}^2(E, \mu; \mathbb{H})$, we deduce by Birkhoff's ergodic theorem,

$$\frac{1}{n}\max_{k\le n}(\|\mathbf{Gf}(X_k) - \mathbf{Gf}(X_0)\|_{\mathbb{H}})^2 \to 0, \quad \text{both in} \quad \mathbf{L}^1(\mathbb{P}_\mu) \quad \text{and} \quad \mathbb{P}_\mu - a.s. \tag{2.13}$$

Applying the classical FCLT to $\mathbf{M}_n^{\rightarrow}(\mathbf{f})$ and $\mathbf{M}_n^{\leftarrow}(\mathbf{f})$, we get immediately the compactness criteria (b).

(c) Assume at first $\mathbf{f} \in \mathbb{D}_{\mathbb{H}}(R_0)$. Then

$$S_n(\mathbf{f}) = \sum_{k=0}^{n-1}(\mathbf{1} - P)R_0\mathbf{f}(X_k) = \mathbf{M}_n(\mathbf{f}) + \Delta_n(\mathbf{f}), \tag{2.14}$$

where

$$\mathbf{M}_n(\mathbf{f}) := \sum_{k=1}^{n}\left(R_0\mathbf{f}(X_k) - PR_0\mathbf{f}(X_{k-1})\right) \tag{2.15}$$

is an additive square integrable $(\mathcal{F}_n)$-martingale satisfying

$$\mathbb{E}\|\mathbf{M}_n(\mathbf{f})\|_{\mathbb{H}}^2 \le 2n\langle R_0\mathbf{f}, \mathbf{f}\rangle_{\mathbb{H}} \le 2n\langle \mathbf{f}, \mathbf{f}\rangle_{\mathbb{W}_{-1}^{\mathbb{H}}} \tag{2.16}$$

by (2.4); and

$$\Delta_n(\mathbf{f}) := R_0\mathbf{f}(X_0) - R_0\mathbf{f}(X_n) \tag{2.17}$$

verifies

$$\frac{1}{n}\max_{k\le n}\|\Delta_n(\mathbf{f})\|_{\mathbb{H}}^2 \longrightarrow 0, \quad \text{both in } \mathbf{L}^1(\mathbb{P}_\mu) \text{ and } \mathbb{P}_\mu - a.s. \tag{2.18}$$

by the classical Birkhoff's ergodic theorem. Hence [2.10)+(2.11)] hold. (**Note:** the argument above is well known, originated from Gordin[4]).

Now for a general $\mathbf{f} \in \mathbb{W}_{-1}^{\mathbb{H}}$ belonging to the closure of $\mathbb{D}_{\mathbb{H}}(R_0) \subset \mathbb{W}_{-1}^{\mathbb{H}}$, we take $\mathbf{f}_\varepsilon \in \mathbb{D}_{\mathbb{H}}(R_0)$ so that $\|\mathbf{f} - \mathbf{f}_\varepsilon\|_{\mathbb{W}_{-1}^{\mathbb{H}}} \to 0$ as $\varepsilon \to 0$. This implies by (2.16) that $\forall\, n$,

$$\mathbf{M}_n(\mathbf{f}_\varepsilon) \longrightarrow \mathbf{M}_n(\mathbf{f}), \quad \text{in } \mathbf{L}^2(\mathbb{P}_\mu), \tag{2.19}$$

as ε goes to zero, and $\mathbb{E}\|\mathbf{M}_1(\mathbf{f}) - \mathbf{M}_1(\mathbf{f}_\varepsilon)\|_{\mathbb{H}}^2 \le 2(\|\mathbf{f} - \mathbf{f}_\varepsilon\|_{\mathbb{W}_{-1}^{\mathbb{H}}})^2$. $\mathbf{M}.(\mathbf{f})$ is obviously additive and satisfies again (2.16). Consequently for each $n \in \mathbb{N}$,

$$\Delta_n(\mathbf{f}_\varepsilon) = S_n(\mathbf{f}_\varepsilon) - \mathbf{M}_n(\mathbf{f}_\varepsilon) \longrightarrow S_n(\mathbf{f}) - \mathbf{M}_n(\mathbf{f}) =: \Delta_n(\mathbf{f}) \quad \text{in } \mathbf{L}^2(\mathbb{P}_\mu). \tag{2.20}$$

It remains to show that $(\Delta_n(\mathbf{f}))$ so defined satisfies the crucial (2.11). For this, by the maximal inequality (2.19),

$$\begin{aligned}
&\mathbb{E}\max_{k\le n}\left(\|\Delta_k(\mathbf{f}) - \Delta_k(\mathbf{f}_\varepsilon)\|_{\mathbb{H}}\right)^2 \\
&= \mathbb{E}\max_{k\le n}\|(S_k(\mathbf{f}) - S_k(\mathbf{f}_\varepsilon)) - (\mathbf{M}_k(\mathbf{f}) - \mathbf{M}_k(\mathbf{f}_\varepsilon))\|_{\mathbb{H}}^2 \\
&\le 2\mathbb{E}\max_{k\le n}\left(\|S_k(\mathbf{f}) - S_k(\mathbf{f}_\varepsilon)\|_{\mathbb{H}}\right)^2 + 2\mathbb{E}\max_{k\le n}\left(\|\mathbf{M}_k(\mathbf{f}) - \mathbf{M}_k(\mathbf{f}_\varepsilon)\|_{\mathbb{H}}\right)^2 \\
&\le 2\,(24n+3)\,\langle \mathbf{f} - \mathbf{f}_\varepsilon, \mathbf{f} - \mathbf{f}_\varepsilon\rangle_{\mathbb{W}_{-1}^{\mathbb{H}}} + 8n\langle \mathbf{f} - \mathbf{f}_\varepsilon, \mathbf{f} - \mathbf{f}_\varepsilon\rangle_{\mathbb{W}_{-1}^{\mathbb{H}}},
\end{aligned}$$

where Doob's maximal inequality and (2.16) are used. It implies (2.16) by (2.18) for $\mathbf{f}_\varepsilon$.

