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DIFFUSIVE-DISPERSIVE TRAVELING WAVES AND KINETIC RELATIONS IV. COMPRESSIBLE EULER EQUATIONS

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Abstract

The authors consider the Euler equations for a compressible fluid in one space dimension when the equation of state of the fluid does not fulfill standard convexity assumptions and viscosity and capillarity effects are taken into account. A typical example of nonconvex constitutive equation for fluids is Van der Waals' equation. The first order terms of these partial differential equations form a nonlinear system of mixed (hyperbolic-elliptic) type. For a class of nonconvex equations of state, an existence theorem of traveling waves solutions with arbitrary large amplitude is established here. The authors distinguish between classical (compressive) and nonclassical (undercompressive) traveling waves. The latter do not fulfill Lax shock inequalities, and are characterized by the so-called kinetic relation, whose properties are investigated in this paper.

Keywords Elastodynamics, Phase transitions, Hyperbolic conservation law, Diffusion, Dispersion, Shock wave, Undercompressive, Entropy inequality, Kinetic relation

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§1. Introduction

This is the fourth paper of a series by the authors (see [3–5]) devoted to traveling wave solutions of diffusive-dispersive models arising in continuum physics. Our previous works were restricted to single equations and systems of two equations. We attempt here to generalize the techniques and results to the (rather challenging) Euler equations for compressible fluids in one-space dimension. We consider the following three conservation laws (mass,

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momentum, total energy) in Lagrangian coordinates:

$$
\begin{aligned}
&v_t - u_x = 0,\\
&u_t - \varepsilon_v(v,S)_x = \alpha\,\big(\nu\,u_x\big)_x - \beta\,(\mu\,v_x)_{xx} + \frac{\beta}{2}\,\big(\mu_v\,v_x^2\big)_x,\\
&E_t - \big(\varepsilon_v(v,S)\,u\big)_x = \alpha\,\big(\nu\,u\,u_x\big)_x + \beta\,\Big(\frac{\mu_v}{2}u\,v_x^2 - u\,(\mu\,v_x)_x\Big)_x + \beta\,\big(\mu\,u_x\,v_x\big)_x.
\end{aligned}
\tag{1.1}
$$

Note here that Lagrangian coordinates are chosen for simplicity in the presentation only, and all the results in this paper extend straightforwardly to the formulation in Eulerian coordinates. The main unknowns are the specific volume $v > 0$, the fluid velocity u, and the specific entropy $S > 0$. The total energy E is given by

$$
E = \varepsilon(v,S) + \frac{u^2}{2}. \tag{1.2}
$$

The coefficients $\nu = \nu(v,S)$ and $\mu = \mu(v,S)$ are non-negative functions of the specific volume and the specific entropy, representing the viscosity and capillarity coefficients of the fluid, respectively. The non-negative parameters α and β serve to measure the "strength" of the viscosity and capillarity terms. To close the system of the equations (1.1), we must prescribe the internal energy $\varepsilon = \varepsilon(v,S)$. We will be interested in the situation where the first-order terms in (1.1), that is,

$$
\begin{aligned}
&v_t - u_x = 0,\\
&u_t - \varepsilon_v(v,S)_x = 0,\\
&E_t - \big(\varepsilon_v(v,S)\,u\big)_x = 0,
\end{aligned}
\tag{1.3}
$$

form an hyperbolic-elliptic system. The typical example of interest in this paper is the (nonconvex) equation of state for van der Waals' fluids

$$
\varepsilon(v,S) = \frac{8\,a}{3}\,(3\,v-1)^{-1/a}\,e^{3\,S/(8a)} - \frac{3}{v}, \tag{1.4}
$$

where a is some positive parameter. (Note that (1.4) requires $v > 1/3$.)

In this paper we will study the existence of traveling wave solutions of the system (1.1) for a class of nonconvex equations of state. That is, we search for solutions depending only on the variable $y := x - \lambda\,t$ for some constant speed λ and converging to constant states at $y = \pm\infty$. We arrive at an ordinary differential equations, and must determine which equilibrium points can be connected by trajectories of the system. Recall that the case when the capillarity is neglected, that is $\beta = 0$, has a long history in the literature (see for instance [6,13,21] and the references therein). Our main focus in this paper is $\beta \neq 0$ and the limit behavior $\beta \to 0$ (vanishing capillarity) and $\beta \to \infty$ (vanishing viscosity).

Some general remarks on the system (1.1) are now made. A simple calculation shows that (1.1) can be rewritten in the variables (v,u,S):

$$
\begin{aligned}
&v_t - u_x = 0,\\
&u_t - \varepsilon_v(v,S)_x = \alpha\,\big(\nu\,u_x\big)_x - \beta\,(\mu\,v_x)_{xx} + \frac{\beta}{2}\,\big(\mu_v\,v_x^2\big)_x\\
&\big(\varepsilon_S + \frac{\beta}{2}\,\mu_S\,v_x^2\big)\,S_t = \alpha\,\nu\,u_x^2.
\end{aligned}
\tag{1.5}
$$

In the special case that the coefficients ν and μ depend upon the specific volume v only, which is often realized in the applications, we obtain

$$\begin{aligned}
&v_t - u_x = 0,\\
&u_t - \varepsilon_v(v,S)_x = \alpha\left(\nu(v)\,u_x\right)_x - \beta\left(\tilde\mu(v)\left(\tilde\mu(v)\,v_x\right)_x\right)_x,\\
&\varepsilon_S(v,S)\,S_t = \alpha\,\nu(v)\,u_x^2
\end{aligned}\tag{1.6}$$

with $\tilde\mu(v) := \sqrt{\mu(v)}$. Furthermore, it is interesting to point out that, if ε would be independent of the entropy S (which is not realistic in the application) and we would ignore the third equation in (1.6), we would arrive at a system of two equations only, that is,

$$\begin{aligned}
&\partial_t v - \partial_x u = 0,\\
&\partial_t u - \partial_x \varepsilon_v(v) = \alpha\left(\nu(v)\,u_x\right)_x - \beta\left(\tilde\mu(v)\left(\tilde\mu(v)\,v_x\right)_x\right)_x.
\end{aligned}\tag{1.7}$$

The system (1.7) was studied in [3] in the hyperbolic case (ε convex) and in [5] in the hyperbolic-elliptic case (ε not globally convex). This system has drawn a lot of attention in the literature, initiated with the pioneering work by Slemrod[16−18]. The importance of the so-called kinetic relation in characterizing the dynamics of undercompressive waves was recognized by Abeyaratne and Knowles[1,2], Truskinovsky[19,20], LeFloch[10], Shearer[9,14,15], Hayes and LeFloch[7,8], and LeFloch[11,12].

Throughout this paper, our assumptions on the energy function $\varepsilon = \varepsilon(v,S)$ are the following ones. We assume that there exist three smooth mappings $v_\star, \tilde v, v^\star : (0,\infty) \to (0,\infty)$ such that $v_\star(S) < \tilde v(S) < v^\star(S)$,

$$\varepsilon_{vv}(v_\star(S),S) = 0, \quad \varepsilon_{vvv}(\tilde v(S),S) = 0, \quad \varepsilon_{vv}(v^\star(S),S) = 0, \tag{1.8a}$$

$$\varepsilon_{vvvv} > 0, \tag{1.8b}$$

and, uniformly with respect to the variable S in every set of the form $[\underline{S}, +\infty)$,

$$\lim_{v\to 0} \varepsilon_v = -\infty, \qquad \lim_{v\to+\infty} \varepsilon_{vv} = +\infty. \tag{1.8c}$$

We also assume that

$$\varepsilon_S > 0, \quad \varepsilon_{vS} < 0, \tag{1.8d}$$

$$\mu_S \ge 0. \tag{1.8e}$$

Finally, some technical condition on the behavior at ∞ will be needed: for every set $[\underline{V}, \overline{V}] \times [\underline{S}, +\infty) \subset (0,\infty)\times(0,\infty)$ there exist constants $c_0, C_0 > 0$ such that

$$\varepsilon_S \ge c_0, \quad \mu_S \le C_0 \quad \text{and } c_0 \le \mu \le C_0, \quad (v,S) \in [\underline{V}, \overline{V}] \times [\underline{S}, +\infty), \tag{1.9}$$

and we assume also that

$$\inf\{(\varepsilon_v(v^\star(S),S), S > 0\} > -\infty. \tag{1.10}$$

It is easy to check that van der Waals' equation of state (1.4) satisfies the conditions above, at least in some interval of v. In particular, we have

$$\begin{aligned}
\varepsilon_S(v,S) &= (3\,v-1)^{-1/a}\,e^{3\,S/(8a)} > 0,\\
\varepsilon_{vS}(v,S) &= \frac{-3}{a}\,(3\,v-1)^{-1-1/a}\,e^{3\,S/(8a)} < 0.
\end{aligned}$$

Under our assumptions (1.8) the first-order system

$$\begin{aligned} &v_t - u_x = 0, \\ &u_t - \varepsilon_v(v, S)_x = 0, \\ &E_t - \big(\varepsilon_v(v, S)\, u\big)_x = 0, \end{aligned}$$

is of elliptic type in the region

$$\mathcal{E} := \big\{(v, S) \,/\, v_\star(S) < v < v^\star(S)\big\}$$

and of hyperbolic type in $\mathcal{H}_\star \cup \mathcal{H}^\star$, with

$$\mathcal{H}_\star := \big\{(v, S) \,/\, v < v_\star(S)\big\}, \quad \mathcal{H}^\star := \big\{(v, S) \,/\, v^\star(S) < v\big\}.$$

Finally, we close this section by casting the system (1.1) in the more compact form. We define the total internal energy by

$$\overline{e} = \overline{e}(v, S, v_x) = \varepsilon(v, S) + \frac{\beta}{2}\, \mu(v, S)\, v_x^2 \tag{1.11}$$

and the thermodynamics pressure p and the total pressure P by

$$\begin{aligned} &p = -\overline{e}_v, \\ &P = -\overline{e}_v + \big(\overline{e}_w\big)_x = p - \frac{\beta}{2}\, \mu_v(v, S)\, v_x^2 + \beta\, \big(\mu(v, S)\, v_x\big)_x, \end{aligned} \tag{1.12}$$

where w denotes the derivative v_x. Then, we can rewrite (1.1) in the form

$$\begin{aligned} &v_t - u_x = 0, \\ &u_t + P_x = \alpha\, \big(\nu\, u_x\big)_x, \\ &E_t + \big(u\, P\big)_x = \alpha\, \big(\nu\, u\, u_x\big)_x + \beta\, \big(e_w(v, S, v_x)\, u_x\big)_x. \end{aligned} \tag{1.13}$$

An outline of this paper follows. In Section 2, we display basic properties of the system under consideration, including some discussion of the Rankine-Hugoniot relations. The existence of the kinetic function is established in Section 3 which is the central part of this paper. Finally, in Section 4, we reformulate our conclusions in terms of wave curves and we discuss important limiting cases when the viscosity or the capillarity coefficients are taken to vanish.

As the specific entropy S must remain positive, throughout this paper we tacitly assume that the ranges of parameters under consideration are restricted by this condition. For instance, the specific entropy S might reach zero along a Rankine-Hugoniot curve.

§2. Basic Properties

In this section we discuss some general properties of traveling wave solutions of the system (1.1), together with basic properties of the Rankine-Hugoniot jump relations. Any traveling wave $y \mapsto u(y), v(y), S(y)$ with speed λ must satisfy

$$\begin{aligned} &\lambda\, v' + u' = 0, \\ &\lambda\, u' + \varepsilon_v(v, S)' = -\alpha\, (\nu\, u')' + \beta\, (\mu\, v')'' - \frac{\beta}{2}\, \big(\mu_v\, v'^2\big)', \\ &\lambda\, E' + \big(\varepsilon_v(v, S)\, u\big)' = -\alpha\, \big(\nu\, u\, u'\big)' + \beta\, \Big(\frac{\mu_v}{2} u\, v'^2 - u\, (\mu\, v')'\Big)' + \beta\, \big(\mu\, u'\, v'\big)', \end{aligned} \tag{2.1}$$

and the following condition at infinity:

$$\begin{aligned}
&v(y) \to v_-, \quad u(y) \to u_-, \quad S(y) \to S_-, \quad y \to -\infty, \\
&v(y) \to v_+, \quad u(y) \to u_+, \quad S(y) \to S_+, \quad y \to +\infty, \\
&u'(y),\ v'(y),\ S'(y),\ v''(y) \to 0, \qquad\qquad y \to \pm\infty,
\end{aligned} \tag{2.2}$$

where $u_-, v_-, S_-, u_+, v_+, S_+$ are constants. It will be convenient to define

$$u_0 := u_-, \qquad v_0 := v_-, \qquad S_0 := S_- \tag{2.3}$$

and to search for all right-hand states (v_+, u_+, S_+) attainable via a traveling wave initiating at (v_0, u_0, S_0). For definiteness we will assume that

$$\lambda > 0,$$

that is, we restrict attention to waves of the third characteristic family, and we also assume that the left-hand state satisfies

$$v_0 < v_\star(S).$$

On one hand, as was already pointed out in (1.5), the third equation in (2.1) can be replaced with

$$\lambda \left(\varepsilon_S + \frac{\beta}{2}\, \mu_S\, {v'}^2 \right) S' = -\alpha\, \nu\, {u'}^2.$$

For $\lambda \neq 0$, using the first equation in (2.1) to eliminate u, we get

$$\left(\varepsilon_S + \frac{\beta}{2}\, \mu_S\, {v'}^2 \right) S' = -\alpha\, \nu\, \lambda\, {v'}^2. \tag{2.4}$$

On the other hand, we can also eliminate the variable u from the second equation in (2.1):

$$-\lambda^2\, v' + \varepsilon_v(v, S)' = \lambda\, \alpha\, (\nu\, v')' + \beta\, (\mu\, v')'' - \frac{\beta}{2} \left(\mu_v\, {v'}^2 \right)'. \tag{2.5}$$

The equations (2.4) and (2.5) form a (second-order) system of two equations for the functions $v = v(y)$ and $S = S(y)$.

It will be convenient to recast the problem as a first-order system of three equations. Setting

$$w = \mu(v, S)\, v', \tag{2.6}$$

we re-formulate the equations (2.4) and (2.5) in terms of the variables (v, w, S):

$$\begin{aligned}
v' &= \frac{w}{\mu(v, S)}, \\
w' &= -\frac{\alpha}{\beta}\, \frac{\nu(v, S)\, \lambda}{\mu(v, S)}\, w + \frac{\mu_v(v, S)}{2\, \mu(v, S)^2}\, w^2 + \frac{1}{\beta}\, g(v_0, S_0, v, S, \lambda^2), \\
S' &= \frac{-2\, \nu(v, S)\, \lambda\, w^2}{2\, \mu(v, S)^2\, \varepsilon_S(v, S) + \beta\, \mu_S(v, S)\, w^2},
\end{aligned} \tag{2.7}$$

where

$$g(v_0, S_0, v, S, \lambda^2) := \varepsilon_v(v, S) - \varepsilon_v(v_0, S_0) - \lambda^2\, (v - v_0). \tag{2.8}$$

The boundary conditions (2.2) now read

$$\begin{aligned} &v(y) \to v_-, \quad w(y) \to 0, \quad S(y) \to S_- \quad &&\text{when } y \to -\infty, \\ &v(y) \to v_+, \quad w(y) \to 0, \quad S(y) \to S_+ \quad &&\text{when } y \to +\infty, \\ &v'(y),\, w'(y),\, S'(y) \to 0 &&\text{when } y \to \pm\infty. \end{aligned} \tag{2.9}$$

For a traveling solution to (2.7)–(2.9) to exist, the states $v_- = v_0$, $S_- = S_0$, v_+, and S_+, and the speed λ must satisfy some compatibility conditions, the Rankine-Hugoniot relations, which are now derived. By integration of the (conservative!) equation (2.5) over some interval $(-\infty, y)$ and in view of (2.9) we obtain

$$\varepsilon_v(v,S) - \varepsilon_v(v_0,S_0) - \lambda^2\,(v - v_0) = \alpha\,\nu\,\lambda\,v' + \beta\,(\mu\,v')' - \frac{\beta}{2}\,\mu_v\,v'^2. \tag{2.10}$$

Additionally, multiplying (2.10) by v' and using the third equation in (2.7) we get

$$\frac{\beta}{2}\,(\mu\,v'^2)' = \varepsilon(v,S)' - \varepsilon_v(v_0,S_0)\,v' - \lambda^2\,(v - v_0)\,v'. \tag{2.11}$$

By integrating (2.11) over some interval $(-\infty, y)$ we find

$$\begin{aligned} \frac{\beta}{2}\,\mu\,v'^2 &= \varepsilon(v,S) - \varepsilon(v_0,S_0) - \varepsilon_v(v_0,S_0)\,(v - v_0) - \frac{\lambda^2}{2}\,(v - v_0)^2 \\ &=: f(v_0, S_0, v, S, \lambda^2). \end{aligned} \tag{2.12}$$

(Of course, (2.12) can also be derived from the third equation in (2.1).) In conclusion, if a traveling wave exists, its propagation speed must satisfy the two Rankine-Hugoniot relations:

$$\lambda^2 = \frac{\varepsilon_v(v_+,S_+) - \varepsilon_v(v_0,S_0)}{v_+ - v_0} = 2\,\frac{\varepsilon(v_+,S_+) - \varepsilon(v_0,S_0) - (v_+ - v_0)\,\varepsilon_v(v_0,S_0)}{(v_+ - v_0)^2}. \tag{2.13}$$

We investigate first the limiting case when the viscosity vanishes ($\nu = 0$) which, in view of the third equation in (2.7), implies that the entropy remains constant: $S_+ = S_0$. In particular, it is interesting also to study the case when both the viscosity $\nu = 0$ and the speed λ^2 vanish. This is the subject of the following statement. Given the hypotheses made on the function ε, it is not difficult to see geometrically that:

Lemma 2.1. (Definition of Maxwell states). *For every (v,S) satisfying $v < \tilde v(S)$ there exists a unique point $(\varphi_0^\flat(v,S), S_0^\flat(v,S))$ which lies on the Hugoniot curve (for the third characteristic family) leaving from (v,S) and satisfies*

$$S_0^\flat(v,S) = S$$

and, denoting here by $\Lambda_0^\flat(v,S)$ the corresponding (square of the) shock speed,

$$\Lambda_0^\flat(v,S) = \frac{\varepsilon_v(\varphi_0^\flat(v,S),S) - \varepsilon_v(v,S)}{\varphi_0^\flat(v,S) - v} = 2\,\frac{\varepsilon(\varphi_0^\flat(v,S),S) - \varepsilon(v,S) - (\varphi_0^\flat(v,S) - v)\,\varepsilon_v(v,S)}{(\varphi_0^\flat(v,S) - v)^2}. \tag{2.14}$$

Moreover, we have

$$\varphi_0^\flat(\varphi_0^\flat(v,S),S) = v. \tag{2.15}$$

In particular, for every value $S > 0$ there exist two states $\underline{v}(S) < \tilde v(S)$ and $\overline{v}(S) > \tilde v(S)$ such that $\varphi_0^\flat(\underline{v}(S),S),S) = \overline{v}(S)$ and the shock speed vanishes: $\Lambda_0^\flat(\underline{v}(S),S) = 0$. These points are characterized by the relations

$$\begin{aligned} &\varepsilon_v(\underline{v}(S),S) = \varepsilon_v(\overline{v}(S),S), \\ &\varepsilon(\overline{v}(S),S) - \varepsilon(\underline{v}(S),S) - (\overline{v}(S) - \underline{v}(S))\,\varepsilon_v(\underline{v}(S),S) = 0. \end{aligned} \tag{2.16}$$

Note that the value $\varphi_0^\flat(v,S)$ is given geometrically by the so-called "equal area rule". We refer to the values $\underline{v}(S)$ and $\overline{v}(S)$ as the Maxwell states at the entropy level S.

The Rankine-Hugoniot relations in (2.13) lead to the implicit equation

$$\mathbb{H}(v_0,S_0;v,S) := \varepsilon(v,S) - \varepsilon(v_0,S_0) - \frac{1}{2}\big(\varepsilon_v(v,S) + \varepsilon_v(v_0,S_0)\big)(v-v_0) = 0. \tag{2.17}$$

Thanks to the assumptions (1.8), we have

$$\mathbb{H}_S(v_0,S_0;v,S) = \varepsilon_S(v,S) - \frac{1}{2}\,\varepsilon_{vS}(v,S)\,(v-v_0) > 0, \quad \text{for} \quad v \geq v_0. \tag{2.18}$$

Therefore, the implicit function theorem allows us to determine a smooth function $v \mapsto S = \mathbb{S}(v_0,S_0;v)$ for $v \geq v_0$. The (square of the) shock speed along the Hugoniot curve, denoted by $v \mapsto \mathbb{L}(v_0,S_0;v)$, is

$$\mathbb{L}(v_0,S_0;v) := \begin{cases} \dfrac{\varepsilon_v(v,\mathbb{S}(v_0,S_0;v)) - \varepsilon_v(v_0,S_0)}{v-v_0}, & v \neq v_0, \\ \varepsilon_{vv}(v_0,S_0), & v = v_0. \end{cases} \tag{2.19}$$

In the following lemma we investigate the properties of the shock speed along the Hugoniot curve.

Lemma 2.2 (Definition of Characteristic States). *There exist two functions $\varphi^\natural = \varphi^\natural(v_0, S_0)$ and $S^\natural = S^\natural(v_0,S_0)$ defined for all $v_0 < \tilde{v}(S_0)$ and all $S_0 > 0$ such that $\varphi^\natural(v_0,S_0) > \tilde{v}(S_0)$ and*

$$\begin{aligned} \varepsilon_{vv}\big(\varphi^\natural(v_0,S_0), S^\natural(v_0,S_0)\big) &= \mathbb{L}\big(v_0,S_0;\varphi^\natural(v_0,S_0)\big), \\ S^\natural(v_0,S_0) &= \mathbb{S}\big(v_0,S_0;\varphi^\natural(v_0,S_0)\big). \end{aligned} \tag{2.20}$$

Similarly, there exist two functions $\varphi^{-\natural} = \varphi^{-\natural}(v_0,S_0)$ and $S^{-\natural} = S^{-\natural}(v_0,S_0)$ defined for all $v_0 < \tilde{v}(S_0)$ and all $S_0 > 0$ such that $\varphi^{-\natural}(v_0,S_0) > \tilde{v}(S_0)$ and

$$\begin{aligned} \varepsilon_{vv}(v_0,S_0) &= \mathbb{L}\big(v_0,S_0;\varphi^{-\natural}(v_0,S_0)\big), \\ S^{-\natural}(v_0,S_0) &= \mathbb{S}\big(v_0,S_0;\varphi^{-\natural}(v_0,S_0)\big). \end{aligned} \tag{2.21}$$

Moreover, the function $\varphi_0^\flat$ satisfies

$$\varphi^\natural(v_0,S_0) \leq \varphi_0^\flat(v_0,S_0) < \varphi^{-\natural}(v_0,S_0), \quad v_0 < \tilde{v}(S_0). \tag{2.22}$$

Furthermore, the function $v \mapsto \mathbb{L}(v_0,S_0;v)$ is strictly monotone decreasing when $v \in (v_0, \varphi^\natural(v_0,S_0))$ and strictly monotone increasing when $v \in (\varphi^\natural(v_0,S_0), +\infty)$, with

$$\lim_{v\to\infty} \mathbb{L}(v_0,S_0;v) = +\infty. \tag{2.23}$$

In the following we will say that $\big(\varphi^\natural(v_0,S_0), S^\natural(v_0,S_0)\big)$ and $\big(\varphi^{-\natural}(v_0,S_0), S^{-\natural}(v_0,S_0)\big)$ are the characteristic states associated with the point (v_0,S_0).

Proof. In this proof, we do not indicate the dependence upon v_0 and S_0. Define the function

$$\mathbb{D}(v) := \varepsilon_v(v_0,S_0) - \varepsilon_v(v,\mathbb{S}(v)) + \varepsilon_{vv}(v,\mathbb{S}(v))\,(v-v_0). \tag{2.24}$$

By differentiating (2.17) and (2.19) with respect to v we get

$$\mathbb{S}'(v) = \frac{\mathbb{D}(v)}{2\,H_S(v,\mathbb{S}(v))}, \tag{2.25}$$

$$\mathbb{L}'(v) = \frac{\varepsilon_S(v,\mathbb{S}(v))\,\mathbb{S}'(v)}{(v-v_0)^2} = \frac{\varepsilon_S(v,\mathbb{S}(v))\,\mathbb{D}(v)}{2\,H_S(v,\mathbb{S}(v))\,(v-v_0)^2}. \tag{2.26}$$

The function $\mathbb{D}$ satisfies

$$\mathbb{D}(v_0) = \mathbb{D}'(v_0) = 0, \quad \mathbb{D}''(v_0) = \varepsilon_{vvv}(v_0, S_0) < 0.$$

This implies that $\mathbb{D}(v) < 0$ on some interval of the form $(v_0, v_0 + \eta)$ with $\eta > 0$, at least. In view of the definition of $\mathbb{L}$ and $\mathbb{D}$, this is completely equivalent to saying that $\mathbb{L}(v) > \varepsilon_{vv}(v, \mathbb{S}(v))$ in the same interval.

Now, assume that $\mathbb{D}$ remains strictly negative for all $v > v_0$. Then, on one hand, from (1.8c) we get

$$\lim_{v \to +\infty} \mathbb{L}(v) = +\infty$$

and, on the other hand, (2.26) would give $\mathbb{L}' < 0$ for all $v > v_0$ which is a contradiction. We conclude that there exists $\varphi^\natural$ such that $\mathbb{D}(\varphi^\natural) = \mathbb{L}'(\varphi^\natural) = 0$ and $\mathbb{D}(v) < 0$ on the interval $[v_0, \varphi^\natural)$.

Now, let us prove that $\varepsilon_{vvv}(\varphi^\natural, S^\natural) > 0$. First it is clear that $\mathbb{D}'(\varphi^\natural) \geq 0$ and a simple calculation gives

$$\mathbb{D}'(\varphi^\natural) = \varepsilon_{vvv}(\varphi^\natural, S^\natural)(\varphi^\natural - v_0).$$

We deduce that $\varepsilon_{vvv}(\varphi^\natural, S^\natural) \geq 0$. Assume by contradiction that $\varepsilon_{vvv}(\varphi^\natural, S^\natural) = 0$. Necessarily, $\mathbb{D}''(\varphi^\natural) \leq 0$ which is contradictory since $\mathbb{D}''(\varphi^\natural) = \varepsilon_{vvvv}(\varphi^\natural, S^\natural)(\varphi^\natural - v_0)$ and $\varepsilon_{vvvv} > 0$ by (1.8).

We claim now that $\mathbb{D}$ remains strictly positive for $v > \varphi^\natural$. Otherwise, there would exist another value $\varphi_2^\natural$ for which $\mathbb{D}(\varphi_2^\natural) = 0$ and $\mathbb{D}'(\varphi_2^\natural) \leq 0$. This would imply that $\varepsilon_{vvv}(\varphi_2^\natural, S_2^\natural) \leq 0$. We must add here the assumption that the Hugoniot curves associated with (1.1) are transverse to the curve defined by $\tilde{v}$ in the sense that any Hugoniot curve intersects the critical curve $\big\{(\tilde{v}(S), S), S > 0\big\}$ at one point. Thus, we obtain a contradiction.