Fix now the initial measure $\nu \ll \mu$. By the known FCLT for $\mathbb{H}$-valued martingales, the law of

$$\Big(\frac{1}{\sqrt{n}}\mathbf{M}_{[nt]}(\mathbf{f})\Big)_{t\in[0,1]}$$

under $\mathbb{P}_\nu$ converges weakly in $\mathbb{D}([0,1],\mathbb{H})$ to the law of an $\mathbb{H}$-valued Brownian Motion $(\mathbf{B}_t)_{t\in[0,1]}$, where the covariance of the limit BM is given by: $\forall \mathbf{h}_1, \mathbf{h}_2 \in \mathbb{H}$,

$$\begin{aligned}
&\mathbb{E}(\langle \mathbf{B}_1, \mathbf{h}_1\rangle_{\mathbb{H}}\langle \mathbf{B}_1, \mathbf{h}_2\rangle_{\mathbb{H}}) = \mathbb{E}(\langle \mathbf{M}_1(\mathbf{f}), \mathbf{h}_1\rangle_{\mathbb{H}}\langle \mathbf{M}_1(\mathbf{f}), \mathbf{h}_2\rangle_{\mathbb{H}}) \\
&= \frac{1}{n}\mathbb{E}(\langle \mathbf{M}_n(\mathbf{f}), \mathbf{h}_1\rangle_{\mathbb{H}}\langle \mathbf{M}_n(\mathbf{f}), \mathbf{h}_2\rangle_{\mathbb{H}}) = \lim_{n\to\infty}\frac{1}{n}\mathbb{E}(\langle S_n(\mathbf{f}), \mathbf{h}_1\rangle_{\mathbb{H}}\langle S_n(\mathbf{f}), \mathbf{h}_2\rangle_{\mathbb{H}}),
\end{aligned}$$

where the last equality follows from [(2.10)+(2.11)].

Moreover since

$$\frac{\max_{1\le k\le n}\|\Delta_k(\mathbf{f})\|_{\mathbb{H}}}{\sqrt{n}} \to 0$$

in probability $\mathbb{P}_\mu$, then in probability $\mathbb{P}_\nu$ (for $\mathbb{P}_\nu \ll \mathbb{P}_\mu$), we see that the FCLT of $S_n(\mathbf{f})$ in part (c) follows from that of $(\mathbf{M}(\mathbf{f}))$ by the martingale decomposition (2.10).

Corollary 2.1. *If* $\mathbf{f} \in \mathbb{W}_{-1}^{\mathbb{H}}$ *and*

$$\liminf_{\varepsilon\to 0}\|\varepsilon R_\varepsilon \mathbf{f}\|_{\mathbb{W}_{-1}^{\mathbb{H}}} < +\infty \quad or \quad \sup_{k\ge 0}\|P^k\mathbf{f}\|_{\mathbb{W}_{-1}^{\mathbb{H}}} < +\infty,$$

then $\mathbf{f} \in \overline{\mathbb{D}_{\mathbb{H}}(R_0)}^{\mathbb{W}_{-1}^{\mathbb{H}}}$. *In particular* $S_n(\mathbf{f})$ *satisfies the FCLT in Theorem* 2.1 (c).

Proof. Let $\mathbf{f}_\varepsilon := \mathbf{f} - \varepsilon R_\varepsilon \mathbf{f} = (I-P)R_\varepsilon \mathbf{f} \in \mathbb{D}_{\mathbb{H}}(R_0)$, where $R_\varepsilon = (\varepsilon + I - P)^{-1}$ is the resolvent. Note that

$$\|\varepsilon R_\varepsilon \mathbf{f}\|_{\mathbb{W}_{-1}^{\mathbb{H}}} = \varepsilon \|\sum_{k=0}^{\infty}(1+\varepsilon)^{-k-1}P^k\mathbf{f}\|_{\mathbb{W}_{-1}^{\mathbb{H}}}$$
$$\le \varepsilon \sum_{k=0}^{\infty}(1+\varepsilon)^{-k-1}\|P^k\mathbf{f}\|_{\mathbb{W}_{-1}^{\mathbb{H}}} \le \sup_{k\ge 0}\|P^k\mathbf{f}\|_{\mathbb{W}_{-1}^{\mathbb{H}}}.$$

Hence it is enough to prove this corollary under

$$\liminf_{\varepsilon\to 0}\|\varepsilon R_\varepsilon \mathbf{f}\|_{\mathbb{W}_{-1}^{\mathbb{H}}} < +\infty.$$

In the last case, we can find $\varepsilon(k) \to 0$ so that $\mathbf{f}_{\varepsilon(k)}$ converges weakly in $\mathbb{W}_{-1}^{\mathbb{H}}$. But as $\varepsilon R_\varepsilon \mathbf{f} \to 0$ in $\mathbb{W}_0^{\mathbb{H}}$, the weak limit of $\mathbf{f}_{\varepsilon(k)}$ in $\mathbb{W}_{-1}^{\mathbb{H}}$ must be $\mathbf{f}$. Finally as the weak closure of $\mathbb{D}_{\mathbb{H}}(R_0)$ in $\mathbb{W}_{-1}^{\mathbb{H}}$ coincides with its strong closure (by Hahn-Banach), then $\mathbf{f}$ satisfies the condition in (c).

Remark 2.1. The results of this section does not depend on the quasi-symmetry assumption (1.5).

Remark 2.2. Up to our knowledge, for a stationary and ergodic $\mathbb{H}$-valued $\mathbf{L}^2$-martingale differences $(\mathbf{m}_k)_{k\ge1}$, the functional law of iterated logarithm for $\left(\mathbf{M}_n := \sum_{k=1}^{n}\mathbf{m}_k\right)$ is unknown, neither for the backward martingale $(\mathbf{M}_n^{\leftarrow})$ here. If one can prove the functional law of iterated logarithm (FLIL in short) for $\mathbf{M}_n^{\rightarrow}(\mathbf{f})$ and $\mathbf{M}_n^{\leftarrow}(\mathbf{f})$ here, we can obtain the FLIL for $S_n(\mathbf{f})$.

§3. Forward-Backward Martingale Decomposition and Invariance Principle: the Continuous Time Case

Let $\mathbb{T} = \mathbb{R}^+$. The situation is more delicate because of the unboundedness of the generator $\mathcal{L}$ of (P_t) in $L^2(E,\mu)$. Our first assumption allows us to define the symmetrization $\mathcal{L}^\sigma$ of the generator $\mathcal{L}$.

(H1) $\mathcal{D}$ is a sub-algebra of $C(E)$ contained in $\mathbb{D}(\mathcal{L})\cap\mathbb{D}(\mathcal{L}^*)$, so that $(\frac{\mathcal{L}+\mathcal{L}^*}{2},\mathcal{D})$ is essentially self-adjoint in $L^2(E,\mu)$. Here $C(E)$ is the space of real continuous functions on E, and $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$ in $L^2(E,\mu)$ (then the generator of (P_t^*) in $L^2(\mu)$ by [5]). Let $\mathcal{L}^\sigma$ be the closure of $(\frac{\mathcal{L}+\mathcal{L}^*}{2},\mathcal{D})$, which is self-adjoint by (H1) and definite nonpositive. Let $(\mathcal{E}^\sigma,\mathbb{D}(\mathcal{E}^\sigma))$ be the symmetric form associated to $-\mathcal{L}^\sigma$. It is the closure of

$$\mathcal{E}^\sigma(u,v) = \frac{1}{2}\left(\langle -\mathcal{L}u,v\rangle + \langle u,-\mathcal{L}v\rangle\right) = \Big\langle -\frac{\mathcal{L}+\mathcal{L}^*}{2}u,v\Big\rangle, \quad \forall u,v\in\mathcal{D}. \tag{3.1}$$

We assume in further

(H2) for any $u\in\mathbb{D}(\mathcal{E}^\sigma)\cap\mathbb{W}_0$, $\mathcal{E}^\sigma(u,u)=0 \implies u=0$, $\mu-$ a.s.

This condition means that $-\mathcal{L}^\sigma : \mathbb{D}(-\mathcal{L}^\sigma)\cap\mathbb{W}_0 \to \mathbb{W}_0$ is injective. Then

$$R_0^\sigma := (-\mathcal{L}^\sigma)^{-1} : Ran(-\mathcal{L}^\sigma) = \mathbb{D}(R_0^\sigma)(\subset \mathbb{W}_0) \longrightarrow \mathbb{D}(-\mathcal{L}^\sigma)\cap\mathbb{W}_0 \tag{3.2}$$

is a well-defined self-adjoint operator on $\mathbb{W}_0$.

Definition 3.1.[17] *$\mathbb{W}_1$ is defined as the completion of the pre-Hilbert space $\mathbb{D}(-\mathcal{L}^\sigma)\cap\mathbb{W}_0$ w.r.t. the inner product $\langle u,v\rangle_1 := \mathcal{E}^\sigma(u,v)$ or the norm $\|u\|_1 := \sqrt{\mathcal{E}^\sigma(u,u)}$. $\mathbb{W}_{-1}$ is defined as the completion of the pre-Hilbert space $\mathbb{D}(R_0^\sigma)$ w.r.t. the inner product $\langle f,g\rangle_{-1} = \langle R_0^\sigma f,g\rangle$ or w.r.t. the norm $\|f\|_{-1} = \sqrt{\langle R_0^\sigma f,f\rangle}$.*

For $\mathbf{f} \in \mathbb{W}_0^{\mathbb{H}} := L_0^2(E,\mu;\mathbb{H})$, we say that $\mathbf{f} \in \mathbb{W}_{-1}^{\mathbb{H}}$, if

$$\|\mathbf{f}\|_{\mathbb{W}_{-1}^{\mathbb{H}}} := \Big(\sum_{i\in I}\|f_i\|_{-1}^2\Big)^{\frac{1}{2}} < +\infty,$$

where $f_i := \langle \mathbf{f}, \mathbf{e}_i\rangle_{\mathbb{H}}$ and $(e_i)_{i\in I}$ is an ONB of $\mathbb{H}$ specified at the beginning of §2.

Given $\mathbf{f} \in \mathbb{W}_0^{\mathbb{H}}$, by the same proof as that of Lemma 2.3, we have that $\mathbf{f} \in \mathbb{W}_{-1}^{\mathbb{H}}$ iff $\exists C \ge 0$,

$$\langle\!\langle \mathbf{f}, \mathbf{g}\rangle\!\rangle \le C\sqrt{\langle A\mathbf{g}, \mathbf{g}\rangle_{\mathbb{H}}}, \quad \forall\, \mathbf{g} \in \mathbb{D}_{\mathbb{H}}(A) \tag{3.3}$$

(recalling that $A = -\mathcal{L}$) and

$$\|\mathbf{f}\|_{\mathbb{W}_{-1}^{\mathbb{H}}} = \inf\{C \ge 0 \mid \quad (3.3) \text{ holds for } C\}.$$

Note a difference from the discrete time case: the space $\mathbb{W}_{-1}^{\mathbb{H}}$, defined as the completion of the pre-Hilbert space $(\{f \in \mathbb{W}_0^{\mathbb{H}};\ \|f\|_{\mathbb{W}_{-1}^{\mathbb{H}}} < +\infty\}, \|\cdot\|_{\mathbb{W}_{-1}^{\mathbb{H}}})$, is in general not contained in $\mathbb{W}_0^{\mathbb{H}}$ (see [17]), unlike Lemma 2.1.

Theorem 3.1. *Assume* **(H1)** *and* **(H2)**. *For any $\mathbf{f} \in \mathbb{W}_0^{\mathbb{H}} \bigcap \mathbb{W}_{-1}^{\mathbb{H}}$, the forward-backward martingale decomposition below holds:*

$$2S_t(\mathbf{f}) = 2\int_0^t \mathbf{f}(X_s)ds = \mathbf{M}_t^{\rightarrow}(\mathbf{f}) + \mathbf{M}_t^{\leftarrow}(\mathbf{f}), \quad \forall t \ge 0, \mathbb{P}-a.s., \tag{3.4}$$

where $\mathbf{M}_\cdot^{\rightarrow}(\mathbf{f})$ and $\mathbf{M}_\cdot^{\leftarrow}(\mathbf{f})$ are additive and càdlàg in time t, linear in $\mathbf{f} \in \mathbb{W}_0^{\mathbb{H}} \cap \mathbb{W}_{-1}^{\mathbb{H}}$, verifying

$$\mathbb{E}^\mu(\|\mathbf{M}_t^{\rightarrow}(\mathbf{f})\|_{\mathbb{H}})^2 = \mathbb{E}(\|\mathbf{M}_t^{\leftarrow}(\mathbf{f})\|_{\mathbb{H}})^2 = 2t\|\mathbf{f}\|_{\mathbb{W}_{-1}^{\mathbb{H}}}^2, \tag{3.5}$$

and $(\mathbf{M}_t^{\rightarrow}(\mathbf{f}))$ is an $\mathbb{H}$-valued $(\mathcal{F}_t)$-martingale, and $(\mathbf{M}_\cdot^{\leftarrow}(\mathbf{f}))$ is an $\mathbb{H}$-valued backward martingale, i.e., for each $T > 0$, $(\mathbf{M}_{T-t}^{\leftarrow}(\mathbf{f}))_{t\in[0,T]}$ is a $(\mathcal{G}_{T-t})_{t\in[0,T]}$-martingale (though it is left continuous).