Now, $\mathbb{L}$ is strictly monotone increasing for $v > \varphi^\natural$ and, thanks to (1.8c) and (2.19), we easily obtain

$$\lim_{v \to \infty} \mathbb{L}(v_0, S_0; v) = +\infty.$$

This implies the existence of the function $\varphi^{-\natural}$ satisfying (2.22).

Thanks to Lemma 2.2 and the formulas (2.25) and (2.26), we obtain the following result:

Lemma 2.3 (Monotonicity Properties of the Specific Entropy). *The function $v \mapsto \mathbb{S}(v_0, S_0; v)$ is strictly monotone decreasing when $v \in (v_0, \varphi^\natural(v_0, S_0))$ and strictly monotone increasing when $v \in (\varphi^\natural(v_0, S_0), +\infty)$, with*

$$\mathbb{S}(v_0, S_0; v) \leq S_0 \text{ if and only if } v \leq \varphi_0^\flat(v_0, S_0). \tag{2.27}$$

§3. Nonclassical Trajectories

In this section, we study the existence of traveling waves solutions of (2.7). The following preliminary lemma determines the range of values where the (square of the) speed $\mathbb{L}$ is positive.

Lemma 3.1. *With each $S_0 > 0$ we can associate a value $v^{-\star}(S_0)$ such that $v^{-\star}(S) < \underline{v}(S)$ and that the shock speed between the points $(v^{-\star}(S_0), S_0)$ and its associated characteristic* *value* $(\varphi^\natural(v^{-\star}(S_0),S_0), S^\natural(v^{-\star}(S_0),S_0))$ *vanishes, that is,*

$$\begin{aligned}
&\varepsilon_v\big(\varphi^\natural(v^{-\star}(S_0),S_0), S^\natural(v^{-\star}(S_0),S_0)\big) = \varepsilon_v(v^{-\star}(S_0),S_0),\\
&\varepsilon_v(\varphi^\natural(v_0,S_0), S^\natural(v_0,S_0)) - \varepsilon_v(v_0,S_0) > 0, \quad v_0 < v^{-\star}(S_0),\\
&\varepsilon_v(\varphi^\natural(v_0,S_0), S^\natural(v_0,S_0)) - \varepsilon_v(v_0,S_0) < 0, \quad v^{-\star}(S_0) < v_0 < \underline{v}(S_0).
\end{aligned}$$

We deduce from Lemma 3.1 that the condition that $\mathbb{L}$ vanishes, together with (2.20), gives

$$\varepsilon_{vv}\big(\varphi^\natural(v^{-\star}(S_0),S_0), S^\natural(v^{-\star}(S_0),S_0)\big) = 0.$$

Proof. Setting $\varepsilon^\natural_{vv}(v_0,S_0) = \varepsilon_{vv}(\varphi^\natural(v_0,S_0), S^\natural(v_0,S_0)) = \mathbb{L}(v_0,S_0;\varphi^\natural(v_0,S_0))$, we will prove that

$$\frac{\partial \varepsilon^\natural_{vv}}{\partial v_0}(v_0,S_0) < 0.$$

By differentiating (2.20) with respect to v_0 we obtain

$$\frac{\partial \varepsilon^\natural_{vv}}{\partial v_0}(v_0,S_0) = \frac{\partial \mathbb{L}}{\partial v_0} + \frac{\partial \mathbb{L}}{\partial v}\frac{\partial \varphi^\natural}{\partial v_0} = \frac{\partial \mathbb{L}}{\partial v_0}(v_0,S_0,\varphi^\natural(v_0,S_0)),$$

since $\frac{\partial \mathbb{L}}{\partial v}(v_0,S_0,\varphi^\natural(v_0,S_0)) = 0$. Now, using (2.19) we obtain

$$\begin{aligned}
&\frac{\partial \mathbb{L}}{\partial v_0}(v_0,S_0,\varphi^\natural(v_0,S_0))\\
&= \frac{\Big(\varepsilon_{vS}(\varphi^\natural,S^\natural)\,\frac{\partial \mathbb{S}}{\partial v_0}(v_0,S_0,\varphi^\natural) - \varepsilon_{vv}(v_0,S_0)\Big)(\varphi^\natural - v_0) + \varepsilon_v(\varphi^\natural,S^\natural) - \varepsilon_v(v_0,S_0)}{(\varphi^\natural - v_0)^2}.
\end{aligned}$$

By differentiating (2.17) with respect to v_0 we obtain

$$\frac{\partial \mathbb{S}}{\partial v_0}(v_0,S_0,\varphi^\natural) = \frac{(\varepsilon_{vv}(v_0,S_0) - \varepsilon_{vv}(\varphi^\natural,S^\natural))(\varphi^\natural - v_0)}{2\varepsilon_S(\varphi^\natural,S^\natural) - \varepsilon_{vS}(\varphi^\natural,S^\natural)(\varphi^\natural - v_0)}.$$

Finally, combining the two last equations we get

$$\frac{\partial \varepsilon^\natural_{vv}}{\partial v_0}(v_0,S_0) = 2\,\frac{\big(\varepsilon_{vv}(v_0,S_0) - \varepsilon_{vv}(\varphi^\natural,S^\natural)\big)\big(\varepsilon_S(\varphi^\natural,S^\natural) - \varepsilon_{vS}(\varphi^\natural,S^\natural)(\varphi^\natural - v_0)\big)}{(v_0 - \varphi^\natural)\,\big(2\,\varepsilon_S(\varphi^\natural,S^\natural) - \varepsilon_{vS}(\varphi^\natural,S^\natural)(\varphi^\natural - v_0)\big)}$$

which is clearly negative by assumptions. On the other hand,

$$\varepsilon^\natural_{vv}(\underline{v}(S_0),S_0) < \Lambda^\flat_0(\underline{v}(S_0),S_0) = 0.$$

Now, we have to prove that $\lim_{v_0\to 0} \varepsilon^\natural_{vv}(v_0,S_0) > 0$. By contradiction, assume that $\varepsilon^\natural_{vv}(v_0,S_0) \le 0$ for all $v_0 \le \underline{v}(S_0)$. Then, from (2.19) and (2.20) we get $\varepsilon_v(v_0,S_0) \ge \varepsilon_v(\varphi^\natural,S^\natural)$. But $\varepsilon_v(\varphi^\natural,S^\natural) \ge \varepsilon_v(v^\star(S^\natural),S^\natural)$. Then, combining the two previous inequalities and using (1.10) we obtain that $\lim_{v_0\to 0} \varepsilon_v(v^\star(S^\natural),S^\natural) = -\infty$ which contradicts (1.10).

Now, thanks to Lemmas 2.2 and 3.1, the (square of) the speed is positive for all $v_0 < v^{-\star}(S_0)$ and we can define the quantities

$$\lambda^\natural(v_0,S_0) = \sqrt{\mathbb{L}(v_0,S_0;\varphi^\natural(v_0,S_0))}, \quad v_0 < v^{-\star}(S_0), \tag{3.1}$$

$$\lambda^\flat_0(v_0,S_0) = \sqrt{\Lambda^\flat_0(v_0,S_0)} = \sqrt{\mathbb{L}(v_0,S_0;\varphi^\flat_0(v_0,S_0))}, \quad v_0 < \underline{v}(S_0), \tag{3.2}$$

$$\lambda_{\min}(v_0,S_0) = \begin{cases} \lambda^\natural(v_0,S_0), & v_0 < v^{-\star}(S_0),\\ 0, & v^{-\star}(S_0) < v_0 < \underline{v}(S_0). \end{cases} \tag{3.3}$$

Thanks to Lemmas 2.2 and 2.3, given λ in the range $(\lambda_{\min}(v_0,S_0),\lambda_0^\flat(v_0,S_0)]$, the Rankine-Hugoniot relations have two (non-trivial) solutions $(v_+,S_+)\neq(v_0,S_0)$, with $S_+\le S_0$, which we denote by

$$(v_1,S_1) \text{ and } (v_2,S_2) \text{ with } v_0<v_1<v_2.$$

To study the O.D.E. system (2.7), it is convenient to regard it as a system of only two equations. Namely, the equation (2.12) derived earlier can be written in the form

$$f(v_0,S_0,v,S,\lambda^2)=\frac{\beta}{2\mu}\,w^2, \tag{3.4}$$

where w and f are defined in (2.7) and (2.12), respectively. Since μ_S and ε_S are positive by assumption, we have

$$\frac{\partial}{\partial S}\Big(f(v_0,S_0,v,S,\lambda^2)-\frac{\beta}{2\mu}\,w^2\Big)=\frac{1}{2\,\mu^2}\,(2\mu^2\,\varepsilon_S+\beta\mu_S\,w^2)>0.$$

Therefore, using the implicit function theorem, the equation (3.4) allows us to express the variable S as a smooth algebraic function $(v,w)\mapsto S:=\mathbf{S}(v,w)$, which leads us to a reduced form of the system (2.7):

$$\begin{aligned} v'&=\frac{w}{\mu(v,\mathbf{S})},\\ w'&=-\frac{\alpha}{\beta}\,\frac{\nu(v,\mathbf{S})\,\lambda}{\mu(v,\mathbf{S})}\,w+\frac{\mu_v(v,\mathbf{S})}{2\,\mu(v,\mathbf{S})^2}\,w^2+\frac{1}{\beta}\,g(v_0,S_0,v,\mathbf{S},\lambda^2). \end{aligned} \tag{3.5}$$

The eigenvalues of the system (3.5) at an equilibrium point are found to be

$$\begin{aligned} \underline{\sigma}(v,S,\lambda,\alpha,\beta)&=\frac{1}{2\beta\,\mu}\Big(-\alpha\nu\,\lambda-\sqrt{\alpha^2\nu^2\,\lambda^2+4\,\beta\,\mu\,(\varepsilon_{vv}-\lambda^2)}\Big),\\ \overline{\sigma}(v,S,\lambda,\alpha,\beta)&=\frac{1}{2\beta\,\mu}\Big(-\alpha\nu\,\lambda+\sqrt{\alpha^2\nu^2\,\lambda^2+4\,\beta\,\mu(\varepsilon_{vv}-\lambda^2)}\Big). \end{aligned} \tag{3.6}$$

The following lemma covers the general case $\alpha\ge 0$ and $\beta\in I\!R\setminus\{0\}$. It applies to any of the three equilibria (v_0,S_0), (v_1,S_1), and (v_2,S_2).