In particular we have

(a) *the maximal inequality below holds:*

$$\mathbb{E}^\mu \sup_{0\le t\le T}\|S_t(\mathbf{f})\|_{\mathbb{H}}^2 \le 14T(\|\mathbf{f}\|_{\mathbb{W}_{-1}^{\mathbb{H}}})^2,\ \forall T > 0; \tag{3.6}$$

(b) *the family of the laws of $t \longrightarrow \frac{1}{\sqrt{n}}S_{nt}(\mathbf{f}) \in C([0,1],\mathbb{H}),\ n \ge 1$, on the Banach space $C([0,1],\mathbb{H})$ under $\mathbb{P}_\mu$ is precompact;*

(c) *assume moreover that $\mathbf{f}$ belongs to the closure of $\mathbb{W}_{-1}^{\mathbb{H}} \cap \mathbb{D}_{\mathbb{H}}(R_0)$ in $\mathbb{W}_{-1}^{\mathbb{H}}$, then there are an additive square integrable càdlàg martingale $(\mathbf{M}_t(\mathbf{f}))$ and an additive càdlàg functional $(\Delta_t(\mathbf{f}))$ such that*

$$S_t(\mathbf{f}) = \mathbf{M}_t(\mathbf{f}) + \Delta_t(\mathbf{f}), \quad \forall t \ge 0, \quad \mathbb{P}_\mu - a.s. \tag{3.7}$$

$$\frac{1}{t}\mathbb{E}^\mu \sup_{s\le t}|\Delta_s(\mathbf{f})|^2 \to 0 \quad (as\ t \to +\infty). \tag{3.8}$$

In particular, for any initial measure $\nu \ll \mu$, as T goes to infinity, the law of $\big(\frac{1}{\sqrt{T}}S_{Tt}(\mathbf{f})\big)_{t\in[0,1]}$ under $\mathbb{P}_\nu$ converges weakly in $C([0,1],\mathbb{H})$ to the law of an $\mathbb{H}$-valued BM $(\mathbf{B}_t)$, where the covariance of $\mathbf{B}_1$ is given by

$$\mathbb{E}(\langle \mathbf{B}_1, \mathbf{h}_1\rangle_{\mathbb{H}}\langle \mathbf{B}_1, \mathbf{h}_2\rangle_{\mathbb{H}}) = \langle \Gamma\mathbf{h}_1, \mathbf{h}_2\rangle_{\mathbb{H}}, \quad \forall\ \mathbf{h}_1, \mathbf{h}_2 \in \mathbb{H},$$

with

$$\begin{aligned}\langle \Gamma\mathbf{h}_1, \mathbf{h}_2\rangle_{\mathbb{H}} &:= \mathbb{E}(\langle \mathbf{M}_1(\mathbf{f}), \mathbf{h}_1\rangle_{\mathbb{H}}\langle \mathbf{M}_1(\mathbf{f}), \mathbf{h}_2\rangle_{\mathbb{H}}) \\ &= \lim_{T\to\infty}\frac{1}{T}\mathbb{E}\langle S_T(\mathbf{f}), \mathbf{h}_1\rangle_{\mathbb{H}}\langle S_t(\mathbf{f}), \mathbf{h}_2\rangle_{\mathbb{H}}.\end{aligned} \tag{3.9}$$

Here $C([0,1],\mathbb{H})$ is the Banach space of all continuous $\mathbb{H}$-valued functions γ on $[0,1]$, equipped with norm $\|\gamma\|_{\sup} := \sup\limits_{t\in[0,1]} \|\gamma(t)\|_{\mathbb{H}}$.

Proof. We prove only the forward-backward martingale decomposition and only in the case where $\dim\mathbb{H} = \infty$. The remaining parts can be established similarly as in Theorem 2.4.

For real $f \in \mathbb{W}_{-1}$, by [17] (Theorem 3.3) the forward-backward martingale decomposition below holds:

$$2S_t(f) = M_t^{\rightarrow}(f) + M_t^{\leftarrow}(f),$$

where $M_t^{\rightarrow}(f)$ (resp. $M_t^{\leftarrow}(f)$) is a forward (resp. backward) additive martingale verifying $\mathbb{E}(M_t^{\rightarrow}(f))^2 = \mathbb{E}(M_t^{\leftarrow}(f))^2 = 2t\langle f,f\rangle_{-1}$.

Now for $\mathbf{f} \in \mathbb{W}_0^{\mathbb{H}} \bigcap \mathbb{W}_{-1}^{\mathbb{H}}$, let $\mathbf{M}_t^{\rightarrow n}(\mathbf{f}) := \sum\limits_{i=1}^{n} M_t^{\rightarrow}(f_i)\mathbf{e}_i$. By Doob's inequality we have for all $m < n$,

$$\mathbb{E}\Big(\sup_{t\le T} \|\mathbf{M}_t^{\rightarrow m}(\mathbf{f}) - \mathbf{M}_t^{\rightarrow n}(\mathbf{f})\|_{\mathbb{H}}^2\Big) \le 4\mathbb{E}\|\mathbf{M}_T^{\rightarrow m}(\mathbf{f}) - \mathbf{M}_T^{\rightarrow n}(\mathbf{f})\|_{\mathbb{H}}^2$$
$$\le 4\sum_{k=m+1}^{n} \mathbb{E}(M_T^{\rightarrow}(f_k))^2 = 8T\sum_{k=m+1}^{n} \|f_k\|_{-1}^2 \to 0,$$

as m,n go to infinity. Therefore by the Cauchy criterion and a triangular argument, there exists an $\mathbb{H}$-valued càdlàg forward martingale $\mathbf{M}_t^{\rightarrow}(\mathbf{f})$ in $\mathbf{L}^2(E,\mu;\mathbb{H})$ such that

$$\sup_{t\le T} \|\mathbf{M}_t^{\rightarrow n}(\mathbf{f}) - \mathbf{M}_t^{\rightarrow}(\mathbf{f})\|_{\mathbb{H}} \longrightarrow 0 \quad (n\to\infty),\ \forall T>0.$$

For the same reason there exists an $\mathbb{H}$-valued càdlàg backward martingale $(\mathbf{M}_t^{\leftarrow}(\mathbf{f}))$ such that

$$\sup_{t\le T} \|\mathbf{M}_t^{\leftarrow n}(\mathbf{f}) - \mathbf{M}_t^{\leftarrow}(\mathbf{f})\|_{\mathbb{H}} \longrightarrow 0\ (n\to\infty),\ \forall T>0.$$

Thus, for each $t\ge 0$ we have $\mathbb{P}_\mu$-a.s.,

$$\mathbf{M}_t^{\rightarrow}(\mathbf{f}) = \sum_{i=1}^{\infty} M_t^{\rightarrow}(f_i)\mathbf{e}_i, \quad \mathbf{M}_t^{\leftarrow}(\mathbf{f}) = \sum_{i=1}^{\infty} M_t^{\leftarrow}(f_i)\mathbf{e}_i,$$

which satisfy

$$\mathbb{E}\|\mathbf{M}_t^{\rightarrow}(\mathbf{f})\|_{\mathbb{H}}^2 = \mathbb{E}\sum_{i=1}^{+\infty}(M_t(f_i))^2 = 2t\sum_{i=1}^{\infty}\|f_i\|_{-1}^2 < +\infty,$$
$$\mathbb{E}\|\mathbf{M}_t^{\rightarrow}(\mathbf{f})\|_{\mathbb{H}}^2 = \mathbb{E}\sum_{i=1}^{\infty}(M_t(f_i))^2 = 2t\sum_{i=1}^{+\infty}\|f_i\|_{-1}^2 < +\infty.$$

Moreover, for each $t\in\mathbb{R}^+$, we have $\mathbb{P}_\mu$-a.s.,

$$2S_t(\mathbf{f}) = \sum_{i=1}^{+\infty} 2S_t(f_i)\mathbf{e}_i = \sum_{i=1}^{+\infty}(M_t^{\rightarrow}(f_i) + M_t^{\leftarrow}(f_i))\mathbf{e}_i = \mathbf{M}_t^{\rightarrow}(\mathbf{f}) + \mathbf{M}_t^{\leftarrow}(\mathbf{f}).$$

By the right continuity of $t\to S_t(f)$, $\mathbf{M}_t^{\rightarrow}(\mathbf{f}) + \mathbf{M}_t^{\leftarrow}(\mathbf{f})$, we get with $\mathbb{P}_\mu$-probability one, $2S_t(\mathbf{f}) = \mathbf{M}_t^{\rightarrow}(\mathbf{f}) + \mathbf{M}_t^{\leftarrow}(\mathbf{f}), \forall t\in\mathbb{R}^+$.

With the same proof as that of Corollary 2.5, we get

Corollary 3.1. *If $\liminf\limits_{\varepsilon\to 0} \varepsilon\left\|\int_0^\infty e^{-\varepsilon t}P_t f\right\|_{\mathbb{W}_{-1}^{\mathbb{H}}} < +\infty$, then $\mathbf{f}$ belongs to the closure of $\mathbb{W}_{-1}^{\mathbb{H}} \cap \mathbb{D}_{\mathbb{H}}(R_0)$ in $\mathbb{W}_{-1}^{\mathbb{H}}$. And all conclusions in Theorem* 3.1 (c) *hold.*

§4. Quasi-Symmetric Case

Throughout this section, the Markov process $(X_t)_{t\in\mathbf{T}}$ is quasi-symmetric, i.e., it satisfies (1.5). In the continuous time case, we shall assume that our process (X_t) is Hunt (see [10]). The main result is Theorem 4.1 below, which is stated in [17] in the real valued case without detailed proof.

Let $A = (I-P)$ or $A = -\mathcal{L}$ according to $\mathbf{T} = \mathbb{N}$ or $= \mathbb{R}^+$, and $R_0 := A^{-1}$. Applying (1.4) to $u = R_0 f, v = R_0 g$, we see that R_0 is still sectorial. Hence the bilinear form

$$\mathcal{E}_{-1}(f,g) := \langle R_0 f, g\rangle, \quad \forall f,g \in \mathbb{D}(R_0) \tag{4.1}$$

is closable, and its closure will be denoted by $(\mathcal{E}_{-1}, \mathbb{D}(\mathcal{E}_{-1}))$. By [5], $\mathbb{D}(R_0^*) \subset \mathbb{D}(\mathcal{E}_{-1})$ is also a form core of $(\mathcal{E}_{-1}, \mathbb{D}(\mathcal{E}_{-1}))$.

Lemma 4.1. *If*

$$\liminf_{\varepsilon\to 0} \langle f, R_\varepsilon f\rangle < +\infty, \tag{4.2}$$

then

$$f \in \mathbb{D}(\mathcal{E}_{-1}), \quad \textit{and} \quad \mathcal{E}_{-1}(f,f) \le \liminf_{\varepsilon\to 0}\langle f, R_\varepsilon f\rangle. \tag{4.3}$$

Proof. In fact, take $f_\varepsilon = f - \varepsilon R_\varepsilon f = AR_\varepsilon f$, $R_0 f_\varepsilon = R_\varepsilon f$ and then

$$\liminf_{\varepsilon\to 0} \mathcal{E}_{-1}(f_\varepsilon, f_\varepsilon) = \liminf_{\varepsilon\to 0}\langle f - \varepsilon R_\varepsilon f, R_\varepsilon f\rangle \le \liminf_{\varepsilon\to 0}\langle f, R_\varepsilon f\rangle < +\infty.$$

By [5] (Chapter VI, Theorwm 1.15, p. 314), it follows that

$$f \in \mathbb{D}(\mathcal{E}_{-1}), \quad \mathcal{E}_{-1}(f,f) \le \liminf_{\varepsilon\to 0}\langle f, R_\varepsilon f\rangle.$$

Lemma 4.2. *For any $f \in \mathbb{D}(R_0)$, we have $f \in \mathbb{W}_{-1}$ and*

$$\sqrt{\mathcal{E}_{-1}(f,f)} \le \|f\|_{-1} \le K\sqrt{\mathcal{E}_{-1}(f,f)}. \tag{4.4}$$

Proof. The left inequality in (4.4) follows from

$$(\|R_0 f\|_1)^2 = \langle R_0 f, AR_0 f\rangle = \langle R_0 f, f\rangle \le \|R_0 f\|_1 \cdot \|f\|_{-1}$$

and the fact that $(\|R_0 f\|_1)^2 = \mathcal{E}_{-1}(f,f)$. For the right inequality in (4.4), let $g = R_0 f$. We have for any $u \in \mathbb{D}(\mathcal{L})$,

$$\langle f, u\rangle = \langle Ag, u\rangle \le K\sqrt{\langle Ag, g\rangle}\cdot\sqrt{\langle Au, u\rangle} = K\sqrt{\mathcal{E}_{-1}(f,f)}\cdot\|u\|_1.$$

Thus $\|f\|_{-1} \le K\sqrt{\mathcal{E}_{-1}(f,f)}$, the desired right side inequality in (4.4).

Lemma 4.3. $\mathbb{D}(R_0) \subset \mathbb{W}_{-1}$ *and is dense in* $(\mathbb{W}_{-1}, \|\cdot\|_{-1})$.