Lemma 3.2 (Properties of Equilibria). *Some values v_- and λ being fixed, let $(v,0)$ be an equilibrium point of* (2.4).

(a) *If $\beta\mu\,(\varepsilon_{vv}(v,S)-\lambda^2)>0$, then $(v,0)$ is a saddle point having two real eigenvalues: $\underline{\sigma}<0<\overline{\sigma}$.*

(b) *If $\beta\mu<0$ and $\varepsilon_{vv}(v,S)-\lambda^2>0$, then* $\mathrm{Re}(\underline{\sigma})$ *and* $\mathrm{Re}(\overline{\sigma})$ *are both positive and $(u,0)$ is referred to as an* unstable equilibrium. *If furthermore $\alpha^2\nu^2\,\lambda^2+4\,\beta\,\mu(\varepsilon_{vv}-\lambda^2)>0$ then it is called an* unstable node *as it corresponds to two real positive eigenvalues $0<\underline{\sigma}<\overline{\sigma}$. Otherwise, if $\alpha^2\nu^2\,\lambda^2+4\,\beta\,\mu(\varepsilon_{vv}-\lambda^2)<0$, it is called an* unstable spiral *since it corresponds to two complex conjugate eigenvalues with positive real parts.*

(c) *If $\beta\mu>0$ and $\varepsilon_{vv}(v,S)-\lambda^2<0$, then* $\mathrm{Re}(\underline{\sigma})$ *and* $\mathrm{Re}(\overline{\sigma})$ *are both negative and $(v,0)$ is referred to as* a stable point. *If furthermore $\alpha^2\nu^2\,\lambda^2+4\,\beta\,\mu(\varepsilon_{vv}-\lambda^2)>0$ then it corresponds to a* stable node *with two real negative eigenvalues $\underline{\sigma}<\overline{\sigma}<0$. Otherwise, if $\alpha^2\nu^2\,\lambda^2+4\,\beta\,\mu(\varepsilon_{vv}-\lambda^2)>0$, it is a* stable spiral *with two complex conjugate eigenvalues with negative real parts.*

The dependence of these eigenvalues with respect to their arguments will be essential in several monotonicity arguments below.

Lemma 3.3 (Monotonicity Properties of Eigenvalues). In the case $\beta>0$ and in the range of parameters where $\underline{\sigma}(v,S,\lambda,\alpha,\beta)$ and $\overline{\sigma}(v,S,\lambda,\alpha,\beta)$ remain real-valued, more specifically when $\varepsilon_{vv} - \lambda^2 > 0$ we have

$$\frac{\partial \underline{\sigma}}{\partial \alpha}(v, S, \lambda, \alpha, \beta) < 0, \quad \frac{\partial \overline{\sigma}}{\partial \lambda}(v, S, \lambda, \alpha, \beta) < 0, \quad \frac{\partial \overline{\sigma}}{\partial \alpha}(v, S, \lambda, \alpha, \beta) < 0. \tag{3.7}$$

For $\beta > 0$, without loss of generality and by a straightforward rescaling of the traveling wave, we can now assume

$$\beta = 1. \tag{3.8}$$

Theorem 3.1 (Existence of Nonclassical Trajectories). *Given (v_0, S_0) with $v_0 \leq \underline{v}(S_0)$ and λ in the range $(\lambda_{\min}(v_0, S_0), \lambda_0^\flat(v_0, S_0)]$, there exists a unique $\alpha \geq 0$ such that (v_0, S_0) is connected to (v_2, S_2) by a traveling wave solution of* (2.1) – (2.2).

By Lemma 2.1, we have $\overline{\sigma}(v_0, S_0, \lambda, \alpha, 1) > 0$ and it is well-known that there are two orbits leaving from v_0 at $y = -\infty$ and satisfying

$$\lim_{y \to -\infty} \frac{w(y)}{v(y) - v_0} = \mu(v_0, S_0)\, \overline{\sigma}(v_0, S_0, \lambda, \alpha, 1). \tag{3.9}$$

A direct verification from (2.3) (by (2.3i), w and v_y have the same sign) shows that one orbit approaches this point in the quadrant $Q_1 = \{v > v_0,\, w > 0\}$ while the other approaches it in the quadrant $Q_2 = \{v < v_0,\, w < 0\}$. On the other hand, there are two orbits reaching u_2 at $y = +\infty$ and satisfying

$$\lim_{y \to +\infty} \frac{w(y)}{v(y) - v_2} = \mu(v_2, S_2)\, \underline{\sigma}(v_2, S_2, \lambda, \alpha, 1). \tag{3.10}$$

One orbit approaches this point in the quadrant $Q_3 = \{v > v_2,\, w < 0\}$, the other approaches in the quadrant $Q_4 = \{v < v_2,\, w > 0\}$.

Since v_0, S_0 and λ are fixed, for simplification we denote $f(v, S) = f(v_0, S_0, v, S, \lambda^2)$ and $g(v, S) = g(v_0, S_0, v, S, \lambda^2)$.

We now need some additional properties of the functions f and g, introduced earlier in Section 2 for our investigation of the Rankine-Hugoniot relations.

Lemma 3.4 (Key Properties of the Functions f and g). (1) *There exist two smooth functions $v \mapsto F(v)$ and $v \mapsto G(v)$ such that*

$$\Big\{(v, S) \,/\, f(v, S) = 0\Big\} = \Big\{(v, S) \,/\, S = F(v)\Big\} = \mathcal{C}_F,$$

$$\Big\{(v, S) \,/\, g(v, S) = 0\Big\} = \Big\{(v, S) \,/\, S = G(v)\Big\} = \mathcal{C}_G.$$

(2) *We have $f(v, S) > 0$ if and only if $S > F(v)$.*

(3) *We have $g(v, S) > 0$ if and only if $S < G(v)$.*

(4) *The two curves $\mathcal{C}_F$ and $\mathcal{C}_G$ intersect at the three equilibrium points:*

$$\mathcal{C}_F \cap \mathcal{C}_G = \Big\{(v, S) \,/\, F'(v) = 0\Big\} = \Big\{(v_0, S_0), (v_1, S_1), (v_2, S_2)\Big\}.$$

(5) *In the (v, S)-plane, the graph of $\mathcal{C}_F$ is "above" the one of C_G for $v < v_0$ and $v \in (v_1, v_2)$, and "below" for $v \in (v_0, v_1)$ and $v > v_2$.*

Proof. The first item is a direct consequence of the implicit function theorem, since

$$\frac{\partial f}{\partial S}(v, S) = \varepsilon_S(v, S) > 0 \quad \text{and} \quad \frac{\partial g}{\partial S}(v, S) = \varepsilon_{vS}(v, S) < 0. \tag{3.11}$$

The second item is also an immediate consequence of (3.11).

On one hand, the function F satisfies

$$F'(v) = -\frac{g(v, F(v))}{\varepsilon_S(v, F(v))}. \tag{3.12}$$

On the other hand, for the function G we have

$$G'(v) = \frac{\lambda^2 - \varepsilon_{vv}(v, G(v))}{\varepsilon_{vS}(v, G(v))} > 0 \quad \text{at } v = v_1 \text{ and } v = v_2. \tag{3.13}$$

The conclusions of the lemma are clear.

Lemma 3.5. *In the phase plane, a traveling wave solution connecting v_0 to v_2 necessarily approaches the equilibrium $(v_0, 0)$ at $y = -\infty$ through the quadrant Q_1, and the equilibrium $(v_2, 0)$ at $y = +\infty$ through the quadrant Q_4.*

Proof. Suppose that such a traveling wave satisfies $v < v_0$ and $w < 0$ in a neighborhood of the point $(v_0, 0)$. By continuity, since $\lim\limits_{y\to+\infty} v(y) = v_2 > v_0$, there would exist some value y_0 achieving a local minimum, that is, such that

$$v(y_0) < v_0, \quad v_y(y_0) = 0, \quad v_{yy}(y_0) \geq 0.$$

Then, $f(v(y_1), S(v(y_1))) = 0$. By Claims 2 and 4 in Lemma 3.4 this implies that $g(v(y_1), S(v(y_1))) < 0$. But we may have a contradiction by the second equation in (3.5). The proof of the second statement concerning the point $(v_2, 0)$ is similar to the first one.