Proof. By Lemma 4.1, for any f satisfying (4.2), $f \in \mathbb{D}(\mathcal{E}_{-1})$. As $\mathbb{D}(R_0)$ is a form core of $\mathcal{E}_{-1}$, we can find $f_k \in \mathbb{D}(R_0), k \ge 1$ so that $\langle f_k - f, f_k - f\rangle + \mathcal{E}_{-1}(f_k - f, f_k - f) \to 0$. By Lemma 4.2, we have

$$\|f_k - f_l\| + \|f_k - f_l\|_{-1} \longrightarrow 0$$

as $k, l \to +\infty$. Consequently $f \in \mathbb{W}_{-1}$ and $f_k \to f$ in $\mathbb{W}_{-1}$. Then $\mathbb{D}(R_0)$ is dense in $\mathbb{W}_{-1}$. Since $\mathbb{D}(R_0)$ is dense both in $\mathbb{W}_{-1}$ and $\mathbb{D}(\mathcal{E}_{-1})$, we have $\mathbb{D}(\mathcal{E}_{-1}) = \mathbb{W}_{-1}$.

Lemma 4.4. *For $\mathbf{f} \in \mathbf{L}^2_0(E,\mu;\mathbb{H})$, if $\liminf\limits_{\varepsilon\to 0}\langle\langle \mathbf{f}, R_\varepsilon \mathbf{f}\rangle\rangle < +\infty$, then*

$$\|\mathbf{f}\|^2_{\mathbb{W}^{\mathbb{H}}_{-1}} \le K^2 \liminf_{\varepsilon\to 0}\langle\langle \mathbf{f}, R_\varepsilon \mathbf{f}\rangle\rangle.$$

Proof. Write $\mathbf{f} = \sum\limits_{i\in I} f_i \mathbf{e}_i,\ f_i = \langle \mathbf{f}, \mathbf{e}_i\rangle_{\mathbb{H}}$. By Fatou's lemma,

$$\sum_{i\in I}\liminf_{\varepsilon\to 0}\langle f_i, R_\varepsilon f_i\rangle \le \liminf_{\varepsilon\to 0}\langle\langle \mathbf{f}, R_\varepsilon \mathbf{f}\rangle\rangle < +\infty.$$

Then $f_i \in \mathbb{D}(\mathcal{E}_{-1})$ by Lemma 4.1. We have by Lemmas 4.2 and 4.1 that for each $N \geq 1$,

$$\begin{aligned}\|\mathbf{f}\|^2_{\mathbb{W}^{\mathbb{H}}_{-1}} &= \sum_{i\in I}\langle f_i, f_i\rangle_{-1} \leq K^2 \sum_{i\in I}\mathcal{E}_{-1}(f_i, f_i)\\ &\leq K^2 \sum_{i\in I}\liminf_{\varepsilon\to 0}\langle f_i, R_\varepsilon f_i\rangle \leq K^2 \liminf_{\varepsilon\to 0}\langle\langle \mathbf{f}, R_\varepsilon \mathbf{f}\rangle\rangle.\end{aligned}$$

Lemma 4.5. *$\mathbb{D}_{\mathbb{H}}(R_0) \subset \mathbb{W}^{\mathbb{H}}_{-1}$ and $\mathbb{D}_{\mathbb{H}}(R_0)$ is dense in $\mathbb{W}^{\mathbb{H}}_0 \bigcap \mathbb{W}^{\mathbb{H}}_{-1}$.*

Proof. (1) If $\mathbf{f} \in \mathbb{D}_{\mathbb{H}}(R_0)$, then there exists $\mathbf{g} \in L^2_0(E, \mu; \mathbb{H})$, such that $A\mathbf{g} = \mathbf{f}$. Then

$$\begin{aligned}\langle\langle \mathbf{f}, R_\varepsilon \mathbf{f}\rangle\rangle &= \langle\langle \mathbf{f}, R_\varepsilon(-\mathcal{L})\mathbf{g}\rangle\rangle = \langle\langle \mathbf{f}, (\mathbf{1} - \varepsilon R_\varepsilon)\mathbf{g}\rangle\rangle\\ &\leq \|\mathbf{f}\|(\|\mathbf{g}\| + \|\varepsilon R_\varepsilon \mathbf{g}\|) \leq 2\|\mathbf{f}\| \cdot \|\mathbf{g}\|.\end{aligned}$$

By Lemma 4.4, $\mathbf{f} \in \mathbb{W}^{\mathbb{H}}_{-1}$.

(2) Given $\mathbf{f} \in \mathbb{W}^{\mathbb{H}}_0 \bigcap \mathbb{W}^{\mathbb{H}}_{-1}$, let $\mathbf{f}^N := \sum\limits_{i\leq N,\ i\in I} f_i \mathbf{e}_i$. Then $\|\mathbf{f}^N - \mathbf{f}\|_{\mathbb{W}^{\mathbb{H}}_{-1}} \to 0$, as $N \to \infty$. By Lemma 4.3, $\forall \varepsilon > 0, \exists\ f_{i,\varepsilon} \in \mathbb{D}(R_0)$ such that

$$\|f_i - f_{i,\varepsilon}\|_{-1} < \frac{\varepsilon}{2^{i+1}}.$$

Choose $\mathbf{f}_{\varepsilon,N} = \sum\limits_{i=1}^{N} f_{i,\varepsilon}\mathbf{e}_i \in \mathbb{D}_{\mathbb{H}}(R_0)$. We have

$$\|\mathbf{f}^N - \mathbf{f}_{\varepsilon,N}\|^2_{\mathbb{W}^{\mathbb{H}}_{-1}} = \sum_{i=1}^{N}\|f_i - f_{i,\varepsilon}\|^2_{-1} < \varepsilon^2.$$

Then

$$\|\mathbf{f} - \mathbf{f}_{\varepsilon,N}\|_{\mathbb{W}^{\mathbb{H}}_{-1}} \leq \|\mathbf{f} - \mathbf{f}^N\|_{\mathbb{W}^{\mathbb{H}}_{-1}} + \|\mathbf{f}^N - \mathbf{f}_{\varepsilon,N}\|_{\mathbb{W}^{\mathbb{H}}_{-1}} \leq \|\mathbf{f} - \mathbf{f}^N\|_{\mathbb{W}^{\mathbb{H}}_{-1}} + \varepsilon^2,$$

which is arbitrarily small for N large enough and ε small enough.