Next, we show that any traveling wave is monotone in some range.

Lemma 3.6. (1) *If $v = v(y)$ is a solution of the system* (3.5) *defined on some interval $(-\infty, \bar{y})$ such that*

$$\lim_{y\to-\infty} v(y) = v_0 \ \textit{ and } \ v_0 < v(y) < v_1$$

for all $y < \bar{y}$, then $v_y > 0$ on the interval $(-\infty, \bar{y})$.

(2) *Similarly, if $v = v(y)$ is a solution of the system* (3.5) *defined on some interval $(\bar{y}, +\infty)$ such that*

$$\lim_{y\to+\infty} v(y) = v_2 \ \textit{ and } \ v_1 < v(y) < v_2$$

for all $y > \bar{y}$, then $v_y > 0$ on the interval $(\bar{y}, +\infty)$.

(3) *If $v = v(y)$ is a trajectory connecting (v_0, S_0) to (v_2, S_2), then $v_y(y) > 0$ for all y such that $v(y) \in (v_1, v_2)$.*

Proof. First, let us prove the first statement. Assume by contradiction that there exists $y_1 \in (-\infty, y_0)$ such that $v(y_1) \in (v_0, v_1)$ with

$$v_y(y_1) = 0, \quad v_{yy}(y_1) \leq 0.$$

Then, we have $f(v(y_1), S(v(y_1))) = 0$, which implies—thanks to Claims 2 and 4 in Proposition 3.4—that $g(v(y_1), S(v(y_1))) > 0$. But we may have a contradiction by the second equation in (3.2). The proof of the second statement is similar to the first one. Now, let us prove the third statement. First, note that for any trajectory connecting (v_0, S_0) to (v_2, S_2) we have $S_y(y) < 0$. Using Lemma 3.4, noting $S_1 \leq S_2 \leq S_0$ and $f(v, S) > 0$ for $v \in (v_1, v_2)$, and studying the intersection between this trajectory and $\mathcal{C}_G$ for $v \in (v_0, v_1)$, one can easily see that along the trajectory we have necessarily $v_y(y) > 0$ if $v(y) \in (v_1, v_2)$.

Proof of Theorem 3.1. For each $\alpha \geq 0$, we consider the orbit leaving from v_0 and satisfying $v > 0$ and $w > 0$ in a neighborhood of $(v_0, 0)$. This trajectory crosses the v-axis for the "first time" at some point $(v_1, w_1^-(\alpha))$. In view of Lemma 3.6, this part of trajectory is the graph of a function

$$[v_0, v_1] \ni v \mapsto w_-(v, \lambda, \alpha).$$

Moreover, by standard theorems on differential equations, w_- is a smooth function with respect to its argument $(v, \lambda, \alpha) \in [v_0, v_1] \times \big(\lambda_{\min}(v_0, S_0), \lambda_0^\flat(v_0, S_0)\big] \times [0, +\infty)$.

Similarly, for each $\alpha \geq 0$, we consider the orbit arriving at v_2 and satisfying $v < v_2$ and $w > 0$ in a neighborhood of $(v_2, 0)$. This trajectory crosses the w-axis for the "last time" at some point $(v_1, w_1^+(\alpha))$ (or equivalently for the "first time" as y decreases from $+\infty$). By Lemma 3.5, this trajectory is the graph of a function

$$[v_1, v_2] \ni v \mapsto w_+(v, \lambda, \alpha).$$

The function w_+ depends smoothly upon $(v, \lambda, \alpha) \in [v_1, v_2] \times \big(\lambda_{\min}(v_0, S_0), \lambda_0^\flat(v_0, S_0)\big] \times [0, +\infty)$.

Now, combining (2.3i) and (2.3ii), we can write for each of these curves $v \mapsto w_-(v)$ and $v \mapsto w_+(v)$ a differential equation in the (v, w) plane:

$$w \frac{dw}{dv}(v) + \alpha\nu\lambda\, w(v) - \frac{\mu_v}{2\mu}\, w^2 = \mu\, g(v, \mathbf{S}(v, w)). \tag{3.14}$$

The continuous function

$$[0, +\infty) \ni \alpha \mapsto z(\alpha) := w_+(v_1, \lambda, \alpha) - w_-(v_1, \lambda, \alpha) = w_1^+(\alpha) - w_1^-(\alpha) \tag{3.15}$$

measures the distance (in the phase plane) between the two trajectories at $v = v_1$. Therefore, thanks to Lemma 3.6, we deduce that the condition $z(\alpha) = 0$ characterizes the traveling wave solution of interest connecting v_0 to v_2. The existence of such a root α is obtained as follows.

Case 1. Suppose first that $\alpha = 0$. In this case $S_y(y) = 0$ along the two semi-trajectories. Then by (3.4) we get in one hand

$$\frac{1}{2}(w_1^-(\alpha))^2 = \mu(v_1, S_0)\, f(v_1, S_0),$$

and

$$\frac{1}{2}(w_1^+(\alpha))^2 = \mu(v_1, S_2)\, f(v_1, S_2)$$

on the other hand. Since $\mu_S > 0$, $f_S = \varepsilon_S > 0$ and $S_2 \le S_0$, we conclude that $z(0) < 0$.

Case 2. Consider next the limit $\alpha \to \infty$. On one hand, for $\alpha > 0$, since $w_1 > 0$, we get in the same way as above

$$\frac{1}{2}(w_1^-(\alpha))^2 = \mu(v_1, S^-(v_1))\, f(v_1, S^-(v_1))$$

with $S^-(v_1) \le S_0$. But since $\mu_S \ge 0$, $f_S = \varepsilon_S > 0$, we get

$$\frac{1}{2}(w_1^-(\alpha))^2 \le \mu(v_1, S_0)\, f(v_1, S_0).$$

On the other hand, consider the function w^+. Dividing (3.9) by w we obtain

$$\frac{dw}{dv}(v) - \frac{\mu_v}{2\mu}\, w + \alpha\nu\lambda = \mu\, \frac{g}{w}.$$

Inspiring from the compact form given in (1.7), we can rewrite this equation in the form

$$\sqrt{\mu}\frac{d}{dv}\Big(\frac{w}{\sqrt{\mu}}\Big) + \alpha\nu\lambda + \frac{\mu_S}{2\mu}\,\frac{dS}{dv}\, w = \mu\, \frac{g}{w}.$$

Now, setting $h = \frac{w}{\sqrt{\mu}}$ and using the first and the third equations in (2.7), we obtain

$$\frac{dh}{dv} + 2\,\frac{\alpha\nu\lambda}{\sqrt{\mu}}\,\frac{1}{1 + \frac{\mu_S}{2\mu\varepsilon_S}h^2} = \sqrt{\mu}\,\frac{g}{w}. \tag{3.16}$$

Since $f \ge 0$ along the trajectory, and thanks to Lemma 3.4, $g(v, S) \le 0$ for $v \in [v_1, v_2]$. On the other hand, thanks to (1.8) and (1.9) we deduce that h satisfies an inequality in the form

$$\frac{dh}{dv} \le -\frac{\alpha\, c_1}{1 + c_2\, h^2} \tag{3.17}$$

where c_1 and c_2 are two positive constants.

Now, integrating (3.17) over the interval $[v_1, v_2]$, recalling that $w_1^+(\alpha) = w_+(v_1)$, and setting $h_1^+(\alpha) = \frac{w_1^+(\alpha)}{\sqrt{\mu_1^+}}$, we find

$$h_1^+(\alpha) + \frac{1}{3} c_2\, h_1^+(\alpha)^3 \geq \alpha\, c_1\,(v_2 - v_1). \tag{3.18}$$

It is clear from (3.18) that $\lim_{\alpha \to 0} h_1^+(\alpha) = +\infty$. Using that $w_1^+(\alpha) = \sqrt{\mu^+}\, h_1^+ \geq c\, h_1^+$ and combining (3.16) and (3.18), for large values of α we get $w_1^+(\alpha) > w_1^-(\alpha)$, and so $z(\alpha) > 0$.

Henceforth, by the intermediate value theorem, there exists at least one value α such that

$$z(\alpha) = 0,$$

which establishes the existence of a trajectory connecting (v_0, S_0) to (v_2, S_2). Thanks to Lemma 3.5, it satisfies $v_y > 0$ globally.

The uniqueness of the solution α of $z(\alpha) = 0$ is checked as follows. Suppose that there would exist two orbits $w = w(v)$ and $w^* = w^*(v)$ associated with distinct values α and $\alpha^* > \alpha$, respectively. Then, Lemma 3.3 would give

$$\overline{\sigma}(v_0, S_0, \lambda, \alpha^*, 1) < \overline{\sigma}(v_0, S_0, \lambda, \alpha, 1), \qquad \underline{\sigma}(v_2, S_2, \lambda, \alpha^*, 1) < \underline{\sigma}(v_2, S_2, \lambda, \alpha, 1).$$

So, in the (v, w) plane, there would exist $v_3 \in (v_0, v_2)$ satisfying

$$w(v_3) = w^*(v_3), \qquad \frac{dw^*}{dv}(v_3) \geq \frac{dw}{dv}(v_3).$$

Comparing the equations (3.14) valid for both w and w^*, we get

$$w(u_3)\Big(\frac{dw}{dv}(v_3) - \frac{dw^*}{dv}(v_3)\Big) = (\alpha^* - \alpha)\, \nu\, \lambda\, w(v_3). \tag{3.19}$$

Now, since $w(v_3) \neq 0$ (the connection with the third critical point $(v_1, 0)$ is impossible), we obtain a contradiction, as the two sides of (3.19) have opposite signs. This completes the proof of Theorem 3.1.

Remark 3.1. It is not difficult to see also that the functions $w_1^\pm$ introduced in the proof of Theorem 3.1 satisfy

$$\alpha \mapsto w_1^-(\alpha) \text{ is monotone decreasing,} \tag{3.20}$$

$$\alpha \mapsto w_1^+(\alpha) \text{ is strictly monotone increasing.} \tag{3.21}$$

In particular, the function $z(\alpha) := w_1^+(\alpha) - w_1^-(\alpha)$ is strictly monotone increasing.

Theorem 3.2 (Definition of the Critical Diffusion). *Given (v_0, S_0) with $v_0 \leq \underline{v}(S_0)$, consider the function*

$$\lambda \in (\lambda_{\min}(v_0, S_0), \lambda_0^\flat(v_0, S_0)] \mapsto \alpha(\lambda, v_0, S_0)$$

which, to a speed λ, associates the unique value α such that there is a nonclassical traveling wave trajectory of (2.1) – (2.2) connecting (v_0, S_0) to (v_2, S_2). Then, $\alpha(\lambda, v_0, S_0)$ is a strictly monotone decreasing function of λ, mapping the interval $(\lambda_{\min}(v_0, S_0), \lambda_0^\flat(v_0, S_0)]$ onto some interval of the form $[0, \alpha_{\max}(v_0, S_0))$, where

$$\begin{aligned} &\alpha_{\max}(v_0, S_0) = +\infty \quad \text{if } v^{-\star}(S_0) < v_0 < \underline{v}(S_0), \\ &\alpha_{\max}(v_0, S_0) < +\infty \quad \text{if } v_0 < v^{-\star}(S_0). \end{aligned} \tag{3.22}$$

Then, taking (v_0, S_0) with $v_0 < v^{-\star}(S_0)$ and $\alpha = \alpha_{\max}(v_0, S_0)$, there exists a traveling wave solution of (2.7) connecting (v_0, S_0) to $(v_2, S_2) = (v_1, S_1) = (\varphi^\natural, S^\natural)$.

Proof. Setting

$$\lambda_s = \lambda^2, \quad \beta_s(\lambda_s) = \sqrt{\lambda_s}\, \alpha(\sqrt{\lambda_s}, v_0, S_0),$$

the proof of Theorem 3.7 in [4] can be immediately adapted to the situation under consideration. First, we see that β_s is a strictly monotone decreasing function of λ_s. Since

$$\alpha(\lambda, v_0, S_0) = \frac{\beta_s(\lambda^2)}{\lambda},$$

we deduce the monotonicity property of α with respect to λ. Second, using Theorem 4.1 in [4], we can derive (3.22). Finally, it is also easy to see that the proof given for the second statement in Theorem 4.1 in [4] can be easily adapted and we can establish the existence of a traveling wave solution to our system, in the case $v_0 < v^{-\star}(S_0)$ for $\alpha = \alpha_{\max}(v_0, S_0)$.

The value $\alpha_{\max}(v_0, S_0)$ will be called the critical diffusion at (v_0, S_0): Nonclassical trajectories leaving from (v_0, S_0) exists only when $\alpha \leq \alpha_{\max}(v_0, S_0)$ and $\alpha_{\max}(v_0, S_0)$ is finite.

§4. Kinetic Functions and Shock Sets

Thanks to Theorem 3.2, given (v_0, S_0) with $v_0 < \underline{v}(S_0)$ and $0 \leq \alpha < \alpha_{\max}(v_0, S_0)$, there exists a unique real $\lambda = \lambda^\flat(v_0, S_0)$ in the interval $(\lambda_{\inf}(v_0, S_0), \lambda_0^\flat(v_0, S_0)]$ such that $\alpha = \alpha(\lambda^\flat(v_0, S_0), v_0, S_0)$. We now discuss the existence of classical traveling wave solutions of (2.7).

Theorem 4.1 (Classical Trajectories). *Given (v_0, S_0) and a real $\alpha > 0$ we have the following properties:*

Case 1. $\underline{v}(S_0) < v_0 < v_\star(S_0)$. *For $0 < \lambda < \varepsilon_{vv}(v_0, S_0))$ there exists a traveling wave solution of (2.7) connecting (v_0, S_0) to (v_1, S_1).*

Case 2. $v^{-\star}(v_0) \leq v_0 < \underline{v}(S_0)$.

- *If $\lambda^\flat(v_0, S_0) < \lambda < \varepsilon_{vv}(v_0, S_0))$, there exists a traveling wave solution of (2.7) connecting (v_0, S_0) to (v_1, S_1).*
- *If $0 < \lambda < \lambda^\flat(v_0, S_0)$, there is no traveling wave solution of (2.7) connecting (v_0, S_0) to (v_1, S_1).*

Case 3. $v_0 < v^{-\star}(S_0)$.

- *If $\alpha < \alpha_{\max}(v_0, S_0)$ and $\lambda^\flat(v_0, S_0) < \lambda < \varepsilon_{vv}(v_0, S_0)$, there exists a traveling wave solution of (2.7) connecting (v_0, S_0) to (v_1, S_1).*
- *If $\alpha < \alpha_{\max}(v_0, S_0)$ and $\lambda^\natural(v_0, S_0) \leq \lambda < \lambda^\flat(v_0, S_0)$, there is no traveling wave solution of (2.7) connecting (v_0, S_0) to (v_1, S_1)*
- *If $\alpha \geq \alpha_{\max}(v_0, S_0)$ and $\lambda^\natural(v_0, S_0) \leq \lambda < \varepsilon_{vv}(v_0, S_0)$, there exists a traveling wave solution of (2.7) connecting (v_0, S_0) to (v_1, S_1).*

The proof of this theorem is the same as the one given in Theorems 5.1 and 5.2 in [4], with some minor modifications to adapt the arguments to our system along the lines of what was discussed earlier. We omit these details. Combining together the results in Theorems 3.2 and 4.1 we deduce:

Theorem 4.2. *Given (v_0, S_0), $v_0 < v^{-\star}(S_0)$ and $\alpha \geq \alpha_{\max}(v_0, S_0)$, there exists a traveling wave solution of (2.7) connecting (v_0, S_0) to $(v_2, S_2) = (v_1, S_1) = (v^\natural, S^\natural)$.*

Given $\alpha > 0$, thanks to the monotonicity of the function $v \mapsto \mathbb{L}(v_0, S_0, v)$ along the Rankine-Hugoniot curve on $[\varphi^\natural(v_0, S_0), \infty)$, one can define the kinetic function associated with nonclassical shocks,

$$(v_0, S_0) \mapsto v_2 := \varphi_\alpha^\flat(v_0, S_0),$$

where $v_2 = \varphi_\alpha^\flat(v_0, S_0)$ is the unique point satisfying $v_2 \geq \varphi^\natural(v_0, S_0)$ with

$$\lambda^\flat(v_0, S_0) = \sqrt{\frac{\varepsilon_v(v_2, \mathbb{S}(v_0, S_0, v_2)) - \varepsilon_v(v_0, S_0)}{v_2 - v_0}}.$$

This definition makes sense for all $v^{-\star}(S_0) \le v_0 < \underline{v}(S_0)$. Also, in the range $v_0 < v^{-\star}(S_0)$, if $\alpha \ge \alpha_{\max}(v_0, S_0)$, we can set by continuity (thanks to Theorem 4.2)

$$vf(v_0, S_0) = \varphi^{\natural}(v_0, S_0).$$

In the same manner, for a given (v_0, S_0) with $v_0 < \underline{v}(S_0)$ and $\lambda = \lambda^{\flat}(v_0, S_0)$, we may introduce the function (thanks to Lemma 2.2)

$$(v_0, S_0) \mapsto v_1 := \varphi^{\sharp}(v_0, S_0)$$

such that

$$\mathbb{L}(v_0, S_0, \varphi^{\sharp}(v_0, S_0)) = \mathbb{L}(v_0, S_0, \varphi^{\flat}_{\alpha}(v_0, S_0)) = \lambda^{\flat}(v_0, S_0)^2$$

with

$$v_0 \le \varphi^{\sharp}(v_0, S_0) \le \varphi^{\natural}(v_0, S_0) \le \varphi^{\flat}_{\alpha}(v_0, S_0).$$

Concerning the case $\underline{v}(S_0) < v_0 < v_\star(S_0)$, we introduce the quantity $\varphi^{\sharp}_0(v_0, S_0)$ where

$$\varphi^{\sharp}_0(v_0, S_0) = \inf\Big\{v \ge v_0 / \mathbb{L}(v_0, S_0, \varphi^{\sharp}_0(v_0, S_0))\Big\} = 0.$$

Finally, using these functions and Theorems 4.1 and 4.2, we obtain

Corollary 4.1. *Given (v_0, S_0) with $v_0 < v_\star(S_0)$ and $\alpha > 0$, the shock set*

$$\mathcal{S}(v_0, S_0) := \{(v, S)/\text{there exists a traveling solution of (2.7) connecting } (v_0, S_0) \text{ to } (v, S)\}$$

is given by

$$\mathcal{S}(v_0, S_0) := \begin{cases} \Big\{(v, \mathbb{S}(v))/v_0 \le v < \varphi^{\sharp}(v_0, S_0)\Big\} \cup \Big\{\big(\varphi^{\flat}_{\alpha}(v_0, S_0), S^{\flat}(v_0, S_0)\big)\Big\}, & v_0 < \underline{v}(S_0), \\ \Big\{(v, \mathbb{S}(v))/v_0 \le v < \varphi^{\sharp}_0(v_0, S_0)\Big\}, & \underline{v}(S_0) < v_0 < v_\star(S_0). \end{cases}$$

Now, we generalize a formula for the kinetic function which was derived in [5] for the 2×2 model.