Theorem 4.1. *Let $\mathbb{T} = \mathbb{N}$ or $\mathbb{R}^+$, and assume* (1.5). *For each $\mathbf{f} \in \mathbf{L}^2_0(E, \mu; \mathbb{H})$, the following properties are equivalent:*

(a) $\mathbf{f} \in \mathbb{W}^{\mathbb{H}}_{-1}$;

(b) $\lim\limits_{t\to\infty} \frac{1}{t}\mathbb{E}\|S_t(\mathbf{f})\|^2_{\mathbb{H}} \in [0, \infty)$ *exists;*

(c) $\sigma^2_{\mathbb{H}}(\mathbf{f}) := \limsup\limits_{t\to\infty} \frac{1}{t}\mathbb{E}\|S_t(\mathbf{f})\|^2_{\mathbb{H}} < \infty$;

(d) $\liminf\limits_{\varepsilon\to\infty}\langle\langle \mathbf{f}, R_\varepsilon \mathbf{f}\rangle\rangle < \infty$, *where $R_\varepsilon = (\varepsilon + A)^{-1}, A = I - P$ or $-\mathcal{L}$.*

In that case, $\sigma^2_{\mathbb{H}}(\mathbf{f}) := \lim\limits_{t\to\infty} \frac{1}{t}\mathbb{E}\|S_t(\mathbf{f})\|^2_{\mathbb{H}} = 0 \Leftrightarrow \mathbf{f} = \mathbf{0}$; and all the conclusions in Theorem 2.1 *and Theorem* 3.1 *hold* (*without assumption* (**H1**) *and* (**H2**) *in the continuous time case*).

Proof. (a)$\Rightarrow$[(b)+ all conclusions in Theorems 2.1 and 3.1]. Let $\mathbf{f} \in \mathbb{W}^{\mathbb{H}}_{-1}$. If $\mathbb{T} = \mathbb{N}$, Theorem 2.1 holds since

$$\overline{\mathbb{D}_{\mathbb{H}}(R_0)}^{\mathbb{W}^{\mathbb{H}}_{-1}} = \mathbb{W}^{\mathbb{H}}_{-1}$$

by Lemma 4.5. Thus all the conclusions in Theorem 2.1 are valid.

If $\mathbb{T} = \mathbb{R}^+$, forward-backward martingale decomposition for real $f \in \mathbb{W}_{-1}$ is established in [17, Theorem 3.3]. For $\mathbb{H}$-valued $\mathbf{f} \in \mathbb{W}^{\mathbb{H}}_{-1}$, by following the same argument as in the proof of Theorem 3.1, the forward-backward martingale decomposition in Theorem 3.1 holds for $S_t(\mathbf{f})$. Thus as consequences of that decomposition, the parts (a), (b), (c) in Theorem 3.1 all hold. By Lemma 4.5, the condition in Theorem 3.1 (c) is verified. Thus all conclusions in Theorem 3.1 are true.

By part (c) of Theorem 2.1 or Theorem 3.1, we have

$$S_t(\mathbf{f}) = M_t(\mathbf{f}) + \Delta_t(\mathbf{f}),\ \forall t \in \mathbb{T},\ \lim_{t\to\infty} \frac{1}{t}\mathbb{E}\sup_{s\le t} \|\Delta_s(\mathbf{f})\|_{\mathbb{H}}^2 = 0,$$

then

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\|S_t(\mathbf{f})\|_{\mathbb{H}}^2 = \lim_{t\to\infty} \frac{1}{t}\mathbb{E}\|\mathbf{M}_t(\mathbf{f})\|_{\mathbb{H}}^2 = \mathbb{E}\|\mathbf{M}_1(\mathbf{f})\|_{\mathbb{H}}^2 < \infty, \tag{4.5}$$

which is part (b).

(b)⇒(c) is trivial.

(c)⇒(d) We only prove it in the continuous time case.

$$\begin{aligned}\mathbb{E}\|S_t(\mathbf{f})\|_{\mathbb{H}}^2 &= 2\mathbb{E}\int_0^t\int_0^v \langle \mathbf{f}(X_u), \mathbf{f}(X_v)\rangle_{\mathbb{H}} du dv \\ &= 2\int_0^t\int_0^v \mathbb{E}\langle \mathbf{f}(X_u), P_{v-u}\mathbf{f}(X_u)\rangle_{\mathbb{H}} du dv = 2\int_0^t\int_0^v \langle\langle \mathbf{f}, P_s\mathbf{f}\rangle\rangle ds dv.\end{aligned}$$

Thus $\limsup_{t\to\infty} \frac{1}{t}\mathbb{E}\|S_t(\mathbf{f})\|_{\mathbb{H}}^2$ is the lim sup in the Césaro sense of $\int_0^v \langle\langle \mathbf{f}, P_s\mathbf{f}\rangle\rangle ds$, which is greater than the lim sup in the Abel sense below

$$\limsup_{\varepsilon\to 0}\int_0^\infty e^{-\varepsilon t}\langle \mathbf{f}, P_t\mathbf{f}\rangle dt = \limsup_{\varepsilon\to 0}\langle\langle \mathbf{f}, R_\varepsilon \mathbf{f}\rangle\rangle.$$

(d)⇒(a) It holds by Lemma 4.4.

It remains to show that if $\sigma_{\mathbb{H}}^2(\mathbf{f}) = 0$, then $\mathbf{f} = 0$. By (c) $\Longrightarrow$ (d) above, $\limsup_{\varepsilon\to 0}\langle\langle \mathbf{f}, R_\varepsilon \mathbf{f}\rangle\rangle = 0$. By Lemma 4.4, $\|f\|_{\mathbb{W}_{-1}^{\mathbb{H}}} = 0$. Thus $\mathbf{f} = 0$.

§5. Proof of Theorem 1.1

Proof of Theorem 1.1. We only prove Theorem 1.1 in the discrete time case. In this section the Hilbert space is $\mathbb{H} := L^2(\mathbb{R}, dF)$.

(a) **Step 1** Recall that the distribution function $F(u) := \mu(f \le u) = \mathbb{P}_\mu(f \le u)$ is assumed to be continuous. Then $F(f(X_k))$ is a uniform random variable on [0,1]. Let

$$F^{-1}(r) = \inf\{u | F(u) > r\}.$$

Then

$$[f(X_k) \le u] = [f(X_k) \le F^{-1}(v)] = [F(f)(X_k) \le v], \quad u = F^{-1}(v).$$

Let $h_u(x) := \mathbf{1}_{[f(x)\le u]} - F(u)$. We have (by following [9])

$$\begin{aligned}&\int_{-\infty}^{+\infty} \mathbb{E}h_u(X_0)h_u(X_k))dF(u) \\ &= \int_0^1 \{\mathbb{E}\mathbf{1}_{[F(f(X_0))\le v]}\mathbf{1}_{[F(f(X_k))\le v]}\}dv - \int_0^1 \{\mathbb{E}\mathbf{1}_{[F(f(X_0))\le v]}\mathbb{E}\mathbf{1}_{[F(f(X_k))\le v]}\}dv\ (u = F^{-1}(v)) \\ &= \int_0^1 \{\mathbb{E}\mathbf{1}_{[\max(F(f(X_0)),F(f(X_k)))\le v]} - v^2\}dv = \int_0^1 P[\max(F(f(X_0)), F(f(X_k))) \\ &\le v]dv - \frac{1}{3} = \frac{2}{3} - \mathbb{E}\max(F(f)(X_0), F(f)(X_k)).\end{aligned}$$