Theorem 4.3 (Kinetic Function). *Given $\alpha > 0$ and S_0, the function*

$$v_0 \in (0, \underline{v}(S_0)) \mapsto \varphi^{\flat}_{\alpha}(v_0, S_0)$$

fails to be globally monotone. More precisely, it satisfies

$$\frac{\partial \varphi^{\flat}_{\alpha}}{\partial v_0}(\underline{v}(S_0), S_0) = \frac{\varepsilon_{vv}(\underline{v}(S_0), S_0)}{\varepsilon_{vv}(\overline{v}(S_0), S_0)}\Big(1 - \frac{\varepsilon_{vS}(\overline{v}(S_0), S_0)}{\varepsilon_S(\overline{v}(S_0), S_0)}\,(\overline{v}(S_0) - \underline{v}(S_0))\Big) > 0 \tag{4.1}$$

with

$$\lim_{v_0 \to 0} \varphi^{\flat}_{\alpha}(v_0, S_0) = +\infty. \tag{4.2}$$

Proof. Given, $\alpha > 0$ and a real S_0, the function $v_0 \mapsto \varphi^{\flat}_{\alpha}(v_0, S_0)$ satisfies the implicit relation

$$\varepsilon_v(\varphi^{\flat}_{\alpha}(v_0, S_0), S^{\flat}(v_0, S_0)) - \varepsilon_v(v_0, S_0) = \lambda^{\flat}(v_0, S_0)^2\,(\varphi^{\flat}_{\alpha}(v_0, S_0) - v_0). \tag{4.3}$$

By differentiating the last equation with respect to v_0 we get

$$\begin{aligned} &\varepsilon_{vv}(\varphi^{\flat}_{\alpha}, S^{\flat})\,\frac{\partial \varphi^{\flat}_{\alpha}}{\partial v_0}(v_0, S_0) + \varepsilon_{vS}(\varphi^{\flat}_{\alpha}, S^{\flat})\,\frac{\partial S^{\flat}}{\partial v_0}(v_0, S_0) - \varepsilon_{vv}(v_0, S_0) \\ &= \frac{\partial}{\partial v_0}\Big(\lambda^{\flat}(v_0, S_0)^2\,(\varphi^{\flat}_{\alpha}(v_0, S_0) - v_0)\Big). \end{aligned} \tag{4.4}$$

First, let us prove that $\frac{\partial \lambda^{\flat 2}}{\partial v_0}(\underline{v}(S_0), S_0) = 0$.

Indeed, multiplying (3.16) by h we get

$$h\,\frac{dh}{dv} + 2\,\frac{\alpha\nu\lambda^\flat\,h}{\sqrt{\mu}}\,\frac{1}{1+\frac{\mu_S}{2\mu\varepsilon_S}h^2} = \varepsilon_v(v,\mathbf{S}) - \varepsilon_v(v_0,S_0) - {\lambda^\flat}^2\,(v-v_0)$$

Integrating the last equation over $(v_0, \varphi^\flat_\alpha(v_0,S_0))$ we obtain

$$\begin{aligned}&2\,\alpha\,\lambda^\flat(v_0,S_0)\int_{v_0}^{\varphi^\flat_\alpha(v_0,S_0)} \frac{\nu\,h}{\sqrt{\mu}}\,\frac{1}{1+\frac{\mu_S}{2\mu\varepsilon_S}h^2}\,d\,v\\ &= \int_{v_0}^{\varphi^\flat_\alpha(v_0,S_0)} (\varepsilon_v(v,\mathbf{S}) - \varepsilon_v(v_0,S_0))\,d\,v - \frac{1}{2}\lambda^\flat(v_0,S_0)^2\,(\varphi^\flat_\alpha(v_0,S_0)-v_0)^2.\end{aligned}$$

Now, since $\lambda^\flat(v_0,S_0)\to 0$ when $v_0\to \underline{v}(S_0)$ we can write

$$2\,\alpha\,\lambda^\flat(v_0,S_0) \sim \frac{N(v_0,S_0)}{D(v_0,S_0)},$$

where

$$\begin{aligned}N(v_0,S_0) &:= \int_{v_0}^{\varphi^\flat_\alpha(v_0,S_0)} (\varepsilon_v(v,\mathbf{S}) - \varepsilon_v(v_0,S_0))\,dv,\\ D(v_0,S_0) &:= \int_{v_0}^{\varphi^\flat_\alpha(v_0,S_0)} \frac{\nu\,h}{\sqrt{\mu}}\,\frac{1}{1+\frac{\mu_S}{2\mu\varepsilon_S}h^2}\,dv.\end{aligned}$$

It is clear that $N(\underline{v}(S_0),S_0)=0$ and $\frac{\partial N}{\partial v_0}(\underline{v}(S_0),S_0)$ exists and $D(\underline{v}(S_0),S_0)>0$. We deduce that $\frac{\partial\lambda^\flat}{\partial v_0}(\underline{v}(S_0),S_0)$ is finite and then

$$\frac{\partial{\lambda^\flat}^2}{\partial v_0}(\underline{v}(S_0),S_0) = 2\,\lambda^\flat(\underline{v}(S_0),S_0)\,\frac{\partial\lambda^\flat}{\partial v_0}(\underline{v}(S_0),S_0) = 0. \tag{4.5}$$

Now, by differentiating the identity

$$\begin{aligned}&f(v_0,S_0,\varphi^\flat_\alpha(v_0,S_0),S^\flat(v_0,S_0),\lambda^\flat(v_0,S_0))^2\\ &= \varepsilon(\varphi^\flat_\alpha,S^\flat) - \varepsilon(v_0,S_0) - \varepsilon_v(v_0,S_0)\,(\varphi^\flat_\alpha - v_0) - \frac{1}{2}{\lambda^\flat}^2(v_0,S_0)\,(\varphi^\flat_\alpha - v_0)^2 = 0\end{aligned}$$

with respect to v_0, and using (2.16) and (4.5), we obtain

$$\frac{\partial S^\flat}{\partial v_0}(\underline{v}(S_0),S_0) = \frac{\varepsilon_{vv}(\underline{v}(S_0),S_0)}{\varepsilon_S(\underline{v}(S_0),S_0)}((\overline{v}(S_0),S_0)-(\underline{v}(S_0),S_0)). \tag{4.6}$$

Finally, plugging (4.6) and (4.5) in (4.4) we obtain (4.1).

Now, we have to prove (4.2). First, we know that

$$\lambda^\flat(v_0,S_0)^2 \le \varepsilon_{vv}(\varphi^\flat_\alpha,S^\flat) \quad\text{and}\quad \varepsilon_v(v^\star(S^\flat),S^\flat)\le\varepsilon_v(\varphi^\flat_\alpha,S^\flat).$$

Then

$$\varepsilon_v(v^\star(S^\flat),S^\flat) - \varepsilon_v(v_0,S_0) \le \varepsilon_v(\varphi^\flat_\alpha,S^\flat) - \varepsilon_v(v_0,S_0) \le \varepsilon_{vv}(\varphi^\flat_\alpha,S^\flat)(\varphi^\flat_\alpha - v_0).$$

Finally, using (1.8c) and (1.10) we deduce (4.2).

To conclude, let us point out the following asymptotic properties of the kinetic function which are deduced from Theorem 3.9 and the continuity properties of solutions of (2.7) with respect to parameters α (and β).

Theorem 4.4 (Asymptotic Properties). *For every fixed $S_0>0$ we have the following*:

(1) *There exists a continuous function* $\kappa^\natural : (0, v^{-\star}(S_0)) \mapsto (0, +\infty)$ *such that*

$$\begin{aligned} &\varphi^\flat_\alpha(v_0, S_0) = \varphi^\natural(v_0, S_0) \qquad \textit{provided } \alpha\, \kappa^\natural(v_0, S_0) \geq 1, \\ &\kappa^\natural(v_0, S_0) \to +\infty \qquad \textit{as } v_0 \to v^{-\star}(S_0). \end{aligned} \tag{4.7}$$

(2) *For each* $v < \underline{v}(S_0)$ *we have*

$$\varphi^\flat_\alpha(v_0, S_0) \to \varphi^\flat_0(v_0, S_0) \qquad \textit{as } \alpha \to 0.$$

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