Let $h(x) := (h_u(x))_{u\in\mathbb{R}} \in L^2(\mathbb{R}, dF(u))$ and $(R_\varepsilon h)_u = R_\varepsilon h_u$. We then have

$$\langle R_\varepsilon h, h\rangle_{L^2(E,\mu,\mathbb{H})} = \int_{-\infty}^{+\infty} \langle R_\varepsilon h_u, h_u\rangle dF(u) = \sum_{k=0}^{+\infty} (\varepsilon+1)^{-k-1} \int_{-\infty}^{+\infty} \mathbb{E} h_u(X_0) h_u(X_k) dF(u)$$
$$= \sum_{k=0}^{\infty} (\varepsilon+1)^{-k-1} \Big(\frac{2}{3} - \mathbb{E}\max(F(f)(X_0), F(f)(X_k))\Big),$$

and it follows that $h \in \mathbb{W}_{-1}^{\mathbb{H}}$ by Theorem 4.1 and by our condition. Thus by part (c) of Theorem 2.1,

$$S_n(h) = M_n(h) + \Delta_n(h),\ \mathbb{E}\frac{\max\limits_{k\le n} \|\Delta_k(h)\|_{\mathbb{H}}^2}{n} \to 0, \tag{5.1}$$

where $(M_n(h))$ is an $\mathbb{H}$-valued additive L^2-martingale.

Step 2. Recalling that $\Gamma_n(u,v) := \mathrm{Cov}(\xi_n(u), \xi_n(v))$, then $\forall f, g \in L^2(\mathbb{R}, dF(u))$, we have by (5.1),

$$\lim_{n\to\infty} \iint_{\mathbb{R}\times\mathbb{R}} \Gamma_n(u,v) f(u) g(v) dF(u) dF(v) = \lim_{n\to\infty} \frac{1}{n} \mathbb{E} S_n(\langle h, f\rangle_{\mathbb{H}}) S_n(\langle h, g\rangle_{\mathbb{H}})$$
$$= \lim_{n\to\infty} \frac{1}{n} \mathbb{E}\langle M_n(h), f\rangle_{\mathbb{H}} \cdot \langle M_n(h), g\rangle_{\mathbb{H}} = \mathbb{E}\langle M_1(h), f\rangle_{\mathbb{H}} \cdot \langle M_1(h), g\rangle_{\mathbb{H}},$$

$$+\infty > \mathbb{E}\|M_1(h)\|_{\mathbb{H}}^2 = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}\|S_n(h)\|_{\mathbb{H}}^2$$
$$= \lim_{n\to\infty} \frac{1}{n} \mathbb{E} \int_{\mathbb{R}} (S_n(h_u))^2 dF(u) = \lim_{n\to\infty} \int_{\mathbb{R}} \Gamma_n(u,u) dF(u).$$

Hence by Cauchy-Schwartz's inequality,

$$\sup_{n\ge 1} \iint_{\mathbb{R}\times\mathbb{R}} (\Gamma_n(u,v))^2 dF(u) dF(v) \le \sup_{n\ge 1} \iint_{\mathbb{R}\times\mathbb{R}} \Gamma_n(u,u) \Gamma_n(v,v) dF(u) dF(v)$$
$$= \sup_{n\ge 1} \left[\int_{\mathbb{R}} \Gamma_n(u,u) dF(u)\right]^2 < \infty.$$

Hence $\{\Gamma_n(u,v)\}$ is relatively compact w.r.t. the weak topology $\sigma\left(L^2(\mathbb{R}, dF), L^2(\mathbb{R}, dF)\right)$ and any weak limit Γ of Γ_n must verify

$$\iint_{\mathbb{R}\times\mathbb{R}} \Gamma(u,v) f(u) g(v) dF(u) dF(v) = \lim_{n\to 0} \iint \Gamma_n(u,v) f(u) g(v) dF(u) dF(v),$$

for all $f, g \in \mathbb{H}$, where the existence of the last limit is shown above. Thus Γ is unique. In other words, $\Gamma_n \to \Gamma$ weakly in $L^2(\mathbb{R}, dF)$.

We now see why Γ is of trace on $\mathbb{H}$. Let $(\mathrm{e}_i)_{i\ge 1}$ denote an ONB of $\mathbb{H} = L^2(\mathbb{R}, dF(u))$ and $h_i := \langle h, e_i\rangle_{L^2(\mathbb{R},dF)}$. We have

$$\langle \Gamma e_i, e_i\rangle_{L^2(\mathbb{R},dF)} = \iint_{\mathbb{R}\times\mathbb{R}} \Gamma(u,v) e_i(u) e_i(v) dF(u) dF(v)$$
$$= \lim_{n\to+\infty} \iint_{\mathbb{R}\times\mathbb{R}} \Gamma_n(u,v) e_i(u) e_i(v) dF(u) dF(v)$$
$$= \lim_{n\to+\infty} \frac{1}{n} \mathbb{E} S_n(h_i) S_n(h_i) = \mathbb{E} M_1(h_i) M_1(h_i),$$

$$\sum_{i=1}^{\infty} \langle \Gamma e_i, e_i\rangle_{L^2(\mathbb{R},dF)} = \sum_{i=1}^{+\infty} \mathbb{E} M_1(h_i) M_1(h_i) = \mathbb{E}\|M_1(h)\|_{L^2(\mathbb{R},dF)}^2 < \infty.$$

(b) It follows directly by Theorem 4.1.

Remark. In the discrete time case, let

$$\eta_n(t,u) := \frac{1}{\sqrt{n}} \sum_{k=0}^{[nt]} \left(1_{[f(X_k)\le u]} - F(u)\right)$$

and regard $\eta_n := (t \to (\eta_n(t,u))_{u\in\mathbb{R}})_{t\in[0,1]}$ as a random element in $\mathbb{D}([0,1], L^2(\mathbb{R}, dF))$. Then Theorem 4.1 is applicable (by the proof of part (a) above) and yields the following invariance principle:

for any initial measure $\nu \ll \mu$, $\mathbb{P}_\nu(\eta_n \in \cdot)$ converges weakly on $\mathbb{D}([0,1], L^2(\mathbb{R}, dF))$ to the law of an $L^2(\mathbb{R}, dF)$-valued Brownian Motion with

$$\mathbb{E}\left(\langle B_1, f\rangle_{L^2(\mathbb{R},dF)} \langle B_1, f\rangle_{L^2(\mathbb{R},dF)}\right) = \iint_{\mathbb{R}^2} \Gamma(u,v) f(u) g(v) dF(u) dF(v).$$

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